Extensionality of Morris's + Wadsworth's relations

Morris's relation

$$M \leq N \text{ iff whenever } C[M] \to nf \text{ then } C[N] \to nf$$

(I do not know the reference for this, but recall it was discussed in Wadsworth's thesis)

Wadsworth's relation

$$M \leq N \text{ iff whenever } C[M] \to hnf \text{ then } C[N] \to hnf$$

(I am not sure that this was in Wadsworth's thesis, but think that he was the first to consider it.)
[Of course this is the preorder induced by the standard D_∞ model.]

The extensionality principle is of the form

$$\frac{\forall L \quad ML \leq NL}{M \leq N}$$

Concentrate on Wadsworth's relation to start off with as it involves less computation (i.e. just to hnf).

Deal first with the case of closed terms M and N.

We show the stronger fact

$$\frac{\forall L_1, \ldots L_n . \quad M L_1 \ldots L_n \to \mathrm{hnf} \Rightarrow N L_1 \ldots L_n \to \mathrm{hnf} \quad (n \geq 1)}{M \leq N.}$$

[Is the hypothesis 'applicative bisimulation' in this case?]

<u>The intuition is as follows:</u>

Assume the hypothesis.

Suppose $C[M] \longrightarrow \mathrm{hnf}$. We may as well consider head reduction. There will be a last time in the reduction sequence that a descendent of one of the occurrences of M in $C[M]$ appears 'at the head': at that moment the term looks like

$$\lambda\,(\text{some variables}).\ M X_1 \ldots X_n .$$

Of course there may be occurrences of M in $X_1, \ldots X_n$, but these remain dormant during the rest of the reduction to hnf and so could be replaced (by constants or else) by occurrences of N; let $X_1' \ldots X_n'$ be $X_1 \ldots X_n$ with occurrences of descendents of M replaced by occurrences of N.

Clearly

$$\lambda(\text{some variables}).\ M X_1' \ldots X_n' \longrightarrow \text{hnf}.$$

Hence by hypothesis,

$$\lambda(\text{some variables}.\ N X_1' \ldots X_n' \longrightarrow \text{hnf}.$$

Now go back to the next to last appearance of M at the head; at that moment the term looks

$$\lambda(\text{fewer perhaps variables}).\ M Y_1 \ldots Y_m$$

Consider the head reduction

$$M Y_1 \ldots Y_m \longrightarrow \lambda(\text{a few more variables})\ M X_1 \ldots X_n$$

All the occurrences of descendents of M in $Y_1 \ldots Y_m$ remain dormant & one of them eventually appears as the M at the head of $M X_1 \ldots X_n$. Hence they could be replaced (by constants or else) by N; let $Y_1' \ldots Y_m'$

be $Y_1 \ldots Y_m$ with occurrences of descendants of M replaced by occurrences of N.

Clearly then

$$\lambda(\text{fewer perhaps variables}).\, M Y_1' \ldots Y_m'$$

$$\longrightarrow \lambda(\text{some variables}).\, N X_1' \ldots X_n'$$

$$\longrightarrow \text{hnf}$$

Hence by hypothesis

$$\lambda(\text{fewer perhaps variables}).\, N Y_1' \ldots Y_m' \longrightarrow \text{hnf}.$$

Continuing inductively in this way we see in the end that $C[N] \longrightarrow \text{hnf}$ as required.

[Remark: When we use the hypothesis in ~~the~~ proof the L_i may be open but it doesn't matter one way or the other.]

In case M and N are not closed I guess we aim to show that

$$\frac{\left(\begin{array}{l}\forall \text{ substitutions } (\,)' \text{ and } \forall L_1 \cdots L_n \\ (M)' L_1 \cdots L_n \to hnf \implies (N)' L_1 \cdots L_n \to hnf\end{array}\right)}{M \leq N}$$

The proof goes through as before except we have to deal with the appearance at the head of descendents of occurrences of M which will have the form $(M)'$; and we have to keep track of these substitutions as we work back.

Now consider Morris's relation. We replace hnf by nf throughout the above. Instead of head reduction we consider normal order (?) reduction or indeed any kind of iterated head reduction (which doesn't interleave where its working).

The same strategy of working back from the last ~~ti~~ time a descendent of M appears at the head position where reduction is taking place will. I guess the details look a bit more complicated?