Variations on Realizability: Realizing the Propositional Axiom of Choice

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1 Introduction

1.1 Historical background

Early investigators of realizability were interested in metamathematical questions. In keeping with the traditions of the time they concentrated on interpretations of one formal system in another. They considered an ad hoc collection of increasingly ingenious interpretations to establish consistency, independence and conservativity results. van Oosten's contribution to the Workshop (see van Oosten [46]) gave inter alia an account of these concerns from a modern perspective.

In the early days of categorical logic one considered realizability as providing models for constructive mathematics; while the metamathematics could be retrieved by 'coding' the models, that aspect took a back seat. In the first instance realizability provided toposes, that is models for constructive type theory; but it also can be used to model stronger systems of impredicative constructive set theory. In time it was recognized that the mathematical structures arising from realizability provided models for more exotic non-classical theories of interest. Work then focused in particular on models for impredicative polymorphic calculi such as System F (Girard [12]) and the Calculus of Constructions (Coquand and Huet [7]), and on Synthetic Domain Theory (Hyland [19] and Taylor [41]). The use of realizability in this context has a quite different character from its earlier metamathematical use. For details of realizability models impredicative type theories the reader may consult Crole [8]. For formal expositions of Synthetic Domain Theory informed by the realizability experience see Reus [34] and Reus and Streicher [35].

The focus of this paper is on the axiom of choice

$$\forall x \in X.\exists y \in Y.\phi(x, y) \to \exists f \in Y^X.\forall x \in X.\phi(x, f(x))\,.$$

I shall call this the *propositional axiom of choice* to distinguish it from the axiom of choice as it holds in Martin-Löf type theory (see Martin-Löf [31]), that is in the propositions-as-types sense.[1] We write the instance of the axiom above as $AC(X \to Y)$.

The models I consider are not new. Some are derived from old interpretations, while I lectured on others in the Netherlands in 1982.[2] My motivation for rehearsing the ideas now is generally that the range of possibilities which, on the one hand, may be used for establishing metamathematical results, and on the other, model exotic non-classical mathematical or computational phenomena, deserves to be better known. When I first thought about abstract approaches to realizability I was sure that at the very least the old techniques would benefit from being put in more mathematically elegant form; except for Scott and his students few took this view at the time. Now with luck a new generation will take all that for granted.

1.2 Maietti's question

A more specific motivation for this paper is that I have been stimulated by a question put to me by Maria-Emilia Maietti. Maietti's question arose in the context of type theory but in categorical form[3] it is essentially this.

> How close to the structure of a topos can one get with the propositional axiom of choice holding, but not the law of the excluded middle?

One can make this question more precise by asking for a category $\mathbb{T}$ of types plus a poset fibration $\mathbb{P} \to \mathbb{T}$ giving a notion of proposition satisfying the following.

- The category $\mathbb{T}$ of types is locally cartesian closed.
- $\mathbb{P} \to \mathbb{T}$ is the subobject fibration: propositions correspond to subobjects in $\mathbb{T}$
- There is a weak subobject classifier $\Omega \in \mathbb{T}$: thus in the type theroy, impredicative higher types are expressed via a weakly generic Prop : Type.
- The propositional axiom of choice holds at all types of $\mathbb{T}$.
- $\mathbb{T}$ has coequalizers of equivalence relations: thus there quotient types.
- Finally we want all the above but with constructive or intuitionistic logic.[4]

[1] Of course in Martin-Löf type theory we do not have an axiom but rather a theorem.

[2] Robin Grayson also worked on clean forms of old interpretations around this time; and soon after Jaap van Oosten studied more subtle interpretations from an abstract point of view.

[3] For accounts of Type Theories and the connection with categories and fibrations, consult the recent books Jacobs [24] and Taylor [42].

[4] More precisely the subobject lattices $\mathbb{P}(X)$ for $X \in \mathbb{T}$ should be Heyting algebras but not generally Boolean algebras.

In this paper we get nowhere near satisfying these requirements; but before explaining our more modest aims, it seems worth making some comments on the original problem.

First if coequalizers of equivalence relations are effective then a quotient of the weak subobject classifier will be a subobject classifier in the strong topos theoretic sense. Thus we would have a topos. (Of course conversely coequalizers are effective in any topos.) Now in a topos the axiom of choice implies the law of excluded middle. (This was observed by Diaconescu. For arguments in the internal logic of toposes see Scott [38] or in the constructive set theory tradition Goodman and Myhill [14].)

Secondly the specific formulation given above may not be fair to Maietti. She has certainly considered variations, some of which she may prefer. Note in particular that the assumption that the types form a locally cartesian closed category gives an extensional type theory. So the constructive significance of the propositional axiom of choice is problematic. With extensionality we have the following basic facts (see Troelstra [43]).

- $AC(\mathbb{N}^{\mathbb{N}} \to \mathbb{N})$ (that is, $AC_{1,0}$ in Troelstra's notation) is already incompatible with Church's Thesis. This is just basic recursion theory.
- $AC(\mathbb{N}^{(\mathbb{N}^{\mathbb{N}})} \to \mathbb{N}$ (that is, $AC_{2,0}$ in Troelstra's notation) is incompatible with weak continuity principles.

These and related issues are discussed also in Troelstra and van Dalen [44] ; I shall not go into the question of their significance for any particular constructive point of view. One should perhaps just note note that Martin-Löf type theory,[5] at least as explained in [31] makes sense conceptually only with intensional equality; and that gives a different reason for avoiding effective equivalence relations.

1.3 Aims

I aim in this paper to describe some realizability models (arising from various realizability toposes) in which some form of the propositional axiom of choice holds. The kind of structure which arise easily satisfy the following.

- The category $\mathbb{T}$ of types forms a locally cartesian closed category.
- The propositional axiom of choice holds for a poset fibration $\mathbb{P} \to \mathbb{T}$ of propositions.
- The category $\mathbb{T}$ of types is closed under quotients of propositional equivalence relations.

[5] One should stress the interest of this predicative approach; after all we can very naturally formalize much of Bishop's Constructive Mathematics in it.

This falls so far short of Maietti's requirements as to be almost ridiculous. Indeed if we did not require $\mathbb{P} \to \mathbb{T}$ posetal, we would be satisfied with the standard fibration $\mathbb{T}^2 \to \mathbb{T}$ for a locally cartesian closed $\mathbb{T}$ with coequalizers. We shall have here to put up with non-standard propositions and we certainly find nothing like a weak classifier for these propositions. Of course we can extract some old style metamathematical information out of the models either directly or in conjunction with standard proof theoretic technique. But I do not do that and instead concentrate on conveying the basics.

