Topological spaces, limit spaces and continuous lattices.

(TOP, LIM, CL will denote the respective categories).

Recall the injection $L : TOP \longrightarrow LIM$ with its left adjoint $T : LIM \longrightarrow TOP$. It is easy to show that LIM is cartesian closed.

An object Y of TOP (or any other category) has function spaces iff the functor $(- \times Y)$ has a right adjoint $[Y, -]$ (i.e. for all X, Z, $Mor_{TOP}(X \times Y, Z) \cong Mor_{TOP}(X, [Y, Z])$ in the natural way).

We also have the notion $[Y, Z]$ is a function space iff $[Y, Z]$ is the set of continuous maps from Y to Z so topologized that for all X, $Mor_{TOP}(X \times Y, Z) \cong Mor_{TOP}(X, [Y, Z])$ naturally. (This $[Y, Z]$ exists as function space iff the (contravariant) functor $Mor_{TOP}(- \times Y, Z)$ is "(co)-representable").

Theorem 1. L preserves what function spaces there are in TOP.

Proof:- Take A, B in TOP, X in LIM, and let $[A, B]$ be a function space.

Now, $Mor_{LIM}(X, L([A,B])) \cong Mor_{TOP}(TX, [A,B]) \cong Mor_{TOP}(TX \times A, B)$
$\cong Mor_{LIM}(L(TX \times A), LB) \cong Mor_{LIM}(LTX, [LA, LB])$.

The identity (underlying set) map, $id : L([A,B]) \longrightarrow [LA, LB]$ is clearly continuous (it comes from $ev : [A,B] \times A \longrightarrow B$). So suppose $id : [LA, LB] \longrightarrow L([A,B])$ is not continuous: then there is a converging filter Φ in $[LA, LB]$ which does not converge in $L([A,B])$ i.e. $\Phi \downarrow x$ but $\Phi \not\supseteq N_x$, i.e. there is U open $x \in U$ such that $U \notin \Phi$: for each $W \in \Phi$ pick $x_W \in W \setminus U$. Consider $\{x_W \mid W \in \Phi\} \cup \{x\}$ with topology discrete on all but x and with for any $W \in \Phi$ $\{x_V \mid V \subseteq W \ \& \ V \in \Phi\} \cup \{x\}$ open. The obvious embedding is continuous with respect to $[LA, LB]$, but not with respect to $L([A,B])$. But for topological C, $Mor_{LIM}(LC, L([A,B])) \cong Mor_{LIM}(LC, [LA, LB])$ by above which is a contradiction. Thus $id : [LA, LB] \longrightarrow L([A,B])$ is continuous so L preserves function spaces #.

Remark Theorem 1 has another less concrete proof based on identifying LIM as the category obtained from the category of cribles on TOP by the application of a certain sheafification.

A po-set $\langle P, \leq \rangle$ may be topologized by taking as open sets those $O \subseteq P$ such that (i) $p \geq q \in O \Rightarrow p \in O$, and (ii) if S is directed in P and $\bigvee S$ exists and is in O then some element of S is in O. We refer to this topology as the <u>Scott topology</u>.

Define $x \prec y$ iff y is in the Scott interior of $\{z \mid x \leq z\}$. A <u>continuous lattice</u> is a complete lattice $\langle D, \leq \rangle$ such that for all $y \in D$, $y = \bigvee \{x \mid x \prec y\}$. We shall refer to the corresponding topological spaces as continuous lattices (tho' better would be injective T_0-spaces).

<u>Theorem 2</u> Let $X \in TOP$ and $D \in CL$; then $T([X,D]_{LIM})$ ($[LX, LD]_{LIM}$) is the obvious lattice $[X,D]$ with the Scott topology.

Proof:- The lattice $[X,D]$ has $f \leq g$ iff $(\forall x \in X)(f(x) \leq g(x))$. Trivial facts about continuous lattices show this is a complete lattice.

Suppose H is open in the induced topology on $[X,D]_{LIM}$. To show that H is Scott open, let $f = \bigvee g_\alpha \in H$; we wish to show some finite $g_{\alpha_1} \vee \dots \vee g_{\alpha_n} \in H$. But consider the filter $\mathcal{H}$ generated by $\{g \mid g \geq g_\alpha\}$; we claim that this converges to f. For take $x \in X$ and $V \ni f(x)$; $\bigvee g_\alpha(x) \in V$ so some $g_{\alpha_1} \vee \dots \vee g_{\alpha_n}(x) = \bar{g}(x) \in V$; $\{g \mid g \geq \bar{g}\} \in \mathcal{H}$ & so $[\bar{g}^{-1}(V), V] \in \mathcal{H}$ thus some neighbourhood of x is mapped inside V by $\mathcal{H}$ & this shows that $\mathcal{H}$ converges to f. But now $H \in \mathcal{H}$, so there is $g_{\alpha_1} \vee \dots \vee g_{\alpha_m} = g'$ such that $\{g \mid g \geq g'\} \subseteq H$, so $g' \in H$; which is what was required to show H Scott open.

