Linear Algebra: Example Sheet 4

The first 10 questions cover the course and should give good understanding. The remainder are of varying length and difficulty and may interest some.

1. (i) Let U, V be finite dimensional vector spaces and suppose $\beta : U \times V \to F$ is a bilinear map. Show that for any $X \leq U$ we have
$$\dim X + \dim X^{\perp} \geq \dim V \ .$$
Show that equality holds if β is non-degenerate. (Can you give a necessary and sifficient condition?)
(ii) Suppose that β is a bilinear form on V. Take $U \leq V$ with $U = W^{\perp}$ for some $W \leq V$. Suppose that $\psi|_U$ is non-singular. Show that ψ is non-singular.

2. Which of the following symmetric matrices are congruent to the identity matrix (a) over $\mathbb{C}$, (b) over $\mathbb{R}$ and (c) over $\mathbb{Q}$? (Try to get away with the minimum calculation.)
$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 4 & 4 \\ 4 & 5 \end{pmatrix}.$$

3. Find the rank and signature of the following quadratic forms over $\mathbb{R}$.
$$x^2 + y^2 + z^2 - 2xz - 2yz, \quad x^2 + 2y^2 - 2z^2 - 4xy - 4yz, \quad 16xy - z^2, \quad 2xy + 2yz + 2zx.$$
If B is the matrix of the form then there exists non-singular Q with Q^tBQ diagonal with entries ± 1. Find such a Q in some representative cases.

4. The map $A \to \mathrm{tr}(A^2)$ is a quadratic form on $\mathrm{Mat}_n(\mathbb{R})$, the $n \times n$ matrices. Find its rank and signature.

5. (i) Show that the quadratic form $2(x^2 + y^2 + z^2 + xy + yz + zx)$ is positive definite.
(ii) *Write down* an orthonormal basis for the corresponding inner product on $\mathbb{R}^3$.
(iii) Compute the basis obtained by applying the Gram-Schmidt process to the standard basis.

6. An endomorphism π of a vector space V is *idempotent* just when $\pi^2 = \pi$. Let $W \leq V$ with V an inner product space. Show that the orthogonal projection onto W is a self-adjoint idempotent. Conversely show that any self-adjoint idempotent is orthogonal projection onto its image.

7. Let S be a real symmetric matrix with $S^k = I$ for some $k \geq 1$. Show that $S^2 = I$.

8. Let $a_1, a_2, \ldots , a_n$ be real numbers such that $a_1 + \cdots + a_n = 0$ and $a_1^2 + \cdots a_n^2 = 1$. What is the maximum value of $a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1$?

9. An endomorphism α of a finite-dimensional inner product space V is *positive definite* if and only if it is self-adjoint and satisfies $\langle \mathbf{x}, \alpha(\mathbf{x}) \rangle > 0$ for all non-zero $\mathbf{x} \in V$.
(i) Prove that a positive definite endomorphism has a unique positive definite square root.
(ii) Let α be a non-singular endomorphism of V with adjoint α^*. By considering $\alpha^*\alpha$ show that α can be factored as $\beta\gamma$ with β unitary and γ positive definite.
(iii) Can you say anything for a general endomorphism α?

10. Find a linear transformation which reduces the pair of real quadratic forms
$$2x^2 + 3y^2 + 3z^2 - 2yz, \qquad x^2 + 3y^2 + 3z^2 + 6xy + 2yz - 6zx$$
to the forms
$$X^2 + Y^2 + Z^2, \qquad \lambda X^2 + \mu Y^2 + \nu Z^2$$
for some $\lambda, \mu, \nu \in \mathbb{R}$.
Does there exist a linear transformation which reduces the quadratic forms $x^2 - y^2$ and $2xy$ simultaneously to diagonal form?

11. Suppose that Q is a non-singular quadratic form on V of dimension $2m$. Suppose that Q vanishes on $U \leq V$ with $\dim U = m$. Establish the following.
(i) We can write $V = U \oplus W$ with q also vanishing on W.
(ii) There is a basis with respect to which Q has the form $x_1x_2 + x_3x_4 + \cdots x_{2m-1}x_{2m}$.

12. Find the rank and signature of the form on $\mathbb{R}^n$ with matrix

$$\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & & & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}.$$

13. Let $f_1, \cdots, f_t, f_{t+1}, \cdots, f_{t+u}$ be linear functionals on the finite dimensional real vector space V. Show that $Q(\mathbf{x}) = f_1(\mathbf{x})^2 + \cdots + f_t(\mathbf{x})^2 - f_{t+1}(\mathbf{x})^2 - \cdots - f_{t+u}(\mathbf{x})^2$ is a quadratic form on V. Suppose Q has rank $p+q$ and signature $p-q$. Show that $p \leq t$ and $q \leq u$.

14. Let P_n be the ($n+1$-dimensional) space of real polynomials of degree $\leq n$. Define

$$\langle f, g \rangle = \int_{-1}^{+1} f(t)g(t)dt\,.$$

Show that $\langle\ ,\ \rangle$ is an inner product on P_n and that the endomorphism $\alpha : P_n \to P_n$ defined by

$$\alpha(f)(t) = (1-t^2)f''(t) - 2tf'(t)$$

is self-adjoint. What are the eigenvalues of α?
Let $s_k \in P_n$ be defined by $s_k(t) = \frac{d^k}{dt^k}(1-t^2)^k$. Prove the following.
(i) For $i \neq j$, $\langle s_i, s_j \rangle = 0$.
(ii) $s_0, \dots, s_n$ forms a basis for for P_n.
(iii) For all $1 \leq k \leq n$, s_k spans the orthogonal complement of P_{k-1} in P_k.
(iv) s_k is an eigenvector of α. (Give its eigenvalue.)
What is the relation between the s_k and the result of applying Gram-Schmidt to the sequence 1, x, x^2, x^3 and so on? (Calculate the first few terms?)

15. Prove Hadamard's Inequality: if A is a real $n \times n$ matrix with $|a_{ij}| \leq k$, then

$$|\det A| \leq k^n n^{n/2}\ .$$

16. Consider the space P of polynomials in variables x_1, ... , x_n. We have linear operators $\partial_i = \frac{\partial}{\partial x_i}$; so for any polynomial $f(x_1, \dots, x_n) \in P$ we have a corresponding linear operator $\hat{f} = f(\partial_1, \dots, \partial_n)$. Consider

$$\langle f, g \rangle = \hat{f}(g)(\mathbf{0})\,,$$

that is the result of applying $f(\partial_0, \dots, \partial_n)$ to $g(x_1, \dots, x_n)$ and then evaluating at $(0, \dots, 0)$. Show that $\langle f, g \rangle$ is an inner product on P.
Fix $g \in P$. What is the adjoint of the map $P \to P; h \to gh$?
Now consider the subspaces $P(d)$ of polynomials homogeneous of degree d. Show that the Laplacian $\Delta = \partial_1^2 + \cdots + \partial_n^2 : P(d) \to P(d-2)$ is surjective.

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