This paper arose out of a talk at a Tutorial Workshop. Such events are always more Workshop than Tutorial, but I hope to make redress here by giving some sense of how one thinks about realizability models. Dana Scott first promoted the idea of thinking of realizability in terms of non-standard truth-values. The obvious analogy is with complete Boolean algebras and Boolean-valued models: realizability toposes and the like are what you get when you take that analogy seriously. As they are mathematical structures we can certainly argue about them 'from the outside'. But they can also be regarded as worlds of constructive mathematics; and we get more insight when we can identify (analogues of) standard mathematical arguments which are valid in the internal logic. Often one does not use the full formalism of the internal language; the idea of the internal argument is usually a sufficient guide. I hope to provide an instructive example.

2 Realizability: Variations

2.1 Tripos extensions and geometric morphisms

A *tripos* is a notion of generalized proposition encapsulated in an indexed preordered set $\mathcal{P}$ over a category $\mathcal{S}$ which for the purposes of this paper we take to be the category of sets. The basic notion is due to Pitts [32] and an account of the basic theory is in [20]. The fundamental properties of interest are as follows.

- Each $\mathcal{P}(I)$ models propositional intuitionistic logic. And for $u : J \to I$ in $\mathcal{S}$, reindexing along u gives a map $u^* : \mathcal{P}(I) \to \mathcal{P}(J)$ of preordered sets preserving the propositional operations.

- For $u : J \to I$ in $\mathcal{S}$ we have adjoints $\exists_u \dashv u^* \dashv \forall_u$ satisfying the Beck-Chevalley condition. (The so-called Frobenius reciprocity linking $\exists$ and $\wedge$ is automatic.)

- There is a generic proposition $\gamma \in \mathcal{P}(P)$. Any generalised proposition $\phi \in \mathcal{P}(I)$ is a reindexing of γ along some $I \to P$.

Given a tripos $\mathcal{P}$ over $\mathcal{S}$ one constructs a topos $\mathcal{S}[\mathcal{P}]$, which is obtained from $\mathcal{S}$ by formally adding subobjects to sets $I \in \mathcal{S}$ to represent the (equivalence classes of) elements of $\mathcal{P}(I)$, and then adding quotients of equivalence relations. The objects of the topos $\mathcal{S}[\mathcal{P}]$ are of the form $(X, |\ =\ |)$ where $|\ =\ | \in \mathcal{P}(X \times X)$ is a non-standard equality; elements x of X have a nonstandard extent or degree of existence $|x \in X| = |x = x| \in \mathcal{P}(X)$.

Remarks

1. The basic details are in the original paper[20]. I hope that they are relatively familiar. The first two conditions give the standard notion of a first-order hyperdoctrine. Finer points as to what is needed of the generic object and indeed of the quantification are discussed in Pitts's contribution to the Workshop [33].

2. There other ways to construct realizability toposes. These involve rather different starting points. Generally the issue of quotients can be handled in terms of the fundamental notion of the exact completion (see Carboni[6] and Robinson and Rosolini[37]).

3. As observed independently at least by Scott and Prawitz (see Scott [38]) all the basic operations of intuitionistic logic can be defined in second order logic using just $\rightarrow$ and $\forall$.[6] Hence a tripos is determined up to equivalence of structure by the interpretation of $\rightarrow$, $\forall$ and the generic predicate. This is particularly useful as realizability provides natural interpretations of just this structure.

As a consequence of this last remark the operations $\top$, $\rightarrow$ and $\forall$ play a special role in work on realizability toposes: one should think of them as the basic operations and the others as derived operations. To bring this out I shall refer to $\top$, $\rightarrow$ and $\forall$ as the *functional operations.*

2.2 Geometric morphisms

The notion of geometric morphism of toposes has a natural counterpart for triposes. Suppose that $\mathcal{P}$ and $\mathcal{R}$ are triposes. A geometric morphism $f : \mathcal{R} \rightarrow \mathcal{P}$ consists of an adjoint pair of indexed functors $f_* : \mathcal{R} \rightarrow \mathcal{P}$ and $f^* : \mathcal{P} \rightarrow \mathcal{R}$, $f^* \dashv f_*$, with f^* left exact. (For our preordered sets left exact amounts to preserving $\top$ and $\wedge$.) A geometric morphism of triposes $(f^* \dashv f_*) : \mathcal{R} \rightarrow \mathcal{P}$ induces a geometric morphism $(f^* \dashv f_*) : \mathcal{S}[\mathcal{R}] \rightarrow \mathcal{S}[\mathcal{P}]$ of the corresponding toposes.

The category of sheaves over a locale is given up to equivalence as a tripos extension[7] and tripos extensions share some properties with localic extensions. In particular geometric morphisms between such extensions are localic.

Proposition 2.1 *If $(f^* \vdash f_*) : \mathcal{R} \rightarrow \mathcal{P}$ is a geometric morphism of triposes, then the induced geometric morphism $(f^* \dashv f_*) : \mathcal{S}[\mathcal{R}] \rightarrow \mathcal{S}[\mathcal{P}]$ is localic.*

Proof. This is immediate as a geometric morphism of toposes $(f^* \vdash f_*) : \mathcal{F} \rightarrow \mathcal{E}$ is localic if and only if every object $F \in \mathcal{F}$ is covered by a subobject of a f^*E for some $E \in \mathcal{E}$.[8]

[6]This is also the basis of the construction of data types in 2nd order impredicative type theory (System F). The corresponding idea for classical logic has little computational force and was known to Russell.

[7]This goes back to Higgs [15]: the tripos to topos construction mimics the H-valued sets approach to sheaves.

[8]The proof is given in detail in Awodey, Birkedal and Scott [1].

When dealing with realizability triposes it is important to have a formulation of the left exactness of the inverse image f^* in terms of the functional operations ($\top$, $\rightarrow$ and $\forall$).

Proposition 2.2 *Suppose we have $f_* : \mathcal{R} \rightarrow \mathcal{P}$ and $f^* : \mathcal{P} \rightarrow \mathcal{R}$ with $f^* \dashv f_*$. Then f^* is left exact (that is, f^* preserves $\top$ and $\wedge$ at each $I \in \mathcal{S}$) if and only if the entailments*

- $\top \vdash f^*(\top)$
- $f^*(p \rightarrow f_* r) \vdash f^* p \rightarrow r$

hold in each $\mathcal{R}(I)$.

