HYLAND

Dear Professor Scott,

Here is the lattice you asked for. The meat of the construction is in the definition of the ψ_i, ϕ_i and the lemmas 1–6 (and the theorem they give projections). The rest is a tarted up version of stuff you did in Continuous Lattices. As yet I've given no thought to making it all look nice.

Just to cheer us up the appropriate language is just the one Colin Mann had his original Church-Rosser worries about so maybe I'll think about that again. After that Böhm type arguments and maybe a maximal equality type result? The $D \cong [D \to D]$ part incidently produces just the same equalities on λ-terms as your original models. I hope its not too sketchy.

Martin Hyland.

Construction of a continuous lattice D such that $D \cong D \times D \cong [D \to D]$.

§1 Defn of the Lattice

Given a continuous lattice D_0 define

$$D_{2k+1} = D_{2k} \times D_{2k}$$
$$D_{2k+2} = [D_{2k+1} \longrightarrow D_{2k+1}].$$

Take
$$\begin{array}{ccccc} D_0 & \underset{\psi_0}{\overset{\phi_0}{\rightleftarrows}} & D_1 & \underset{\psi_1}{\overset{\phi_1}{\rightleftarrows}} & D_2 \\ & & \shortparallel & & \shortparallel \\ & & D_0 \times D_0 & & (D_1 \to D_1] \end{array}$$
system of projections where

ϕ_0 is $\lambda x.\,(x,x)$ $\qquad$ ϕ_1 is $\lambda z.\,(\lambda y.\,z)$

ψ_0 is $\lambda(x,y).\ x \sqcap y$ $\qquad$ ψ_1 is $\lambda w.\ w(\bot)$

for the sake of something specific. We use some properties of these projections in the structural lemmas but many others would do just as well.

Define $\psi_{2k+2} : D_{2k+2} \times D_{2k+2} \longrightarrow D_{2k+2}$; $(x_{2k+2}, y_{2k+2}) \longmapsto \langle \psi_{2k} \circ x_{2k+2},\ \psi_{2k} \circ y_{2k+2} \rangle$,

$\phi_{2k+2} : D_{2k+2} \longrightarrow D_{2k+2} \times D_{2k+2}$; $z_{2k+2} \longmapsto (\phi_{2k} \circ p \circ z_{2k+2},\ \phi_{2k} \circ q \circ z_{2k+2})$,

$\psi_{2k+3} : [D_{2k+3} \to D_{2k+3}] \longrightarrow D_{2k+3}$;

$x_{2k+4} \longmapsto (\psi_{2k+1} \circ p \circ x_{2k+4} \circ \phi_{2k+2} \circ \phi_{2k+1},\ \psi_{2k+1} \circ q \circ x_{2k+4} \circ \phi_{2k+2} \circ \phi_{2k+1})$

$\phi_{2k+3} : D_{2k+3} \longrightarrow [D_{2k+3} \to D_{2k+3}]$

$(x_{2k+2}, y_{2k+2}) \longmapsto \langle \phi_{2k+1} \circ x_{2k+2} \circ \psi_{2k+1} \circ \psi_{2k+2},\ \phi_{2k+1} \circ y_{2k+2} \circ \psi_{2k+1} \circ \psi_{2k+2} \rangle$,

where p, q are the obvious projections and

$\langle f, g \rangle = (f \times g) \circ \Delta$.

Theorem The ψ_i, ϕ_i are projections.

Proof:- By induction,

$$\psi_{2k+2} \circ \phi_{2k+2}(z_{2k+2}) = \langle \psi_{2k} \circ \phi_{2k} \circ p \circ z_{2k+2}, \text{———} \rangle$$
$$= \langle p \circ z_{2k+2}, q \circ z_{2k+2} \rangle \quad \text{by ind= hyp.}$$
$$= z_{2k+2}.$$

$$\phi_{2k+2} \circ \psi_{2k+2}((x_{2k+2}, y_{2k+2})) = (\phi_{2k} \circ p \circ \langle \psi_{2k} \circ x_{2k+2}, \psi_{2k} \circ y_{2k+2} \rangle, \text{———})$$
$$= (\phi_{2k} \circ \psi_{2k} \circ x_{2k+2}, \phi_{2k} \circ \psi_{2k} \circ y_{2k+2})$$
$$\sqsubseteq (x_{2k+2}, y_{2k+2}) \quad \text{by ind= hyp.}$$

