PROPOSITIONS FROM TYPES

Methods of Proof Theory in Mathematics

MPIM BONN

JUNE 2007

PROPOSITIONS FROM TYPES

Methods of

Proof Theory in Mathematics

MPIM BONN

JUNE 2007

DEVELOPMENT

of

Proof Theory in the Abstract

REACTION

to

Paulo Oliva's

Unifying Functional Interpretations

(and his talk at

the meeting)

NON-STANDARD LOGICS

Over a category $\mathbb{T}$ of types have a (cloven) fibration

$$\mathbb{P} \longrightarrow \mathbb{T}$$

of propositions, where

- the fibres $\mathbb{P}(T)$ are preordered $\phi \vdash \psi$
- have logical structure
  - connectives
  - quantifiers

(preserved under substitution)

NON-STANDARD PROOFS

Over a category $\mathbb{T}$ of types have a (cloven) fibration

$$\mathbb{P} \longrightarrow \mathbb{T}$$

of $\begin{cases} \text{propositions and proofs,} \\ \text{or types.} \end{cases}$

Have categorical logical structure

- in the fibres (eg products, exponentials)
- relating fibres (eg adjoints to substitution)

(again all preserved under substitution)

PROPOSITIONS FROM TYPES

(Lawvere on Hyperdoctrines)

Non-standard logics come from non-standard type theories: preordered reflection.

- Many interesting cases
- Good methodology

Curry - Howard

Correspondence

REALIZABILITY

- The realizability topos is the preordered reflection of a (not very nice) type theory
  (closed subobjects of the realizers).

- The 'modest sets' is a near-complete small subcategory in realizability. Its externalization is a type theory with preordered reflection extensional realizability.

(Use this to build many more logics.)

DIALECTICA INTERPRETATION

$$\phi \longmapsto \phi^D = \exists u\, \forall x\, \phi_D(u, x)$$

apparently propositional

but $\phi^D \vdash \psi^D$ is NOT

of primary interest.

Rather we ask for existence

of f, F such that

$$\phi_D(u, Fuy) \vdash^{u,y} \psi_D(fu, y)$$

and such f, F are maps

in a category of 'Dialectica

Proofs'.

WARM UP : $\mathbb{C} \times \mathbb{C}^{op}$

Suppose $\mathbb{C}$ is symmetric monoidal closed with finite $\times$s.
(Models multiplicative additive Intuitionistic Linear Logic.)

Then $\mathbb{C} \times \mathbb{C}^{op}$ is $*$-autonomous
(Models multiplicative LL.)

$$(U, X) \otimes (V, Y) = (U \otimes V,\ V \multimap X \times U \multimap Y)$$

$$(V, Y) \multimap (W, Z) = (V \multimap W \times Z \multimap Y,\ V \otimes Z)$$

If in addition $\mathbb{C}$ has $+$s

then $\mathbb{C} \times \mathbb{C}^{op}$ has $\times$s and $+$s.

COMONADS ON $\mathbb{C} \times \mathbb{C}^{op}$

Suppose $\mathbb{C}$ has a linear exponential comonad !

Gödel Dialectica $\quad (u, x) \longmapsto (!u, !u \multimap x)$

Kreisel Modified Realizability $\quad (u, x) \longmapsto (!u, 1)$

Diller-Nahm Dialectica Variant $\quad (u, x) \longmapsto (!u, !u \multimap M(x))$

where $M(-)$ provides a notion of commutative monoid for $\mathbb{C}$, x s: and is well-adapted (H + Schalk) that is, we have distributivity of Gödel' monad over the obvious $(u, x) \longmapsto (u, M(x))$.

GIRARD TRANSLATION

Given $\mathbb{L}$ smc cats with !

take the Kleisli category $\mathbb{K}$:

$$\mathbb{K}(A,B) = \mathbb{L}(!A,B)$$

In good circumstances

$\sim$ $!A \otimes !B \cong !(A \times B)$,

$\mathbb{K}$ is cartesian closed.

- $\times$ as in $\mathbb{L}$
- $\mathbb{K}(A \times B, C) = \mathbb{L}(!(A \times B), C)$

$$\begin{aligned} &\cong \mathbb{L}(!A \otimes !B, C) \\ &\cong \mathbb{L}(!A, !B \multimap C) = \\ &\qquad \mathbb{K}(A, !B \multimap C) \end{aligned}$$

So $B \to C = !B \multimap C$ is the function space.

GOOD for MR and Diller-Nahm.

CHEAP RESULTS

Every cartesian closed caty
(model for the typed λ-calculus)
'appears as' the Kleisli caty
of a $*$-autonomous caty with !
(model for classical LL)

An smc caty with ×s and !
(model for intuitionistic LL)
embeds in a $*$-aut caty with !
(model for classical LL)
with the same Kleisli caty.

(Conservativity of the
Girard translation.)
(There are refined variants.)

General Setting

Base category $\mathbb{T}$ a model for intuitionistic LL and (cloven) fibration

$$\mathbb{P} \longrightarrow \mathbb{T}$$

a fibred model i.e. structure preserved under pullback.

(Other features eg for Dialer-Nelson is needed.)

Special Case

$$\mathbb{P} \longrightarrow \mathbb{T}$$

codomain fibration for a locally cartesian closed $\mathbb{T}$.