2.3 Inclusions

By analogy with the notion for geometric morphisms, say that a geometric morphism $(f^* \vdash f_*) : \mathcal{R} \rightarrow \mathcal{P}$ of triposes is an *inclusion* of triposes and that $\mathcal{R}$ is a *subtripos* of $\mathcal{S}$ just when f_* reflects the order (equivalently when the counit $f^* f_* \rightarrow 1_{\mathcal{R}}$ is an isomorphism). (As usual we shall adopt this terminology even when we do not literally have $\mathcal{R} \subseteq \mathcal{S}$.) In the same vein, $(f^* \vdash f_*) : \mathcal{R} \rightarrow \mathcal{P}$ is a *surjection* just when f^* reflects the order (equivalently the unit $1_{\mathcal{P}} \rightarrow f_* f^*$ is an isomorphism, so also $(f^* \vdash f_*)$ is connected). Any geometric morphism factors as a surjection followed by an inclusion.

If $\mathcal{R} \rightarrow \mathcal{P}$ is an inclusion then it is easy to see that the basic functional operators of $\mathcal{R}$ are the restriction of those in $\mathcal{P}$; and moreover that $\mathcal{R}$ is an *exponential ideal* in $\mathcal{P}$ in the sense that whenever $\phi \in P$ and $\psi \in R$ then $\phi \rightarrow \psi \in \mathcal{R}$.[9] Let us say that a tripos $\mathcal{R}$ is a *functional substructure* of a tripos $\mathcal{P}$ just when each $\mathcal{R}(I) \subseteq \mathcal{P}(I)$ and when the basic functional operations of $\mathcal{R}$ are the restrictions of those of $\mathcal{P}$.

It is an elementary but important fact (important especially for realizability triposes) that if $\mathcal{R}$ is a functional substructure of $\mathcal{P}$ and an exponential ideal in $\mathcal{P}$ then $\mathcal{R}$ is a subtripos of $\mathcal{P}$. We explain this in some detail as it is an example of a proof obtained by consideration of some internal mathematics.

The inclusion map $i_* : \mathcal{R} \rightarrow \mathcal{P}$ respects the basic tripos operations ($\top$, $\rightarrow$ and $\forall$). We ask more significantly whether it *preserves limits*. By this we mean what is generally meant in the case of fibred categories: $h : \mathcal{R} \rightarrow \mathcal{P}$ *preserves limits* just when

- pointwise each $h_I : \mathcal{R}(I) \rightarrow \mathcal{P}(I)$ preserves finite limits; and
- h commutes with the right adjoints $\forall_u$.

There is no problem with the second of these, but the first is clear only in special cases. However we can hope!

[9] If $(f^* \vdash f_*) : \mathcal{R} \rightarrow \mathcal{P}$ is an inclusion then f^* commutes with reindexing u^*, so taking right adjoints f_* commutes with $\forall_u$. The exponential ideal property familiar from locale theory is a special case of the corresponding property for sheaves.

On this basis we try to construct a left adjoint $i^* : \mathcal{P} \to \mathcal{R}$ using the adjoint functor theorem. In its poset or preordered set interpretation that suggests a formula of the form

$$i^*(p) = \bigwedge \{ r \in \mathcal{R} \,|\, p \leq i_*(r) \} \,.$$

We transform this (as in the coding of algebraic or inductive data types in second order λ-calculus) into the formula

$$i^*(p) = \forall r \in \mathcal{R}.(p \to r) \to r$$

in the tripos logic. Formally we calculate this in $\mathcal{P}$; but as $\mathcal{R}$ forms an exponential ideal, the answer is in $\mathcal{R}$. Now we need to check things.

The adjunction $i^*(\phi) \vdash \psi$ if and only if $\phi \vdash i_*(\psi)$.
This translates as

$$\forall r.(\phi \to r) \to r \vdash \psi \quad \text{if and only if} \quad \phi \vdash \psi$$

which follows easily by intuitionistic logic. Just for fun we use Paul Taylor's old proof tree macro to present the proofs. First I give the tree showing that $\forall r.(\phi \to r) \to r \vdash \psi$ implies $\phi \vdash \psi$.

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And now I give the tree showing that $\phi \vdash \psi$ implies $\forall r.(\phi \to r) \to r \vdash \psi$.

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The definition of a tripos says in effect that it is sound for intuitionistic logic; so these proofs establish the adjunction.

Left exactness $i^*(\top) = \top$ and $i^*(\phi \to i_*\psi) \vdash i^*\phi \to \psi$.
This translates into the conditions

$$\top \vdash \forall r.(\top \to r) \to r \quad \text{and} \quad \forall r.((\phi \to \psi) \to r) \to r \vdash (\forall r.(\phi \to r) \to r) \to \psi \,.$$

The proof trees demonstrating these are easy and we omit them.

The above discussion proves the following.

Theorem 2.3 *If a tripos $\mathcal{R}$ is a functional substructure of a tripos $\mathcal{P}$ and an exponential ideal in $\mathcal{P}$, then $\mathcal{R}$ is a subtripos of $\mathcal{P}$.*

2.4 Basic realizability triposes

We take the point of view of categorical logic: so we use realizability (or other functional interpretations) to provide a tripos. We recall the basic set-up. Let $(A, \cdot)$ be a partial combinatory algebra (PCA). We define the indexed family $\mathcal{P}(I)$ of preordered sets as follows. First we define an internal implication on the power set $P(A)$ by setting

$$p \to q = \{c \in A \,|\, \forall a \in p.\, (c \cdot a) \in q\} \in \mathcal{P}(A)$$

for each $p, q \in P(A)$. Then we set $\mathcal{P}(I) = (\mathcal{P}(I), \vdash) = (P(A)^I, \vdash)$ where for $\phi, \psi \in P(A)^I$ we define

$$\phi \vdash \psi \quad \text{if and only if} \quad \bigcap_{i \in I} \phi(i) \to \psi(i) \neq \emptyset\,.$$

The crucial feature of this definition is that while the underlying set of $\mathcal{P}(I)$ is given as the I-indexed power $P(A)^I$, the preorder is not defined pointwise.

The easy properties of this realizability are as follows.

- Each $\mathcal{P}(I)$ models minimal logic with implication defined pointwise. And for $u : J \to I$ in $\mathcal{S}$, reindexing along u gives a map $u^* : \mathcal{P}(I) \to \mathcal{P}(J)$ of preordered sets preserving implication.
- For $u : J \to I$ in $\mathcal{S}$ we have adjoints $\exists_u \dashv u^* \dashv \forall_u$ satisfying the Beck-Chevalley condition. (These are essentially given by union and intersection; but *partial* combinatory algebras force us to be a bit more subtle.)
- There is a generic proposition $\gamma \in \mathcal{P}(P(A))$ given by $\gamma(p) = p$.