$$\psi_{2k+3} \circ \phi_{2k+3}((x_{2k+2}, y_{2k+2})) = (\psi_{2k+1} \circ \phi_{2k+1} \circ x_{2k+2} \circ \psi_{2k+1} \circ \psi_{2k+2} \circ \phi_{2k+2} \circ \phi_{2k+1}, \text{———})$$
$$= (x_{2k+2}, y_{2k+2}) \quad \text{by ind= hyp.}$$

$$\phi_{2k+3} \circ \psi_{2k+3}(x_{2k+4}) = \langle \phi_{2k+1} \circ \psi_{2k+1} \circ p \circ x_{2k+4} \circ \phi_{2k+2} \circ \phi_{2k+1} \circ \psi_{2k+1} \circ \psi_{2k+2}, \text{———} \rangle$$
$$\sqsubseteq \langle p \circ x_{2k+4}, q \circ x_{2k+4} \rangle \quad \text{by ind= hyp.}$$
$$\sqsubseteq x_{2k+4}.$$

As we have the theorem for $i = 0, 1$ this is sufficient.

We have now an inverse system D_i, ψ_i of continuous lattices and projections; thus

$$\cdots\cdots D_{2K} \underset{\psi_{2K}}{\overset{\phi_{2K}}{\rightleftarrows}} \overset{D_{2K+1}}{D_{2K} \times D_{2K}} \underset{\psi_{2K+1}}{\overset{\phi_{2K+1}}{\rightleftarrows}} \overset{D_{2K+2}}{[D_{2K+1} \to D_{2K+1}]} \underset{\psi_{2K+2}}{\overset{\phi_{2K+2}}{\rightleftarrows}} \overset{D_{2K+3}}{D_{2K+2} \times D_{2K+2}} \underset{\psi_{2K+3}}{\overset{\phi_{2K+3}}{\rightleftarrows}} [D_{2K+3} \to D_{2K+3}] \cdots\cdots.$$

Now by the results of Scotts Continuous Lattices there is an inverse limit $D = \{ \langle x_K \rangle \mid \psi_K(x_{K+1}) = x_K \}$ with an induced ordering.

We have (i) D is a continuous lattice

(ii) there are projections from D to D_n ie $D_n \rightleftarrows D$ so that we can regard the D_n as contained in D.

(iii) $\langle x_n \rangle = \bigsqcup_n (x_n)$ ie $\langle x_n \rangle$ is actually the union of its "finite" approximants.

§2 Structural Lemmas

Lemma 1 $\psi_{2k+1} \circ \psi_{2k+2} = (\psi_{2k} \times \psi_{2k}) \circ (\psi_{2k+1} \times \psi_{2k+1})$

(these are maps : $D_{2k+3} \longrightarrow D_{2k+1}$).

Proof:- For $k=0$ we have

$\psi_1 \circ \psi_2 \left((x_2, y_2) \right) = \psi_1 \left(\langle \psi_0 \circ x_2 , \psi_0 \circ y_2 \rangle \right) = \left(\psi_0 \circ x_2(\perp) , \psi_0 \circ y_2(\perp) \right)$ and

$(\psi_0 \times \psi_0) \circ (\psi_1 \times \psi_1) \left((x_2, y_2) \right) = \left(\psi_0 \circ x_2(\perp) , \psi_0 \circ y_2(\perp) \right)$. (so true at $k=0$.)

Now take $k>0$,

$\psi_{2k+1} \circ \psi_{2k+2} \left((x_{2k+2}, y_{2k+2}) \right) = \left(\psi_{2k-1} \circ \psi_{2k} \circ x_{2k+2} \circ \phi_{2k} \circ \phi_{2k-1} , \text{———} \right)$.

$(\psi_{2k} \times \psi_{2k}) \circ (\psi_{2k+1} \times \psi_{2k+1}) \left((x_{2k+2}, y_{2k+2}) \right) = \left(\psi_{2k} \left((\psi_{2k-1} \circ p \circ x_{2k+2} \circ \phi_{2k} \circ \phi_{2k-1} , \text{—}) \right) , \text{———} \right)$

$= \left(\langle \psi_{2k-2} \circ \psi_{2k-1} \circ p \circ x_{2k+2} \circ \phi_{2k} \circ \phi_{2k-1} , \text{——} \rangle , \langle \text{——} , \text{——} \rangle \right)$.

$= \left((\psi_{2k-2} \times \psi_{2k-2}) \circ (\psi_{2k-1} \times \psi_{2k-1}) \circ x_{2k+2} \circ \phi_{2k} \circ \phi_{2k-1} , \text{———} \right)$.

and so we have "true for $k-1$" $\Rightarrow$ "true for k"
and we are home.