Suppose now that H is Scott open. Take $\mathcal{H} \downarrow f \in H$. Now $\mathcal{H} \downarrow f$ iff $(\forall x \in X)(\forall V \ni f(x))(\exists U_{x,V} \subseteq X \text{ nbd of } x)\{[U_{x,V}, V] \in \mathcal{H}\}$:
and since $\{\underbrace{\mathrm{Int}(\{d' \mid d' \geq d\})}_{V_d}\}_{d \in D}$ is a basis for the Scott topology,

$\mathcal{H} \downarrow f$ iff $(\forall x)(\forall d \prec f(x))(\exists U_{x,d} \text{ nbd of } x)\{[U_{x,d}, V_d] \in \mathcal{H}\}$.

In this situation, let $g_{x,d} = \bigwedge [U_{x,d}, V_d]$ and observe that if $y \in U_{x,d}$ $g_{x,d}(y) \geq d$.

Since D is continuous lattice $f(x) = \bigvee \{d \mid d \prec f(x)\}$; so $f \leq \bigvee \{g_{x,d} \mid x \in X \ \&\ d \prec f(x)\}$. Since H is Scott open there are $g_{x_1,d_1} \vee \dots \vee g_{x_n,d_n} = \bar{g} \in H$ so $\{g \mid g \geq \bar{g}\} \subseteq H$. But $\{g \mid g \geq \bar{g}\} \supseteq [U_{x_1,d_1}, V_{d_1}] \wedge \dots \wedge [U_{x_n,d_n}, V_{d_n}] \in \mathcal{H}$. Hence $H \in \mathcal{H}$. This is what was required to show that H is open in the induced topology on $[X,D]_{LIM}$.

Let $\mathbb{O}$ be the two-point Sierpinski space. Then $[X,\mathbb{O}]$ is a complete lattice (Heyting algebra) of open sets of $\mathbb{O}$.

<u>Corollary 3</u> (a) If $[X,\mathbb{O}]$ is a function space it has the Scott topology.

(b) If $[X,\mathbb{O}]$ has the Scott topology, then for any Y, $\mathrm{Mor}_{TOP}(X\times Y,\mathbb{O}) \xrightarrow{\cong}$ (naturally included in) $\mathrm{Mor}_{TOP}(Y,[X,\mathbb{O}])$.

Proof:- ad (a), if $[X,\mathbb{O}]$ is a function space it is $TL([X,\mathbb{O}])$ by Thm 1 which is Scott topology on $[X,\mathbb{O}]$ by Thm 2.
ad (b) $\mathrm{Mor}_{TOP}(X\times Y,\mathbb{O}) \cong \mathrm{Mor}_{LIM}(LX\times LY, L\mathbb{O}) \cong \mathrm{Mor}_{LIM}(LY,[LX,L\mathbb{O}])$

$\subseteq \mathrm{Mor}_{TOP}(Y, T[LX,L\mathbb{O}])$

Scott topology.

<u>Proposition 4</u>. If $[X,\mathbb{O}]$ is a continuous lattice it is a function space.

Proof:- In view of Corol 3(b), it suffices to take continuous $f: Y \to [X,\mathbb{O}]$ and show $\{(x,y) \mid x\in f(y)\}$ is open in $X\times Y$.
Let $x\in f(y)$; $f(y) = \bigvee\{U \mid U \prec f(y)\}$ so there is $U \prec f(y)$ with $x\in U$; but then $(x,y)\in U\times f^{-1}(\mathrm{Int}\{V \mid V\supseteq U\})$ which is included in $\{(a,b)\mid a\in f(b)\}$.

<u>Proposition 5</u>. If $[X,\mathbb{O}]$ is a function space, then X has function spaces.

Proof:- Topologize $[X,Z]$ with coarsest topology such that all $[X,Z]\xrightarrow{[1,V]}[X,\mathbb{O}]$; $f\to f^{-1}(V)$ are continuous. Bare hands shows this works, but more subtly, Z is a limit of $\mathbb{O}$'s (and of the 2 point space with trivial topology which we forget) and we have made $[X,Z]$ the same limit of $[X,\mathbb{O}]$'s which it must be if $[X,-]$ to be right adjoint. For any Y, $\mathrm{Mor}_{TOP}(Y,-)$: TOP $\to$ SETS is a right adjoint. Hence,

$\mathrm{Mor}(Y,[X,Z]) \cong \mathrm{Mor}(Y,\varprojlim[X,\mathbb{O}]) \cong \varprojlim \mathrm{Mor}(Y,[X,\mathbb{O}])$

$\cong \varprojlim \mathrm{Mor}(X\times Y,\mathbb{O}) \cong \mathrm{Mor}(X\times Y, \varprojlim \mathbb{O}) \cong \mathrm{Mor}(X\times Y, Z)$.

Proposition 6. If $ev: [X, \mathcal{O}]_{\text{Scott top.}} \times X \longrightarrow \mathcal{O}$ is continuous, then $[X, \mathcal{O}]$ is a continuous lattice.