By Remark 3 in 2.1, this is enough to generate the full structure of a tripos.

2.5 Some variations

One should regard realizability as just one kind of functional interpretation. So one gets variations on the basic idea of the relizability tripos and topos either by changing the nature of the sets of realizers or by changing the style of functional interpretation. I list a few of these varieties of functional interpretation.

- Relative realizability. Typically this involves variations on how the preorder $\vdash$ is defined.
- Restricted realizability. This involves allowing only a restricted set of truth-values rather than all of $P(A)$.
- Modified realizability. This is a very natural functional interpretation with a clear propositions as types flavour.
- Dialectica interpretations. These are further special forms of functional interpretation.

- Extensional realizability. Here one enriches the collection of truth values.
- Realizabilities formed by iterations of triposes. (These are of all shapes and sizes.)

This is certainly not a complete list. Other possibilities have recently been developed by van Oosten, and some were mentioned in his Workshop contribution [46].

In this paper I shall concentrate on the use of restricted realizability and extensional realizability for giving models for extensional type theory with the propositional axiom of choice; but in the last section I make some brief comments on others in the list.

3 Restricted Realizability

3.1 The subtripos

The idea of restricted realizability is to take some subcollection $\mathcal{R}(I) \subseteq \mathcal{P}(I)$, of the basic collection of truth-values, closed under suitable logical operations. Such subcollections arise naturally when the PCA $(A, \cdot)$ has some additional structure respected by application. For example if A is partially ordered and application $\cdot$ preserves the order, then one could require that the truth-values $\psi \in \mathcal{R}(I)$ be downwards or upwards closed. Of course the result may be pretty trivial.[10] However the basic idea is a good one.

Here I concentrate on one particular kind of example. Suppose that $(D, \cdot)$ is a Scott domain model of the lambda calculus which for simplicity I take to be a complete lattice. Again for simplicity I consider only the restricted realizability tripos

$$R(D) = \{\phi \in P(D) \,|\, \phi \text{ is closed under } \vee\}\,,$$

consisting of the $\vee$-closed subsets of D.[11] The basic functional structure ($\top$, $\rightarrow$ and $\forall$) is inherited from the standard realizability tripos $\mathcal{P}$, and the generic proposition is obvious. Much of the following is then true by definition.

Proposition 3.1 *The restricted realizability tripos $\mathcal{R}$ is a functional substructure of the standard realizability tripos $\mathcal{P}$. And $\mathcal{R}$ forms an exponential ideal in $\mathcal{P}$: if $\phi \in P$ and $\psi \in R$ then $\phi \rightarrow \psi \in \mathcal{R}$.*

Proof. The significant point is that $\mathcal{R}$ is an exponential ideal. For that note that we are dealing with Scott domains, so $\vee$ on a function space is given pointwise. Thus if $f, g : p \rightarrow r$ with r itself $\vee$-closed, then $f \vee g : p \rightarrow r$.

[10] For example if A has a bottom element $\perp$ then all nonempty downclosed sets contain $\perp$. It follows that a proposition $\phi(i)$ indexed over $i \in I$ is determined up to equivalence by the $i \in I$ for which $\phi(i)$ is empty/non-empty. So the tripos is equivalent to the standard power set tripos on $\mathcal{S}$.

[11] It might be computationally more natural to add a density condition.

We can now use Theorem 2.3 to deduce the following basic fact about restricted realizability.

Proposition 3.2 *The restricted realizability tripos $\mathcal{R}$ is a subtripos of the standard realizability tripos $\mathcal{P}$.*

WARNING The naive idea to take for $i^*(p)$ the closure of p under $\vee$ is hopelessly wrong. The correct formula

$$i^*(p) = \forall r \in \mathcal{R}.(p \to r) \to r$$

comes from the proof of the theorem.

3.2 Assemblies and modest sets

We recall some special full subcategories of the standard realizability models. First there the *assemblies*. Concretely these are objects $(X, | = |)$ of the topos $\mathcal{S}[\mathcal{P}]$ such that

$$|x = x'| = \begin{cases} |x \in X| = |x' \in X| & \text{if } x = x', \\ \emptyset & \text{otherwise.} \end{cases}$$

So assemblies are obtained by just adding subobjects but no quotients. Abstractly the category of assemblies is equivalent to that of the separated objects for the double negation topology. So they form a reflective subcategory in the indexed sense. The definition of assemblies still makes sense for restricted realizability: again they form a reflective subcategory equivalent to that of the separated objects.

Next there are the modest sets.[12] Concretely these are objects $(X, | = |)$ of the topos $\mathcal{S}[\mathcal{P}]$ such that

$$|x = x'| = \begin{cases} |x \in X| = |x' \in X| & \text{if } |x \in X| \cap |x' \in X| \neq \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$

It follows that modest sets are assemblies where distinct elements have disjoint realizing sets. Abstractly they are separated objects orthogonal to the codiscrete object $\Delta 2$. For details see Hyland, Robinson and Rosolini [23] where there is an account of the internal and indexed category of such orthogonality classes of objects, the 'discrete objects' in any standard realizability topos. Again the definition of modest sets, their identification with the separated objects orthogonal to $\Delta 2$, and their representation as an internal category carry over without difficulty for restricted realizability.[13]

[12] This is Scott's term. I originally called them effective objects as generalizing the effective operations (see Hyland [17]). Most published work on realizability interpretations is concerned with such obects.

[13] Thomas Streicher has observed that the representation as an internal category needs attention: he has a definite negative result for modifed realizability.

3.3 Flat sets

The motivation for the objects we now consider comes from experience with the continuous functionals (Kreisel [28]) or countable functionals (Kleene [26]). These were first proposed in the 1950s in connection on the one hand with the foundations of analysis[14], and on the other with generalized recursion theory. Kleene's recursion theoretic interests were taken up in the 1970s; and it was also recognized around that time that the continuous functionals are the (global sections of the) higher types in various realizability and sheaf models. Recently the subject has come back into play in connection with Equilogical Spaces (Scott [39]) and related categories (see in particular Birkedal, Carboni, Rosolini and Scott [4]).

For us the main point will be the old but unpublished fact that choice principles hold for finite types in appropriate realizability models based on domains with continuous functions. Here we will give one considerable generalization.

