Michaelmas Term 2013 J. M. E. Hyland

Linear Algebra: Example Sheet 2

The first 12 questions cover the course and should ensure good understanding. The remainder vary in difficulty but cover some instructive points.

1. Show that an $n \times n$ matrix is invertible if and only if it is a product of elementary matrices. Determine which of the following matrices are invertible, and find the inverses when they exist.

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 3 & 0 \end{pmatrix}.$$

2. Let A and B be $n \times n$ matrices over a field $\mathbb{F}$. Show that the $(2n \times 2n)$ matrix

$$C = \begin{pmatrix} I & B \\ -A & O \end{pmatrix} \quad \text{can be transformed into} \quad D = \begin{pmatrix} I & B \\ 0 & AB \end{pmatrix}$$

by elementary row operations. By considering the determinants of C and D, obtain another proof that $\det AB = \det A \det B$.

3. Compute the determinant of the $n \times n$ matrix whose entries are λ down the diagonal and 1 elsewhere.

4. Let A, B be $n \times n$ matrices, where $n \geq 2$. Show that, if A and B are non-singular, then

$$(i)\ \mathrm{adj}(AB) = \mathrm{adj}(B)\mathrm{adj}(A), \quad (ii)\ \det(\mathrm{adj}A) = (\det A)^{n-1}, \quad (iii)\ \mathrm{adj}(\mathrm{adj}A) = (\det A)^{n-2}A.$$

What happens if A is singular?

Show that the rank of the matrix $\mathrm{adj}A$ is $\mathrm{r}(\mathrm{adj}(A)) = \begin{cases} n & \text{if } \mathrm{r}(A) = n; \\ 1 & \text{if } \mathrm{r}(A) = n-1; \\ 0 & \text{if } \mathrm{r}(A) \leq n-2. \end{cases}$

5. (i) Suppose that V is a non-trivial finite dimensional real vector space. Show that there are no endomorphisms α, β of V with $\alpha\beta - \beta\alpha = I$.
(ii) Find endomorphisms of the space of infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$ which do satisfy $\alpha\beta - \beta\alpha = I$.

6. Compute the characteristic polynomials of the matrices

$$\begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Which of the matrices are diagonalizable over $\mathbb{C}$? Which over $\mathbb{R}$?

7. Find the eigenvalues and give bases for the eigenspaces of the following complex matrices:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The second and third matrices commute, so find a basis with respect to which they are both diagonal.

8. Suppose that $\alpha \in \mathcal{L}(V, V)$ is invertible. Describe the characteristic and minimal polynomials and the eigenvalues of α^{-1} in terms of those of α.

9. Let α be an endomorphism of a finite dimensional complex vector space. Show that if λ is an eigenvalue for α then λ^2 is an eigenvalue for α^2. Show further that every eigenvalue of α^2 arises in this way. [*This result fails for real vector spaces. Why is that?*] Are the eigenspaces $\ker(\alpha - \lambda I)$ and $\ker(\alpha^2 - \lambda^2 I)$ necessarily the same?

10. Show that an endomorphism $\alpha : V \to V$ of a finite dimensional complex vector space V has 0 as only eigenvalue if and only if it is *nilpotent*, that is, $\alpha^k = 0$ for some natural number k. Show that the minimum such k is at most $\dim(V)$. What can you say if the only eigenvalue of α is 1?

11. (i) An endomorphism $\alpha : V \to V$ of a finite dimensional vector space is *periodic* just when $\alpha^k = I$ for some k. Show that a periodic matrix is diagonalisable over $\mathbb{C}$.
(ii) Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be a basis for a vector space V over $\mathbb{C}$. For σ a permutation of $\{1, \dots, n\}$, define $\widehat{\sigma} : V \to V$ by $\widehat{\sigma}(\mathbf{e}_i) = \mathbf{e}_{\sigma(i)}$. What are the eigenvalues of $\widehat{\sigma}$?
(iii) Is every periodic endomorphism of the form $\widehat{\sigma}$ for some choice of permutation σ and basis $\mathbf{e}_1, \dots, \mathbf{e}_n$?

12. Show that if two $n \times n$ real matrices P and Q are conjugate when regarded as matrices over $\mathbb{C}$, then they are conjugate as matrices over $\mathbb{R}$.

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13. Let $f(x) = a_0 + a_1 x + \ldots + a_n x^n$, with $a_i \in \mathbb{C}$, and let C be the *circulant* matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ a_n & a_0 & a_1 & \dots & a_{n-1} \\ a_{n-1} & a_n & a_0 & \dots & a_{n-2} \\ \vdots & & & \ddots & \vdots \\ a_1 & a_2 & a_3 & \dots & a_0 \end{pmatrix}.$$

Show that the determinant of C is $\det C = \prod_{j=0}^{n} f(\zeta^j)$, where $\zeta = \exp(2\pi i/(n+1))$.

14. Let A be an $n \times n$ matrix all the entries of which are real. Show that the minimum polynomial of A, over the complex numbers, has real coefficients.

15. Suppose that $\alpha : \mathbb{C}^n \to \mathbb{C}^n$ has eigenvalues $\lambda_1, \dots, \lambda_n$. Regard $\mathbb{C}^n \cong \mathbb{R}^{2n}$ as a $2n$-dimensional real vector space, and consider the corresponding endomorphism $\alpha : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. What are the complex eigenvalues of this α?

16. Let $\alpha : V \to V$ be an endomorphism of a finite dimensional real vector space V with $\mathrm{tr}(\alpha) = 0$.
(i) Show that, if $\alpha \neq 0$, there is a vector $\mathbf{v}$ with $\mathbf{v}, \alpha(\mathbf{v})$ linearly independent. Deduce that there is a basis for V relative to which α is represented by a matrix A with all of its diagonal entries equal to 0.
(ii) Show that there are endomorphisms β, γ of V with $\alpha = \beta\gamma - \gamma\beta$.

Comments, corrections and queries can be sent to me at `m.hyland@dpmms.cam.ac.uk`.