Lemma 2 (Dual to Lemma 1) $\phi_{2k+2} \circ \phi_{2k+1} = (\phi_{2k+1} \times \phi_{2k+1}) \circ (\phi_{2k} \times \phi_{2k})$.

Proof:- For $k=0$,

$\phi_2 \circ \phi_1 \left((x_0, y_0) \right) = \left(\phi_0 \circ p \circ (\lambda z . (x_0, y_0)) , \phi_0 \circ q \circ (\lambda z . (x_0, y_0)) \right) = \left(\lambda z . \phi_0(x_0) , \lambda z . \phi_0(y_0) \right)$.

$(\phi_1 \times \phi_1) \circ (\phi_0 \times \phi_0) \left((x_0, y_0) \right) = \left(\lambda z . \phi_0(x_0) , \lambda z . \phi_0(y_0) \right)$ so true.

Now for $k>0$,

$\phi_{2k+2} \circ \phi_{2k+1} \left(x_{2k}, y_{2k} \right) = \left(\phi_{2k} \circ \phi_{2k-1} \circ x_{2k} \circ \psi_{2k-1} \circ \psi_{2k} , \text{———} \right)$

$(\phi_{2k+1} \times \phi_{2k+1}) \circ (\phi_{2k} \times \phi_{2k}) \left(x_{2k}, y_{2k} \right) = \left(\phi_{2k+1} \left(\phi_{2k-2} \circ p \circ x_{2k} , \phi_{2k-2} \circ q \circ x_{2k} \right) , \text{———} \right)$

$= \left((\phi_{2k-1} \times \phi_{2k-1}) \circ (\phi_{2k-2} \times \phi_{2k-2}) \circ x_{2k} \circ \psi_{2k-1} \circ \psi_{2k} , \text{———} \right)$

so again we have an induction argument.

By applying $p_\circ$ to these 2 lemmas we get the corollaries

Lemma 3 $\quad p \circ \psi_{2k+1} \circ \psi_{2k+2} = \psi_{2k} \circ \psi_{2k+1} \circ p$

Lemma 4 $\quad p \circ \phi_{2k+2} \circ \phi_{2k+1} = \phi_{2k+1} \circ \phi_{2k} \circ p$

(and similarly for q.).

Lemma 5 $\quad [\phi_{2k+3} \circ \phi_{2k+2}(x_{2k+2})]\left(\phi_{2k+2} \circ \phi_{2k+1}(y_{2k+1})\right) = \phi_{2k+2} \circ \phi_{2k+1}(x_{2k+2}(y_{2k+1}))$.

Proof:- $\text{L.H.S.} = \langle \phi_{2k+1} \circ \phi_{2k} \circ p \circ x_{2k+2} \circ \psi_{2k+1} \circ \psi_{2k+2}, \longrightarrow \rangle (\phi_{2k+2} \circ \phi_{2k+1}(y_{2k+1}))$

$= (\phi_{2k+1} \times \phi_{2k+1}) \circ (\phi_{2k} \times \phi_{2k}) \circ x_{2k+2} \circ \psi_{2k+1} \circ \psi_{2k+2} \circ \phi_{2k+2} \circ \phi_{2k+1}(y_{2k+1})$

$=$ R.H.S. using Lemma 2 and the fact that the ψ_i's are projections.

Lemma 6 (Dual to Lemma 5)

$$(\psi_{2k+2} \circ \psi_{2k+3}(x_{2k+4}))(\psi_{2k+1} \circ \psi_{2k+2}(y_{2k+3})) \sqsubseteq \psi_{2k+1} \circ \psi_{2k+2}(x_{2k+4}(y_{2k+3})).$$

Proof:- $\text{L.H.S.} = \langle \psi_{2k} \circ \psi_{2k+1} \circ p \circ x_{2k+4} \circ \phi_{2k+2} \circ \phi_{2k+1}, \longrightarrow \rangle (\psi_{2k+1} \circ \psi_{2k+2}(y_{2k+3}))$

$= \psi_{2k+1} \circ \psi_{2k+2} \circ x_{2k+4} \circ \phi_{2k+2} \circ \phi_{2k+1} \circ \psi_{2k+1} \circ \psi_{2k+2}(y_{2k+3})$ using Lemma 1

$\sqsubseteq$ R.H.S. by projection property.

Remark Lemmas 1–4 are needed for the $D \times D$ structure and Lemmas 5, 6 for $D \to D$ structure.