Recall that the continuous functionals can be represented as quotients of subspaces of domains as described for example in Hyland [16].[15] In that treament the domains vary with the types; however using just one universal domain gives exactly the representation as a modest set in the realizability topos. Now the equivalence classes in these representations have a particularly simple form:

- if $\xi \subseteq D$ is the equivalence class corresponding to a continuous functional x, then $X \subseteq \xi$ implies $\bigvee X \in \xi$ and $a, b \in \xi$ implies $a \wedge b \in \xi$.

Furthermore the equivalence classes are incomparable:

- if $\xi, \eta \subseteq D$ are the equivalence classes for $x \neq y$, then $a \in \xi$ and $b \in \eta$ imply a and b are incomparable.

There are two sides to this: the equivalence classes are extremely well behaved (one can relate this to various notions of filter of limit space); and the induced topology on the object is something like 'Hausdorff'.

We do not need conditions quite as specific as these; but consideration of how the properties of the equivalence classes can be used to prove choice principles for higher types suggests a number of abstract possibilities. We could consider objects X (necessarily modest sets) which are any of the following:

- 2-replete, that is orthogonal to all 2-equable maps;
- 2-separated, that is the canonical map $X \to 2^{2^X}$ is monic;
- (stably) orthogonal to Σ the natural open subobject classifier.[16]

[14] Remarkably Kreisel's paper also gives the first account of his modified realizability.

[15] The basic phenomenon was first observed by Ershov. For a succinct account of his original perspective see [11].

[16] Up to isomorphism there is a unique choice of Σ classifying open subsets of the internalisation of D.

Of these possibilities the first two are pretty restrictive. The first is the most restrictive: it defines what we might reasonably think of as the profinite objects (where 'finite' means 'decidable finite'). The second easily contains the first; but while it is (presumably) less restrictive it still retains some aspects of total disconnectedness: no reasonable representation of the real numbers can have enough maps into 2 in a constructive universe such as ours. We concentrate on the third which is considerably less restrictive: it effectively requires just that the order obtained from the intrinsic Σ 'topology' be discrete.

For the moment let us define an object X to be *flat* just when X is a modest set stably orthogonal to Σ. (Of course we intend this in an internal or indexed sense. A discussion of orthogonality from the indexed point of view is in Hyland and Moggi [21].) The crucial property of flat objects is as the following.

Proposition 3.3 *Let X be flat. Suppose that a realizing $x \in X$ and a' realizing $x' \in X$ are such that $a \leq a'$. Then $x = x'$*

Proof. To represent Σ we take some non-trivial Scott open subset $U \subseteq D$: and we let Σ be the modest set with underlying set $\{\top, \bot\}$ and with

$$|\top \in \Sigma| = U \text{ and } |\bot \in \Sigma| = D - U\,.$$

If $a \leq a'$ then there is a unique map $\Sigma \to X$ tracked by the continuous function mapping U to a' and $D - U$ to a; this map takes $\bot$ to x and $\top$ to x'. But if X is flat then this map must be constant. So $x = x'$.

Finally in this section we record some closure properties of our classes of objects. Recall that as explained in detail in Hyland, Robinson and Rosolini [23], the modest sets form a small internal (and so indexed) category in the relizability topos internally complete and cocomplete for separable diagrams.[17]

Proposition 3.4 *The indexed categories of 2-replete, of 2-separated, and of flat modest sets all give reflective (internal or indexed) subcategories of the (internal or indexed) category of modest sets. These categories are locally cartesian closed categories and complete and cocomplete for separable diagrams.*

Proof. This proposition is essentially obvious by internal category theory. Each of the conditons can be described internally in the separated part of the topos. Intuitively the objects satisfying any of the conditions are closed under limits; so one can use an internal or indexed adjoint functor theorem to show that the inclusion of the full subcategory has a left adjoint. One rather crude way of making this precise avoiding internal aspects is explained in Hyland and Moggi [21]. The local cartesian closedness is also (not very obviously I am afraid) in [21]. By general category theory, the internal completeness and cocompleteness follow from the corresponding properties for modest sets.

[17] There is a good and more accessible account of the issues in Robinson [36].

3.4 Realizing the axiom of choice

We regard $\mathcal{S}[\mathcal{R}]$ as a subtopos of $\mathcal{S}[\mathcal{P}]$, taking objects from the former but using the logic of the latter. Our basic result is the following.

Proposition 3.5 *Suppose that $X \in \mathcal{S}[\mathcal{R}]$ is separable and $Y \in \mathcal{S}[\mathcal{R}]$ is flat. Then*

$$\forall x \in X.\exists y \in Y.\phi(x,y) \to \exists f \in Y^X.\forall x \in X.\phi(x, f(x))$$

holds in the logic of $\mathcal{S}[\mathcal{P}]$.

Proof. The terminology we use is from [18]. First we consider the general issue. Typical realizability models fail to satisfy axioms of choice for all but the simplest modest sets. Suppose that X and Y are modest (or X seprable and Y modest) and that c realizes $\forall x \in X.\exists y \in Y.\phi(x,y)$. If a realizes $x \in X$, then $c \cdot a$ represents a pair $(c_1 \cdot a, c_2 \cdot a)$ say with $c_1 \cdot a$ realizing $y \in Y$ for some y, and then $c_2 \cdot a$ realizing $\phi(x,y)$. However if a and a' are different codes for $x \in X$, then $c_1 \cdot a$ and $c_1 \cdot a'$ may be codes for $y \in Y$ and $y' \in Y$ for distinct y and y'. Thus no function from X to Y need be tracked. However if X is seperable in the subtopos $\mathcal{S}[\mathcal{R}]$ then we have $a \vee a'$ also a code for $x \in X$. So we have $c_1 \cdot a, c_1 \cdot a' \leq c_1 \cdot (a \vee a')$ all codes for some $y \in Y$. Thus there is a function $f : X \to Y$ tracked and so realized by c_1; and then c_2 gives a realizer for $\phi(x, f(x))$. This completes the proof.
It is as well to stress the importance of sticking with the logic of $\mathcal{S}[\mathcal{P}]$. It seems very difficult to calculate in the logic of $\mathcal{S}[\mathcal{R}]$.

We can consider our model of flat objects from two points of view. First we can take the point of view of the topos $\mathcal{S}[\mathcal{P}]$.

Theorem 3.6 *The collection of flat objects of $\mathcal{S}[\mathcal{R}]$ forms an internal locally cartesian closed category in $\mathcal{S}[\mathcal{P}]$ complete and cocomplete for separable diagrams; and the propositional axiom of choice holds in $\mathcal{S}[\mathcal{P}]$ for all such objects.*

Secondly we can relate the flat objects to Maietti's question.