BASIC CONSTRUCTION

Category with

Objects

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ U & & X \end{tikzcd}$$

$(A \in \mathbb{P}(U \otimes X))$

Maps

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] \arrow[dd, dashed, "\phi"] & \\ U \arrow[dd, "f"'] & & X \\ & B \arrow[dl] \arrow[dr] & \\ V & & Y \arrow[uu, "F"'] \end{tikzcd}$$

(-----)

$f : U \to V \quad F : Y \to X$

$\phi : \prod_{uy} \left(A(u, Fy) \to B(fu, y) \right)$

ERROR VARIANT

OF THE DIALECTICA INTERPR$^{\underline{N}}$

comes from comonad

$$
\begin{tikzcd}
 & A \arrow[dl] \arrow[dr] & & & & A^+ \arrow[dl] \arrow[dr] & \\
U & & X & \longmapsto & U & & U \to (X+1)
\end{tikzcd}
$$

where

$$
\begin{tikzcd}
A^+ \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A + U \arrow[d] \\
U \times (U \to (X+1)) \arrow[r] & U \times (X+1)
\end{tikzcd}
$$

$$U \times (X+1) = U \times X + U$$

$$
A^+(u,f) = \begin{cases} \text{case } f(u) \in X & A(u, f(u)) \\ \text{case } f(u) \in 1 & 1 \end{cases}
$$

Kleisli with Errors

Maps

$$\begin{tikzcd}
 & A^+ \arrow[dl, dash] \arrow[dr, dash] \arrow[dd, dotted, dash] & \\
U \arrow[d] & & U \to (X+1) \\
V & B \arrow[l, dash] \arrow[r, dash] & Y \arrow[u]
\end{tikzcd}$$

are

$$\begin{tikzcd}
U \arrow[d, "f"'] & X+1 \\
V & Y \arrow[u]
\end{tikzcd}$$

i.e. $f : U \to V \quad F : U \times Y \longrightarrow X+1$

and

$$\phi : \prod_{uy} \begin{array}{c} |Fuy \in X| \to A(u, Fuy) \to B(fu, y) \\ \times \\ |Fuy \in 1| \longrightarrow B(fu, y) \end{array}$$

PRODUCTS

(in the basic construction and so in the Kleisli)

$$\begin{tikzcd}
 & A \arrow[dl] \arrow[dr] & \\
U \arrow[d, "{(f,g)}"'] & & X \\
V \times W & & Y + Z \arrow[u, "\binom{h}{k}"'] \\
 & B \times W + V \times C \arrow[ul] \arrow[ur] \arrow[uuu, dashed] &
\end{tikzcd}$$

$$\phi : \prod_{uyz} \begin{array}{l} A(u, hy) \longrightarrow B \times W(f(u), g(u), y) \\ \times \\ A(u, kz) \longrightarrow V \times C(f(u), g(u), z) \end{array}$$

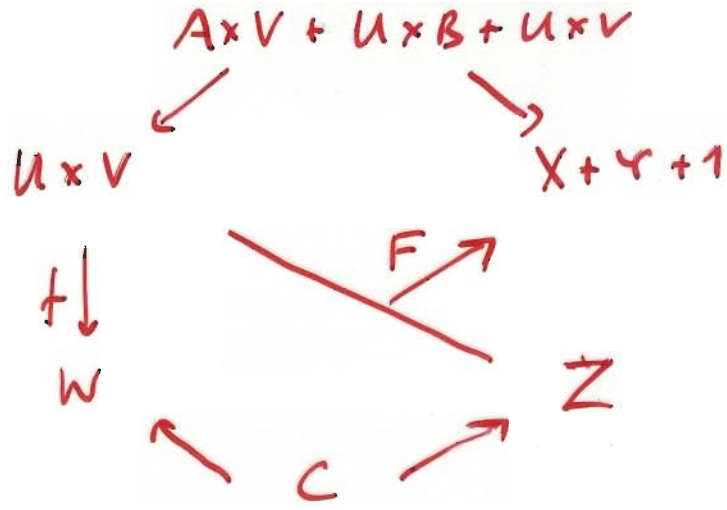
i.e.

$$\phi_1 : \prod_{uy} A(u, hy) \longrightarrow B(f(u), y)$$

$$\phi_2 : \prod_{uz} A(u, kz) \longrightarrow C(g(u), z)$$

MULTI - MAPS

IN THE KLEISLI CATEGORY



$$
\phi : \prod_{uvz} \begin{array}{l} |Fuvz \in X| \longrightarrow A(u, Fuvz) \longrightarrow C(fuv, z) \\ |Fuvz \in Y| \longrightarrow B(v, Fuvz) \longrightarrow C(fuv, z) \\ |Fuvz \in 1| \longrightarrow C(fuv, z) \end{array}
$$

FUNCTION SPACE

(semi-function space in the Kleisli category)

$$
\begin{tikzcd}
 & A+U \arrow[dl] \arrow[dr, dashed] & \\
U \arrow[d] & & X+1 \\
\bullet & & V \times Z \\
 & \bullet \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

$$\left(\Sigma g : V \to W \quad G : V \times Z \to Y+1 . \; \left[\Pi v z .\ |Gvz \in Y| \to B(v, Gvz) \to C(gv, z)\right]\right)$$

$$\left(\Sigma g, G, v, z . \; [\qquad] \times \left(|Gvz \in 1| \to C(gv, z)\right)\right)$$

THEOREM

The Kleisli category $\mathbb{K}$ for the Dialectica-with-errors is semi-cartesian-closed: that is, splitting idempotents gives a cartesian closed category.