Proof:- Take $U \in [X, \mathcal{O}]$. For any $x \in U$ $ev((U,x)) = T$, so there is open $\mathcal{W} \subseteq [X, \mathcal{O}]$, $V \subseteq X$ with $(U,x) \in \mathcal{W} \times V$ and $(\forall W \in \mathcal{W})(V \subseteq W)$. Then $U \succ V \ni x$. Since x was arbitrary, this shows that $U = \bigcup\{V \mid V \prec U\}$, and so $[X, \mathcal{O}]$ is in CL.

Theorem 7. The following are equivalent:

1) $[X, \mathcal{O}]$ is a continuous lattice,
2) $[X, \mathcal{O}]$ can be topologized to be a function space,
3) $[X, \mathcal{O}]_{\text{Scott top.}}$ is a function space,
4) X has function spaces.

Proof:- 1) $\Rightarrow$ 3) by Prop. 4; 3) $\Rightarrow$ 4) by Prop 5; 4) $\Rightarrow$ 1) by Prop. 6; 3) $\Rightarrow$ 2) trivial; 2) $\Rightarrow$ 3) by Corol 3 (a).

A space is <u>properly locally compact</u> iff each point has a "fundamental system of compact neighbourhoods", (i.e. the compact neighbourhoods form a filter base generating the neighbourhood filter). [Note that if a space is Hausdorff or regular, it is properly locally compact iff it is locally compact].

Proposition 8. If X is properly locally compact, then $[X, \mathcal{O}]$ is a continuous lattice (and so X has function spaces by Thm 7).

Proof:- Let U be open in X, $x \in U$. Then x has compact neighbourhood $C_x \subseteq U$. Since $\{U' \mid U' \supseteq C_x\}$ is Scott open, int $C_x \prec U$. Thus $U = \bigcup\{V \mid V \prec U\}$ and $[X, \mathcal{O}]$ is in CL.

Corollary 9. CL is cartesian closed, and L preserves its cartesian closed structure.

Proof:- If a subcategory is closed under the product and function space of a cartesian closed category, it is cartesian closed. So by Theorems 1, 2, 7 it suffices to show $[D, \mathcal{O}]$ in CL whenever D in CL. But for $d \in D$, $\{\{e' \mid e' \sqsupseteq e\} \mid e \prec d\}$ is a fundamental system of compact neighbourhoods at d, so this follows by Prop. 8.

Proposition 10. If X is properly locally compact, then its (guaranteed by Prop. 8) function spaces carry the compact open topology.

Proof:- We first show this for $[X, \mathbb{O}]$. Let $W \in \mathcal{W}$ open in $[X, \mathbb{O}]$. By the argument of Prop. 8,

$$W = \bigcup \{ \mathrm{Int}\, C \mid C \text{ compact}, C \subseteq W \}.$$

Hence there is compact $C \subseteq W$ with $\mathrm{Int}\, C \in \mathcal{W}$. Thus $\mathcal{W} = \bigcup \{ \{V \mid V \supseteq C\} \mid C \text{ compact and } \mathrm{Int}\, C \in \mathcal{W} \}$. On the other hand any $\{V \mid V \supseteq C\}$ is open. Thus $[X, \mathbb{O}]$ carries the compact-open topology.

Now that $[X, Z]$ carries the compact-open topology follows by the characterization of that topology in the proof of Prop. 5.

Corollary 11. Locally compact Hausdorff spaces have function spaces carrying the compact-open topology.

Proposition 12. If X is Hausdorff but not locally compact, then $[X, \mathbb{O}]_{LIM}$ is not topological.

Proof:- In $[X, \mathbb{O}]_{LIM}$, consider $t : X \longrightarrow \{T\}$. Let $\mathbb{H} \downarrow t$. Let $x \in X$ be a point with no compact neighbourhood. Now there exist $W \in \mathbb{H}$ and open $U \ni x$ such that $W(U) = \{T\}$. Let $\{U_\alpha\}$ be a cover of $cl(U)$ with no finite subcover.

Let $\mathbb{H}^*$ be the filter generated by the $[V, \{T\}] = \{f \mid f(V) = \{T\}\}$, where either $V \cap cl(U) = \phi$ and $[V, \{T\}] \in \mathbb{H}$,
or $V \subseteq U_\alpha$ some α and $[V, \{T\}] \in \mathbb{H}$.

Then it is easy to see, (i) that $\mathbb{H}^* \downarrow t$
(ii) $[U, \{T\}] \notin \mathbb{H}^*$

Thus there is no minimal filter converging to t, and so $[X, \mathbb{O}]$ is not topological.

Corollary 13. If X is Hausdorff, the conditions of Theorem 7 are equivalent to "X is locally compact".

Proof:- via Corollary 11, Proposition 12 and Theorem 1.

Remark The form of argument embodied in Prop. 12 can be used to show other results such as the (classical) "If X is completely regular but not locally compact then no topological function space $[X, I]$ exists" (I is $[0,1]$).