Theorem 3.7 *The collection of flat objects of $\mathcal{S}[\mathcal{R}]$ form a locally cartesian closed category $\mathbb{T}$. Equipping each $A \in \mathbb{T}$ with its lattice of subobjects in $\mathcal{S}[\mathcal{P}]$ gives a non-standard poset fibration $\mathbb{P} \to \mathbb{T}$.*

- *The propositional axiom of choice holds in the poset fibration.*
- *$\mathbb{T}$ has quotients for $\mathbb{P}$-equivalence relations.*[18]

Proof. For the last point take the quotient in $\mathcal{S}[\mathcal{P}]$ and then reflect.

[18] The treatment of quotients in Jacobs [24] should make clear what is intended.

4 Extensional Realizability

4.1 The local localic extension

The idea of extensional realizability is old, but should be well known as it provides about the simplest case of Lawvere's 'unity and identity of opposites' in the context of realizability toposes. Suppose that $(A, \cdot)$ is a partial applicative structure. Then we have the standard tripos $\mathcal{P}$ of ordinary realizability :

$$\mathcal{P}(I) = (P(A)^I, \vdash)\,.$$

But we can also consider another tripos:

$$\mathcal{PER}(I) = (\mathbf{PER}(A)^I, \vdash)$$

whose elements are I-indexed families of the PERs or modest sets of ordinary realizability, and whose preorder is the preorder reflection of the indexed category of 'I-indexed PERs'.

Let $f : \mathcal{PER} \to \mathcal{P}$ be defined by $f(\rho) = |\rho|$, the field of definition of the partial equivalence relation ρ; and for a subset ϕ of the PCA let $d(\phi)$ be ϕ with the discrete equivalence relation, and $c(\phi)$ be ϕ with the codiscrete or chaotic equivalence relation.

Proposition 4.1 *We have the adjunctions $d \dashv f \dashv c$ and d is left exact. So we have geometric morphisms of triposes*

$$(d \dashv f) : \mathcal{PER} \to \mathcal{P} \quad \text{and} \quad (f \dashv c) : \mathcal{P} \to \mathcal{PER};$$

and hence we have geometric morphisms of toposes

$$(d \dashv f) : \mathcal{S}[\mathcal{PER}] \to \mathcal{S}[\mathcal{P}] \quad \text{and} \quad (f \dashv c) : \mathcal{S}[\mathcal{P}] \to \mathcal{S}[\mathcal{PER}].$$

Proof. This is more or less obvious.

In fact we can say a bit more about these geometric morphisms.

Proposition 4.2 *The geometric morphism $(d \dashv f) : \mathcal{PER} \to \mathcal{P}$ is connected, $(f \dashv c) : \mathcal{P} \to \mathcal{PER}$ is an inclusion and so $\mathcal{PER}$ is a local extension of $\mathcal{P}$. Hence $(d \dashv f) : \mathcal{S}[\mathcal{R}] \to \mathcal{S}[\mathcal{P}]$ is a connected, $(f \dashv c) : \mathcal{S}[\mathcal{P}] \to \mathcal{S}[\mathcal{R}]$ is an inclusion and so $\mathcal{S}[\mathcal{R}]$ is a local (localic) topos over $\mathcal{S}[\mathcal{P}]$.*

We call the objects of $\mathcal{S}[\mathcal{PER}]$ in the essential image of $d : \mathcal{P} \to \mathcal{PER}$ *discrete*[19] and objects in the essential image of $c : \mathcal{P} \to \mathcal{PER}$ *chaotic*. Note that the chaotic objects form an indexed reflective subcategory of $\mathcal{S}[\mathcal{PER}]$.

[19] Let us not worry about the clash of terminology.

4.2 Extensional assemblies and modest sets

An *extensional assembly* is an object $(X, | = |)$ of the topos $\mathcal{S}[\mathcal{PER}]$ such that

$$|x = x'| = \begin{cases} |x \in X| = |x' \in X| & \text{if } x = x', \\ \emptyset & \text{otherwise.} \end{cases}$$

So extensional assemblies are obtained by just adding subobjects corresponding to $\mathcal{PER}$ but no quotients. Again abstractly the category of assemblies is equivalent to that of the separated objects for the double negation topology. So they form a reflective subcategory in the indexed sense. With ones bare hands one can show that the functors $c : \mathcal{P} \to \mathcal{PER}$, $e : \mathcal{PER} \to \mathcal{P}$ and $d : \mathcal{P} \to \mathcal{PER}$ preserve assemblies; but it is more instructive to see this as an abstract consequence of the easy identification of assemblies with separated objects for the double negation topology. The standard functors $\Delta : \mathcal{S} \to \mathcal{S}[\mathcal{P}]$ and $\Delta : \mathcal{S} \to \mathcal{S}[\mathcal{PER}]$ include $\mathcal{S}$ in $\mathcal{S}[\mathcal{P}]$ and $\mathcal{S}[\mathcal{PER}]$ respectively as sheaves for the double negation topology. And Δ commutes with our functors $d \dashv f \dashv c$. But separated objects are subobjects of sheaves. So as c, f and d are all left exact, it follows that they map separated objects to separated objects.

An *extensional modest set* is an object $(X, | = |)$ of the topos $\mathcal{S}[\mathcal{PER}]$ such that

$$|x = x'| = \begin{cases} |x \in X| = |x' \in X| & \text{if } f(|x \in X|) \cap f(|x' \in X|) \neq \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$

In other words, extensional modest sets are extensional assemblies where the realizing PERs of distinct elements have disjoint fields of definition. Abstractly they are again separated objects orthogonal to the object $\Delta 2 \in \mathcal{S}[\mathcal{PER}]$. Let Mod be the category of modest sets in $\mathcal{S}[\mathcal{P}]$ and EMod the category of extensional modest sets in $\mathcal{S}[\mathcal{PER}]$. One can see either concretely or by abstract orthogonality considerations that we have

$$c : \mathrm{Mod} \to \mathrm{EMod}, \quad f : \mathrm{Emod} \to \mathrm{Mod} \quad d : \mathrm{Mod} \to \mathrm{EMod}\,.$$

In particular $c : \mathrm{Mod} \to \mathrm{EMod}$ embeds the category of modest sets as a reflective subcategory of the category of extensional modest sets. Its essential image are the *chaotic modest sets.*

Let us conclude this section by recording some closure properties of the chaotic modest sets.

Proposition 4.3 *The indexed category of chaotic modest sets reflective (internal or indexed) subcategories of the (internal or indexed) category of extensional modest sets. The category is locally cartesian closed categories and internally complete and cocomplete for separable diagrams.*

Proof. Along the same lines as 3.4.