§3 The Isomorphism $D \cong D \times D$.

Consider $\langle x_n \rangle \longrightarrow (\langle y_n \rangle, \langle z_n \rangle)$ where $y_{2k} = p(x_{2k+1})$, $y_{2k+1} = \psi_{2k+1}(y_{2k+2})$ etc.

and $(\langle y_n \rangle, \langle z_n \rangle) \longrightarrow \langle x_n \rangle$ where $x_{2k+1} = (y_{2k}, z_{2k})$, $x_{2k} = \psi_{2k}(x_{2k+1})$.

We check that these maps really do produce elements of D. For that it is sufficient to check

(a) $\psi_{2k}(y_{2k+1}) = y_{2k}$ in the first line

i.e. $\psi_{2k} \circ \psi_{2k+1} \circ p = p \circ \psi_{2k+1} \circ \psi_{2k+2}$ but this is Lemma 3

and (b) $\psi_{2k+1}(x_{2k+2}) = x_{2k+1}$ in the second line

i.e. $\psi_{2k+1} \circ \psi_{2k+2} = (\psi_{2k} \circ \psi_{2k+1}) \times (\psi_{2k} \circ \psi_{2k+1})$ i.e. Lemma 1.

We have now the projection maps $p : D \longrightarrow D$, $q : D \longrightarrow D$ and a pairing map which are plainly appropriately inverse to one another.

The continuity of these operations follows from that of the corresponding operations on the "finite" lattices D_n e.g. $\left(p\left(\bigsqcup_{r \in I} \langle x_n \rangle^{(r)}\right)\right)_{2k} = p\left(\bigsqcup_{r \in I} x_{2k+1}^{(r)}\right) = \left(\bigsqcup_{r \in I} p\langle x_n \rangle^{(r)}\right)_{2k}$.

and $\left(\left(a, \bigsqcup_{\ell} \langle y_n \rangle^{(\ell)}\right)\right)_{2k+1} = \left(a_{2k}, \left(\bigsqcup_{\ell} \langle y_n \rangle^{(\ell)}\right)_{2k}\right) = \bigsqcup_{\ell} (a_{2k}, (y^{\ell})_{2k}) = \left(\bigsqcup_{\ell} (a, \langle y_n \rangle^{\ell})\right)_{2k+1}$

Thus we have proved the

<u>Theorem</u> $D \cong D \times D$.

Finally we check that the pairing and unpairing of finite elements given by $D_{2k+1} = D_{2k} \times D_{2k}$ is reflected in D. Take $x_{2k}, y_{2k} \in D_{2k}$; as elements of D they are represented as sequences and we check $((x_{2k})_{2k+2}, (y_{2k})_{2k+2}) = (\phi_{2k+1} \circ \phi_{2k}\, x_{2k}, \phi_{2k+1} \circ \phi_{2k}\, y_{2k})$

$$= \phi_{2k+2} \circ \phi_{2k+1} (x_{2k}, y_{2k}) \quad \text{by Lemma 2}$$

$$= ((x_{2k}),(y_{2k}))_{2k+3} \quad \text{as required.}$$

Similarly we get a reflection of the pairing using Lemma 4.

(Note we have done all we need as agreement at one level of the finite structures forces it at all lower levels, and we have an induction to show agreement at arbitrarily high levels).

§4 The Isomorphism $D \cong [D \to D]$.

Define application in the lattice by

$$\langle x_n \rangle (\langle y_n \rangle) = \bigsqcup_k z_{2k+1} \qquad \text{where } z_{2k+1} = x_{2k+2}(y_{2k+1}).$$

Remark Lemma 6 says this is a directed union.

Lemma $x_{2k+2}(y) = x_{2k+2}(y_{2k+1})$

Pf:- Sufficient to show $x_{2k+2}(y_{2k+3}) = x_{2k+2}(y_{2k+1})$

Well, $\phi_{2k+3} \circ \phi_{2k+2}(x_{2k+2})\,(y_{2k+3})$

$$= \langle \phi_{2k+1} \circ \phi_{2k} \circ p \circ x \circ \psi_{2k+1} \circ \psi_{2k+2}, \longrightarrow \rangle \; (y_{2k+3})$$

$$= \phi_{2k+2} \circ \phi_{2k+1}\,(x_{2k+2}(y_{2k+1})) \quad \text{using Lemma 2}$$

which is $x_{2k+2}(y_{2k+1})$ in the space two up.

Theorem (a) The map $D \to D$ $\lambda y . D : \langle x_n \rangle (y)$ is continuous for each $\langle x_n \rangle$.

