INTERACTION OF STRATEGIES

BY

COMPOSITION OF INTERLEAVINGS

\[ A \xrightarrow{\sigma} B \xrightarrow{\tau} C \]
\[ \frac{\quad}{A \xrightarrow{\sigma, \tau} C} \]

\[ \Gamma \xrightarrow{\Delta, A} A, \Pi \xrightarrow{\tau} \Sigma \]
\[ \frac{\quad}{\Gamma, \Pi \xrightarrow{(\sigma, \tau)_A} \Delta, \Sigma} \]

Interaction by plugging

\[ \sigma \begin{array}{c} \tau \end{array} \rho \]

clearly associative

BUT
PARALLEL COMPOSITION
PLUS HIDING

A \xrightarrow{\sigma} B \xrightarrow{\tau} C

defined via

\[
\begin{array}{c|c|c}
A^+ & B & B^+ \\
\hline & & C
\end{array}
\]

intrinsically associative.

However

1. It is much less clear for some restricted kinds of strategies
2. There is the Blaus Problem.
Blass Example: Sequential

Initial move

Response

1

\[ A \xrightarrow{\sigma} B \quad B \xrightarrow{\tau} C \quad C \xrightarrow{\rho} D \]

2

\[ A \xrightarrow{\sigma; \tau} C \]

\[ A \xrightarrow{\sigma; \tau; \rho} D \]

\[ B \xrightarrow{\tau; \rho} D \]

\[ A \xrightarrow{\sigma; (\tau; \rho)} D \]
A strategy $\sigma : A$ is identified with the plays ($= \text{sequences of moves/positions}$) in $A$ to which it gives rise.

So $\sigma : A \rightarrow B$ is a sequence from $A,B$.

Consider sequences $w$ from $A,B,C$. Then

$$\sigma ; w = \{ (w)_{A,B} : (w)_{A,B} \in \sigma \text{ and } (w)_{B,C} \in w \}.$$

For associativity consider sequences from $A,B,C,D$; one has to reconstruct these from various restrictions.
A sequence \( u \) from \( A, B \) is an interleaving of a sequence \( r \) from \( A \) and \( s \) from \( B \):

\[
r \xrightarrow{u} s
\]

**Claim.** There is a category whose maps are (appropriate) interleavings.

Then we have

\[
\sigma; r = \{ r \xrightarrow{u} s \xrightarrow{v} t | u \in \sigma, v \in \tau \}
\]

and this obviously associative.
Merging: Generalities

Consider preordered sets \( P, Q \): a merge is given by embeddings \( P \to P + Q \) into a preorder on the disjoint sum.

Equivalently by relations

\[
(q \leq p) \quad P \times Q^\text{op} \to 2 \\
p \overset{r}{\to} q
\]

\[
(p \leq q) \quad Q \times P^\text{op} \to 2 \\
q \overset{r'}{\to} p
\]

such that

\[
r \circ r' + \text{id}_p (= \leq_p) \quad p \leq q \leq p' + p \leq p' \\
r' \circ r + \text{id}_q (= \leq_q) \quad q \leq p \leq q' + q \leq q'.
\]

(Then the merge on \( P + Q \) is a \( 2 \)-colimit.)

Conditions

Asynchrony \( P, Q \) posets and \( p \leq q \leq p' + p \leq p' \)

\( q \leq p \leq q' + q \leq q' \)

Sequentiality \( P, Q \) linearly ordered and \( p \leq q \) if and only if \( q \not\leq p \).
Composition of Merges

\[ P \xrightarrow{r} Q \xrightarrow{s} R \]

\[ P \xleftarrow{r,s} R \]

where we have e.g.

\[ p < q < r < q' \leq p' \Rightarrow p < q < r < q' \leq p' \Rightarrow p \neq p' \]

So there is a category of preordered sets and merges.

Warning.

Asynchrony is preserved.
Sequentiality is not:

\[ p \]

\[ \quad \]

\[ Q \]

\[ P \]

\[ Q \]

\[ P \]

\[ P \]

\[ P \]

\[ P \]

\[ P \]
In games the identity is a copycat strategy

\[
A \rightarrow A
\]

\[
a_0 \rightarrow a_0 \quad 0\text{-move}
\]

\[
a_1 \rightarrow a_1 \quad P\text{-move}
\]

\[
a_2 \rightarrow a_2 \quad 0\text{-move}
\]

\[
a_3 \rightarrow a_3 \quad P\text{-move}
\]

i.e. we interleave in a twisted sense one way for opponent moves the other for opponent moves.

Simple merging has an identity which synchronizes the moves.
INTERLEAVING OF PLAYS

A play in a game $A$ is already a merge $P = (P_i \leftrightarrow P_0)$ of Opponent moves $P_i$ and Proponent moves $P_0$.