4.3 Realizing the axiom of choice

Extensional realizability is more familiar and easy to calculate with.

Proposition 4.4 *Suppose that X is separable chaotic and Y is modest. Then*

$$\forall x \in X.\exists y \in Y.\phi(x, y) \to \exists f \in Y^X.\forall x \in X.\phi(x, f(x))$$

holds in the extensional realizability logic of $\mathcal{S}[\mathcal{PER}]$.

Proof. Let c realize $\forall x \in X.\exists y \in Y.\phi(x, y)$. So if a realises $x \in X$ then $c \cdot a$ represents a pair $(c_1 \cdot a, c_2 \cdot a)$ say with $c_1. \cdot a$ realizing $y \in Y$ for some y, and then $c_2 \cdot a$ realizing $\phi(x, y)$. But as X is chaotic any two a, a' realizing x are related; so $c_1 \cdot a$ and $c_1 \cdot a'$ are related; so as Y is modest y is uniquely determined by x. Thus there is a function $f : X \to Y$ tracked and so realized by c_1; then c_2 gives a realizer for $\phi(x, f(x))$. This completes the proof.

While extensional realizability gives us more structure (in the form of the left adjoint d) than we had for restricted realizability, we do not exploit that here. Again we can consider our model of chaotic modest sets from two points of view. First we can take the point of view of the topos $\mathcal{S}[\mathcal{PER}]$.

Theorem 4.5 *The collection of chaotic modest sets forms an internal locally cartesian closed category in $\mathcal{S}[\mathcal{PER}]$ complete and cocomplete for separable diagrams; and the propositional axiom of choice holds in $\mathcal{S}[\mathcal{PER}]$ for all such objects.*

Secondly we can relate the chaotic modest sets to Maietti's question.

Theorem 4.6 *The collection of chaotic modest sets form a locally cartesian closed category $\mathbb{T}$. Equipping each $A \in \mathbb{T}$ with its lattice of subobjects in $\mathcal{S}[\mathcal{PER}]$ gives a non-standard poset fibration $\mathbb{P} \to \mathbb{T}$.*

- *The propositional axiom of choice holds in the poset fibration.*
- *$\mathbb{T}$ has quotients for $\mathbb{P}$-equivalence relations.*[20]

Proof. For the last point take the quotient in $\mathcal{S}[\mathcal{PER}]$ and then reflect.

5 A Panorama of Functional Interpretation

5.1 Other variations

Relative realizability This idea goes back to Kleene-Vesley [27]. Suppose that $(B, \cdot)$ is a subPCA of $(A, \cdot)$ that is a subapplicative structure of $(A, \cdot)$ containing k and s. Then one can define a new preorder $\vdash$ on $P(A)^I$ by

$$\phi \vdash^{(B)} \psi \quad \text{if and only if} \quad \bigcap_{i \in I} (\phi(i) \to \psi(i)) \cap B \neq \emptyset\,.$$

[20] Again see the treatment of quotients in Jacobs [24].

The point is that while members of A contribute to the meaning of a proposition, that proposition is valid just when it has members in B. In the original Kleene-Vesley case, A is $\mathbb{N}^{\mathbb{N}}$ under Kleene application, and B is the subalgebra of recursive functions. The general idea is however being exploited for quite new purposes by Scott's group at Carnegie-Mellon (see Awodey, Birkedal and Scott [1]). One should note in particular their extended modal language and their analysis of the local localic extensions in the Workshop contribution.

Relative realizability provides another exercise in calculations based on intuitions about the internal logic. Let $\mathcal{R}(I) = (P(B)^I, \vdash)$ be the standard realizability tripos derived from the PCA $(B, \cdot)$, and $\mathcal{Q}(I) = (P(A)^I, \vdash^{(B)})$ be the relative realizability tripos derived as above from $(B, \cdot) \subseteq (A, \cdot)$. We have an obvious adjoint pair $\Delta : \mathcal{R} \to \mathcal{Q}$ and $\Gamma : \mathcal{Q} \to \mathcal{R}$ with $\Delta \dashv \Gamma$, where $\Delta(\psi) = \psi$ and $\Gamma(\phi) = \phi \cap B$. This clearly a geometric morphism. What is more intuitively Γ preserves colimits: $\exists_u$ is essentially given by union and $- \cap B$ distributes over union; preservation of $\perp$ is obvious and as $\vee$ can be defined ad hoc using k and s to construct codings we could (perhaps not quite convincingly) deal with that too. So we would expect to be able to define a right adjoint ∇ via the adjoint functor theorem. This suggests a formula of the form

$$\nabla\psi = \bigvee \{ q \,|\, \Gamma q \leq \psi \} \, .$$

We transform this (in the style of the coding of coinductive data types) into

$$\nabla\psi = \exists q . q \wedge (\Gamma q \to \psi) \, .$$

This looks as if it needs interpretation: what is that implication? It could either be $\Delta(\Gamma q \to_{\mathcal{R}} \psi)$ where $\to_{\mathcal{R}}$ is the implication in $\mathcal{R}$; or it could be $\Delta\Gamma q \to \psi$ where $\to$ is the implication in $\mathcal{Q}$. These are not the same, but in the context of the formula it does not matter.

Recall from section 2.3 that a geometric morphism $(f^* \dashv f_*) : \mathcal{R} \to \mathcal{P}$ of triposes is *surjective* or a *surjection* just when f^* reflects the preorder: $f^*(\phi) \vdash f^*(\psi)$ implies $\phi \dashv \psi$.

Proposition 5.1 *Suppose that* $(\Delta \dashv \Gamma) : \mathcal{Q} \to \mathcal{R}$ *is a surjective geometric morphism of triposes such that* Γ *preserves existential quantification. Then* Γ *has a right adjoint* ∇*.*

Proof. Set $\Delta\Gamma = \sharp$ for consistency with [1]. Then $\sharp$ is left exact and so gives a monoidal comonad which preserves existential quantification. Arguing in the second order modal logic we show that $\sharp\phi \vdash \psi$ if and only if $\phi \vdash \flat\psi$ (the notation again that of [1]) where we can define

$$\text{either} \quad \flat\psi = \exists q . q \vee \sharp(\sharp q \to \psi) \quad \text{or} \quad \flat\psi = \exists q . q \vee (\sharp q \to \psi) \, .$$

Now for $\chi \in \mathcal{R}$, set $\nabla\chi = \flat\Delta\chi$. Restricting above to $\psi = \Delta\chi$ and using the faithfulness of Δ we deduce $\Gamma\phi \vdash \chi$ if and only if $\phi \vdash \nabla\chi$.