(b) It is (1-1) and continuous from $D \to [D \to D]$.

Proof:- (a) just the sup of the $\lambda y.\, D: x_{2k+2}(y)$ continuous by the above lemma.

(b) (1-1) obvious using Lemma and similarly continuity.

We now check our map $D \longrightarrow [D \to D]$ is onto

Given $F : D \to D$ continuous, set

$$f_{2k+2} = \lambda y_{\cdot} D_{2k+1} : (F(y_{2k+1}))_{2k+1} \in D_{2k+2},$$
$$f_{2k+1} = \psi_{2k+1}(f_{2k+2}).$$

Then (a) $\langle f_n \rangle \in D$

Proof:- Show $f_{2k} = \psi_{2k} \circ \psi_{2k+1}(f_{2k+2})$

$$\psi_{2k} \circ \psi_{2k+1}(f_{2k+2}) = \psi_{2k}(\psi_{2k-1} \circ p_0 (\lambda y_{2k+1} (F(y_{2k+1}))_{2k+1}) \circ \phi_{2k+2}, \phi_{2k+1}, \text{---})$$

$$= (\psi_{2k-2} \circ \psi_{2k-1} \circ p_0 (\lambda y_{2k-1} (F(y_{2k-1}))_{2k+1}, \text{---})$$ (observe the plus & minus.)

$$= \psi_{2k-1} \circ \psi_{2k} (\lambda y_{2k-1} (F(y_{2k-1}))_{2k+1})$$ by Lemma 1.

$$= \lambda y_{2k-1} (F(y_{2k-1}))_{2k-1} = f_{2k}$$ as required.

(b) obviously this map $: [D \to D] \longrightarrow D$ is continuous.

(c) $\lambda y . D ; \langle f_n \rangle (y) = F$.

Proof:- $\langle f_n \rangle (y) = \bigsqcup_k f_{2k+2}(y_{2k+1}) = \bigsqcup_k (F(y_{2k+1}))_{2k+1}$

$$F(y) \cdot \bigsqcup_l \bigsqcup_k (F(y_{2k+1}))_{2l+1}$$

Now as usual $F(y) \sqsupseteq \langle f_n \rangle (y)$ obvious,

while for $l > k$ $(F(y_{2k+1}))_{2l+1} \sqsubseteq (F(y_{2l+1}))_{2l+1}$ and so we get $F(y) \sqsubseteq \langle f_n \rangle (y)$.

We have now established the

<u>Theorem</u> $D \cong [D \to D]$.

<u>Remark</u> Lemma 5 shows that the finite elements act under the $[D \to D]$ structure as expected.

§5 A relation between the two structures.

Theorem $(x,y)(z) = (x(z), y(z))$.

Proof:- Consider $(x,y)_{2k+2}(z)_{2k+1} = \langle \psi_{2k} \circ x_{2k+2}, \psi_{2k} \circ y_{2k+2} \rangle (z_{2k+1})$

$$= (\psi_{2k}\, x_{2k+2}(z_{2k+1}), \psi_{2k}\, y_{2k+2}(z_{2k+1}))$$

Now $\bigsqcup_k (x,y)_{2k+2}(z)_{2k+1} = (x,y)(z)$.

By Lemma 6 $\psi_{2k-1} \circ \psi_{2k}\, x_{2k+2}(z_{2k+1}) \sqsupseteq x_{2k}(z_{2k-1})$

so $\bigsqcup_k \psi_{2k}\, x_{2k+2}(z_{2k+1}) = x(z)$

and so we derive result.

§6 An appropriate language.

Take a countable set of free variables, and have operations of application, λ-abstraction, pairing and projecting i.e.

Terms ① Variables are terms.

② If M, N are terms so are $M(N)$, $\lambda x M$, (M, N) and pM & qM.

Reduction rules Ⓐ

$$(\lambda x P) Q \geq P[Q/x]$$

$$\lambda x\, Mx \geq M \qquad x \notin FV(M).$$

Ⓑ

$$p(M, N) \geq M$$

$$(pM, qM) \geq M$$

Ⓒ

$$(M, N)(P) \geq (M(P), N(P))$$

$$\lambda x\,(M, N) \geq (\lambda x M, \lambda x N)$$

Given the natural definition for an interpretation of terms in the D model, then the reductions Ⓐ hold on account of the $D \cong [D \to D]$ structure those of Ⓑ by the $D \cong [D \times D]$ structure and those of Ⓒ by para 5.