Interleaving is given by Proponent data only:

$$
\begin{align*}
P_i & \leftrightarrow P_0 \\
\uparrow & \\
\downarrow & \\
Q_i & \leftrightarrow Q_0
\end{align*}
$$

such that

$$
\begin{align*}
q_i \leq P_i & \leq P_0 \leq Q_0 \lor q_i \leq Q_0 \\
P_0 \leq Q_0 & \leq Q_i \leq P_i \lor P_0 \leq P_i
\end{align*}
$$

(This induces merges

$$
\begin{align*}
P_i & \leftrightarrow Q_i \\
P_0 & \leftrightarrow Q_0 \\
P_i & \leftrightarrow Q_0 \\
Q & \leftrightarrow P_0
\end{align*}
$$

but that is denied structure.)
COMPOSITION OF INTERLEAVINGS

\[
\begin{align*}
P_1 & \leftrightarrow P_0 \\
\uparrow & \uparrow \\
\Phi_1 & \leftrightarrow \Phi_0 \\
\downarrow & \uparrow \\
R_1 & \leftrightarrow R_0
\end{align*}
\]

composes in the obvious way as e.g.

\[
\forall x_1 \leq P_1 \leq P_0 \leq \Phi_0 \leq \Phi_1 \leq R_0 \leq R_1 \leq \forall x
\]

In the asynchronous (sequential) case
the identity is now COPY-CAT.
SEQUENTIAL GAMES

Take plays \((p_i \leftrightarrow p_0)\) which are (finite)

- asynchronous sequential
- opponent starting \(\exists p_1 \forall p_0 \ p_i < p_0\)
- alternating \(p_i < p_i' + \exists p_0 \ p_i < p_0 < p_i'\), \(p_0 < p_0' + \exists p_1 \ p_0 < p_1 \leq p_0'\)

Take interleavings \(p_i \leftrightarrow p_0\)

\[
\begin{array}{c}
\downarrow \\
(\neg) \\
\uparrow \\
q_i \leftrightarrow q_0
\end{array}
\]

such that in the \(\neg\) game

- opponent starts \(\exists q_1 \forall p_i \ q_i \leq p_i\)
- opponent chooses where to play
  \(\forall p_i q_0\).
  - either \(\exists p_0 \ p_i \leq p_0 \leq q_0\)
  - or \(\exists q_1 \ q_0 \leq q_1 \leq p_i\)

This is the sequential setting.
Observations in Sequential Setting

1. The order of opponent moves now also forced i.e.

\[ \forall q_1, p_0 \quad \text{either } \exists p_1, q_1 \leq p_1 \leq p_0 \quad \text{or } \exists q_0, p_0 \leq q_0 \leq q_1 \]

(If the switching convention is forced)

Moreover the reduced merge totally order \( p_1 + p_0 + q_1 + q_0 \).

2. The interleaving properties are closed under composition.

So we get an alternative view on associativity in the standard cases.
ABSTRACT GAMES

Models of linear logic constructed thus

Geometry of interaction $\Downarrow$ Trace
  $\Downarrow$ mon. cats.

Degenerate models $\Downarrow$ Compact closed cats
  $\Downarrow$ Glueing

"Non-interactive" models
  $\Downarrow$ Orthogonality

"Interactive" models

"Often read" as categories whose objects are a kind of game.

Examples

Non-interactive: GRel, "Concurrent games" Abramsky, Heijne

Interactive: Coherence spaces, Hypercoherences

? G. Heijne?
CONCRETE NON-SEQUENTIAL GAMES
(SIMPLE VERSION: SKETCH)

A game $A$ is a collection of plays $(P_i \leftrightarrow P_o)$ closed under initial segment ($\& \ldots$).

A strategy $\sigma$: $A$ is a collection of plays again closed under initial segment and consistent for Proponent, i.e.

Suppose $p_o$ w i $(P_i \leftrightarrow P_o) \in \sigma$
$p_o'$ w i $(P_i' \leftrightarrow P_o') \in \sigma$

with \{ $p_i : p_i \leq p_o$ \} = \{ $p_i' : p_i' \leq p_o'$ \}
(less ordered sets)

Then there is play $(P_i'' \leftrightarrow P_o'') \in \sigma$

with $p_o, p_o' \in \sigma$ \& \{ $p_i'' : p_i'' \leq p_o''$ \} = \{ $p_i'' : p_i'' \leq p_o'$ \}

[i.e. exact asynchronous case \( \tau_{\text{ini}} \) is $p_o = p_o'$]

(Idea: Petri-net style computation)
MULTIPLICATIVE STRUCTURE  
(SIMPLEST CASE)

Plays $\nu : B \to C$ are given by all interleavings of plays $(\varphi, \epsilon R \varphi_0)$ $\nu : B$ with plays $(R_i \epsilon \to R_0)$ $\nu : C$.

Then plays $\nu : A \to (B \to C)$ correspond exactly to plays $\nu : A \otimes B \to C$ : more symmetrically, plays $\nu : A \otimes B \otimes C$ are interleavings of plays $\nu : A \otimes B \otimes C$ via data satisfying

six conditions of form

$p_i \leq \varphi_0 \leq \varphi_1 \leq \varphi_0 + p_i \leq \varphi_0$

and six conditions of form

$p_i \leq \varphi_0 \leq \varphi_1 \leq \varphi_0 + p_i \leq \varphi_0$

Take maps $A \to B$ to be strategies $\sigma : A \to \mathbf{B}$ and we have a

*-autonomous category

(in fact, model for full linear logic)
BLASS EXAMPLE : NON-SEQUENTIAL

\[ A \xrightarrow{\sigma} B \quad B \xrightarrow{\tau} C \quad C \xrightarrow{\rho} D \]

1. Inhibit more response

2. \[ A \xrightarrow{(\sigma; \tau); \rho} D \]

3. \[ B \xrightarrow{\tau; \rho} D \]

4. \[ A \xrightarrow{\sigma; (\tau; \rho)} D \]