Finally one should note that the global sections of subobject classifiers in relative realizability toposes provide all sorts of curious (incomplete) Heyting algebras. In particular from the Kleene-Vesley set-up one obtains the opposite of the old lattice of Medvedev degrees of mass problems (see [25]). Hartley Rogers gives Medvedev's result that the opposite of his lattice is a Heyting algebra. Here is the reason.

Modified realizability Kreisel [28] introduced typed modified realizability quite explicitly as a functional interpretation. The most obvious feature is that modified realizability refutes Markov's principle (see Troelstra [43] for example). The untyped form of modified realiability leading to toposes comes originally from Troelstra [43]. (In the context of this paper a warning is perhaps in order. Troelstra explains that in the intensional (non-extensional) context (cf HRO and ICF below) modified realizability validates the propositional axiom of choice; but that does not hold for the extensional hierarchy.)

An extremely general setting for modified realizability employing a weakening of the notion of a PCA was developed by Hyland and Ong [22]. And recently Thomas Streicher and Jaap van Oosten have made calculations in this area (see van Oosten [45] for example). Both van Oosten [46] and Longley [29] at the Workshop addressed aspects of the interplay between typed and untyped realizability. One result which is significant in this contex is a theorem of Bezem [3] which we briefly explain. In any standard realizability topos there is a weakly cartesian closed category whose objects are closed subobjects of the natural modest set of realizers. Take either the effective topos or the topos based on Kleene's function application. The natural numbers object is an object of this weakly cartesian closed subcategory so we can take the higher types over it: we get in a natural way the hereditarily recursive operations HRO in one case and the intensional continuos functionals ICF in the other. Now we can take the subquotients HRO^E and ICF^E respectively given by hereditarily requiring extensionality. The standard extensional higher types on the other hand are the hereditarily effective operations HEO and the standard (extensional) continuous functionals ECF respectively. Then we have the following.

Theorem 5.2 (Bezem) $\mathrm{HRO}^E \cong \mathrm{HEO}$ *and* $\mathrm{ICF}^E \cong \mathrm{ECF}$.

It would be good to understand better how continuity is used in the argument.

Dialectica style interpretations I believe that the (idea of the) Dialectica interpretation also gives rise to realizability-like triposes. One idea already in Girard [12] is that of giving a Dialectica interpretation to higher order logic using the impredicative theory of functions F_ω; there is a brief conceptual analysis in Troelstra [43]. This suggests a simple interpretation of which I give a brief outline. Assume $(\mathbb{N}, \cdot)$ is the standard Kleene applicative system in the convenient form used by Troelstra [43] for his version of modified realizability for the theory of species (i.e. second order constructive sets). So certainly $0 \cdot n = 0$ all n, and we may as well also assume a pairing in which $(0, 0) = 0$. Then we consider propositions of the form

$$A \subseteq U \times X$$

where $U, X \subseteq \mathbb{N}$ each contain 0. We let $\mathcal{D}(I)$ be the collection of I-indexed families of such structures; and we set

$$(A(i) \subseteq U(i) \times X(i)) \vdash (B(i) \subseteq V(i) \times Y(i))$$

just when there exists $f, F \in \mathbb{N}$ where in the usual realizability sense

$$f : U(i) \to V(i) \quad \text{and} \quad F : U(i) \times Y(i) \to X(i)\,,$$

and where for $u \in U(i)$ and $y \in Y(i)$,

$$(u, F \cdot (u, y)) \in A(i) \quad \text{implies} \quad (f \cdot u, y) \in B(i)\,.$$

The $\forall$ quantification of an indexed family $(B(i) \subseteq V(i) \times Y(i))$ along the projection $I \to 1$ can easily be described. We take

$$\bar{B} \subseteq \bigcap_i V(i) \times \bigcup_i Y(i)\,,$$

where

$$\bar{B}(v, y) \quad \text{if and only if} \quad B(i)(v, y) \quad \text{for all } i \text{ with } y \in Y(i) \quad .$$

The generic proposition is the natural indexed family of all truth values.

The beautiful and mathematically natural Diller-Nahm [10] variant of the Dialectica interpretation and its variants (for which see Diller [9] and Stein [40] can also be treated in this way, and thus give rise to toposes. Most recently there has been some very interesting work on interpreting constructive set theory by Burr [5] using an extension of the ideas of Diller-Nahm which presents an interesting challenge to the abstract point of view. (The investigations of the Münster school present a challenge to the categorical perspective and deserve to be better known.)

Unfortunately the possibility sketched above is really not faithful to the original Dialectica interpretation: we have lost the decidability of the basic predicates. And my memory is that when I looked at this model in the early 1980s I found its properties disappointing. (I forget the details and have lost my notes!) There are other possibilities more closely following the Girard's use of typed impredicative functions, but I do not know that they have ever been investigated.

Iterations of triposes These can be used in many ways. Most obviously they give an elegant account of old proof theoretically motivated variants of realizability such as q-realizability and mq-realizability (see Troelstra [43]). The resulting topos can also be obtained by glueing. Perhaps the most telling use of iteration is in the proof of Goodman's Theorem (see [13]).

Theorem 5.3 (Goodman) *Consider Heyting arithmetic extended with higher function types. The propositional axiom of choice together with extensionality at these higher types is conservative over Heyting arithmetic.*

The proof as analysed by Beeson [2] uses a realizability extension of a sheaf model: the sheaf model is used to create a generic function $\mathbb{N} \to \mathbb{N}$, and then we use Kleene application for functions recursive in the generic function as oracle.[21] (Of course there is still some proof theory to do.) The technique throws some further light on Maietti's question, but really needs a separate paper.

5.2 Final thoughts

Perhaps the most natural computational perspective on realizability is that insight is obtained by varying the underlying PCA. The idea is that distinct PCAs correspond to distinct computational paradigms and these show up in the logic of the corresponding topos. Compelling support for this point of view was provided by Longley's analysis of exact functors between standard realizability toposes. He gives a convincing computational counterpart in his notion of applicative morphism (see Longley [30]).

By contrast what I have tried to do here is to give some impression of the rich panorama of models which can be obtained from the the general notion of a functional interpretation point by varying the style and flavour of the interpretation. Perhaps it is not clear exactly how to think of this from a computational point of view; but if we look carefully at examples, something may emerge. At least the existence of this range of possibilities challenges any simplistic view of the primacy of the standard realizability models. Anyway there is plenty of work to do. My final hope is to have shown that it helps to treat the models frrom an abstract point of view.

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