Linear Algebra: Example Sheet 3

1. Show that none of the following matrices are conjugate:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   Is the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

   conjugate to any of them? If so, which?

2. Let A be a complex 5×5 matrix with $A^4 = A^2 \neq A$. What are the possible minimum and characteristic polynomials? What are the possible JNFs?

3. Show that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ form a basis for $\mathbb{R}^3$. Find the dual basis for $\mathbb{R}^{3*}$.

4. Let V be a 4-dimensional vector space over $\mathbb{R}$, and let $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ be the basis of V^* dual to the basis $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ for V. Determine, in terms of the ξ_i, the bases dual to each of the following:
   (a) $\{\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_3\}$;
   (b) $\{\mathbf{x}_1, 2\mathbf{x}_2, \frac{1}{2}\mathbf{x}_3, \mathbf{x}_4\}$;
   (c) $\{\mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_2 + \mathbf{x}_3, \mathbf{x}_3 + \mathbf{x}_4, \mathbf{x}_4\}$;
   (d) $\{\mathbf{x}_1, \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_2 + \mathbf{x}_1, \mathbf{x}_4 - \mathbf{x}_3 + \mathbf{x}_2 - \mathbf{x}_1\}$;
   (e) $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4\}$.

5. Show that if $\mathbf{x} \neq \mathbf{y}$ are vectors in the finite dimensional vector space V, then there is a linear functional $\theta \in V^*$ such that $\theta(\mathbf{x}) \neq \theta(\mathbf{y})$.

6. Suppose that V is finite dimensional. Let $A, B \leq V$. Prove that $A \leq B$ if and only if $A^o \geq B^o$. Show that $A = V$ if and only if $A^o = \{\mathbf{0}\}$. Deduce that a subset $F \subset V^*$ of the dual space spans V^* just when $f(\mathbf{v}) = 0$ for all $f \in F$ implies $\mathbf{v} = \mathbf{0}$.

7. Let P_n be the space of real polynomials of degree at most n. For $x \in \mathbb{R}$ define $\varepsilon_x \in P_n^*$ by $\varepsilon_x(p) = p(x)$. Show that $\varepsilon_0, \ldots, \varepsilon_n$ form a basis for P_n^*, and identify the basis of P_n to which it is dual.

8. Suppose that U and V are finite dimensional vector spaces. Take $\theta \in U^*$ and $\phi \in V^*$. Show that $\psi(\mathbf{x}, \mathbf{y}) = \theta(\mathbf{x}).\phi(\mathbf{y})$ defines a bilinear form of rank 0 or 1. When is the rank 0? Show that any bilinear form $\psi : U \times V \to F$ of rank 1 can be expressed as $\psi(\mathbf{x}, \mathbf{y}) = \theta(\mathbf{x}).\phi(\mathbf{y})$ for some θ and ϕ.

9. Let $\phi : U \times V \to F$ and $\psi : U \times V \to F$ be bilinear forms on the finite dimensional vector spaces U and V. Suppose ψ is non-singular. Show that there are linear maps $\alpha : U \to U$ and $\beta : V \to V$ with

$$\phi(\mathbf{u}, \mathbf{v}) = \psi(\alpha(\mathbf{u}), \mathbf{v}) = \psi(\mathbf{u}, \beta(\mathbf{v})) \, .$$

10. Suppose that ψ is a bilinear form on V. Take $U \leq V$ with $U = W^\perp$ some $W \leq V$. Suppose that $\psi|_U$ is non-singular. Show that ψ is non-singular.

11. Let U, V be finite dimensional and suppose $\psi : U \times V \to F$ is a bilinear form. Show that for any $X \leq U$ we have

$$\dim X + \dim X^\perp \geq \dim V \, .$$

    Show that equality holds if ψ is non-degenerate.

12. Now let $\psi : V \times V \to F$ be a bilinear form; take $U \leq V$ and let $\tilde{\psi} = \psi|_U : U \times U \to F$ be the restriction of ψ to U. Show that $\tilde{\psi}$ is non-singular if and only if $U \oplus U^{\perp} = V$.
Is it the case that ψ non-singular implies $\tilde{\psi}$ non-singular?
Is it the case that $\tilde{\psi}$ non-singular implies ψ non-singular?

---

13. Find a basis with respect to which $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$ has JNF. Hence compute $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}^n$.

14. Let θ and ϕ be linear functionals on V with the property that $\theta(\mathbf{x}) = 0$ if, and only if, $\phi(\mathbf{x}) = 0$. Show that θ and ϕ are scalar multiples of each other.

15. Show that the dual of the space P of real polynomials is isomorphic to the space $\mathbb{R}^{\mathbb{N}}$ of all sequences of real numbers, via the mapping which sends a linear form $\xi : P \to \mathbb{R}$ to the sequence $(\xi(1), \xi(t), \xi(t^2), \ldots)$.

    In terms of this identification, describe the effect on a sequence $(a_0, a_1, a_2, \ldots)$ of the linear maps dual to each of the following linear maps $P \to P$:
    (a) The map D defined by $D(p)(t) = p'(t)$.
    (b) The map S defined by $S(p)(t) = p(t^2)$.
    (c) The map E defined by $E(p)(t) = p(t-1)$.
    (d) The composite DS.
    (e) The composite SD.

    Verify that $(DS)^* = S^*D^*$ and $(SD)^* = D^*S^*$.

16. For A an $n \times m$ and B an $m \times n$ matrix over the field $\mathbb{F}$, let $\tau_A(B)$ denote $\mathrm{tr}AB$.
Show that, for each fixed A, τ_A is a linear map $\mathcal{M}_{m\times n} \to \mathbb{F}$.
Now consider the mapping $A \mapsto \tau_A$. Show that it is a linear isomorphism $\mathcal{M}_{n\times m} \to \mathcal{M}^*_{m\times n}$.

17. Let $\alpha : V \to V$ be an endomorphism of a finite dimensional complex vector space and let $\alpha^* : V^* \to V^*$ be its dual. Show that a complex number λ is an eigenvalue for α if, and only if, it is an eigenvalue for α^*. How are the algebraic and geometric multiplicities of λ for α and α^* related? How are the minimal and characteristic polynomials for α and α^* related?

18. Suppose that $\psi : U \times V \to F$ is a bilinear form on U, V finite dimensional vector spaces. Show that there exist bases $\mathbf{e}_1, \ldots, \mathbf{e}_m$ for U and $\mathbf{f}_1, \ldots, \mathbf{f}_n$ for V such that when $\mathbf{x} = \sum_1^m x_i \mathbf{e}_i$ and $\mathbf{y} = \sum_1^n y_j \mathbf{f}_j$ we have $\psi(\mathbf{x}, \mathbf{y}) = \sum_1^r x_k y_k$, where r is the rank of ψ. What are the dimensions of the left and right kernels of ψ?

19. Find the left and right kernels of the bilinear form with matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

    with respect to the standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_4$. Let $V = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$. Find $V^{\perp}$ and ${}^{\perp}V$. Give a basis $\mathbf{f}_1, \ldots, \mathbf{f}_4$ with respect to which the bilinear form has the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

20. Let $P_2 = P_2(x, y)$ be the space of polynomials in x, y of degree ≤ 2 in each variable. (So $\dim P_2 = 9$.)
(i) What is the JNF of the map $P_2 \to P_2; f(x, y) \mapsto \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$?
(ii) What is the JNF of the map $P_2 \to P_2; f(x, y) \mapsto f(x + 1, y + 1)$?
(iii) So what do you think the answers are for P_n?

Comments, corrections and queries can be sent to me at `m.hyland@dpmms.cam.ac.uk`.