The S-replete construction

J.M.E. Hyland
M.Hyland@pmms.cam.ac.uk
DPMMS, Univ. of Cambridge
16 Mill Lane
CB2 1SB Cambridge, UK

E. Moggi
moggi@disi.unige.it
DISI, Univ. di Genova
v. Benedetto XV, 3
16132 Genova, Italy

January 30, 1995

1 Introduction

The replete construction was introduced independently by Hyland and Taylor (see [Hyl91, Tay91]) in the context of Synthetic Domain Theory (SDT). Given a model of SDT, i.e. a topos $\mathcal{E}$ with a *classifier* $t: 1 \to \Sigma$ satisfying certain axioms, the construction was used to define a full reflective subcategory of $\mathcal{E}$, the category of Σ-replete objects, which is *suitable* for Denotational Semantics.

However, the replete construction is of a very general nature and most of its properties are independent of the axioms for Σ and of $\mathcal{E}$ being a topos. We consider the replete construction in the more general setting of a cartesian closed $\mathcal{B}$-fibration $p: \mathcal{C} \to \mathcal{B}$, where $\mathcal{B}$ is a category with finite products. This encompasses the case of a cartesian closed category (i.e. $p: \mathcal{C} \to 1$) and that of a locally cartesian closed category (quasi-topos or topos) fibred over itself (i.e. $p = cod: \mathcal{B}^{\to} \to \mathcal{B}$). The first case is of interest in the context of Classical Domain Theory, while the second one is typical of SDT. There are important conceptual advantages of working in a fibred setting:

- It makes clear that additional properties of the construction rarely depend on further assumptions about the base category $\mathcal{B}$.
- It shows which form of stability is needed in a fibre and which involves reindexing. This clarifies many arguments.
- It enables one to distinguish two notions: replete and fibrewise replete. Both notions correspond to full subcategories of $\mathcal{C}$. The replete objects form a full subfibration of p, and so are the natural focus of interest. However fibrewise replete objects are easier to study and are closed under many categorical constructions when viewed as giving a full subcategory of the fibre $\mathcal{C}_I$ over I. Under mild assumptions about p the two notions of replete coincide, so one gets the best of both.
- It avoids appeals to internal category theory, which many find obscure!

In studying the replete construction, we have focused on those properties which are more interesting in relation to SDT, Axiomatic Domain Theory (see [Fre91, Sim92]) and Evaluation Logic (see [Mog94]).

1.1 Summary of main definitions and results

This section summarizes the main definitions and results, but for the sake of simplicity they are not stated in the most general form. In this section we fix a category $\mathcal{B}$ with binary products, and a cartesian closed $\mathcal{B}$-fibration $p: \mathcal{C} \to \mathcal{B}$ (i.e. each fibre $\mathcal{C}_I$ is cartesian closed and the structure is preserved by reindexing). We shall write as if our fibration has a cleavage: for $\alpha: J \to I$ in $\mathcal{B}$, we write α^* for the corresponding reindexing (or substitution) functor from $\mathcal{C}_I$ to $\mathcal{C}_J$. However the reader will see that nothing really depends on this. We start by choosing an object $S \in \mathcal{C}_1$.

Definition 1.1 (Replete) *We say that:*

- $e: P \to Q$ *in* $\mathcal{C}_I$ *is* S-**iso** $\stackrel{\Delta}{\iff} (!^*S)^e$ *is an iso in* $\mathcal{C}_I$*;*
- $X \in \mathcal{C}_I$ *is* **fibrewise** S**-replete** $\stackrel{\Delta}{\iff} X^e$ *is an iso in* $\mathcal{C}_I$*, for every* $e: P \to Q$ S*-iso in* $\mathcal{C}_I$*;*
- $X \in \mathcal{C}_I$ *is* S**-replete** $\stackrel{\Delta}{\iff} (\pi_1^* X)^{\pi_2^* e}$ *is an iso in* $\mathcal{C}_{I\times J}$*, for every* $J \in \mathcal{B}$ *and every* $e: P \to Q$ S*-iso in* $\mathcal{C}_J$*.*

We write $\mathcal{S}_I^f$ *for the full subcategory of fibrewise* S*-replete objects in* $\mathcal{C}_I$*, and* $\mathcal{S}_I$ *for that of* S*-replete objects.*

Remark 1.2 The above definition could be generalized to the case $S \in \mathcal{C}_U$. The S-replete objects are preserved under reindexing, and so form a subfibration of p. On the other hand, the definition of fibrewise S-replete in $\mathcal{C}_I$ is given *fibrewise*, i.e. (given the notion of S-iso) it depends only on the fibre $\mathcal{C}_I$ instead of the whole fibration. However, a few additional properties of the ambient fibration p (see below) ensure stability under reindexing of properties such as: "X is fibrewise S-replete" and "$r_X: X \to R(X)$ is the reflection of $X \in \mathcal{C}_I$ in $\mathcal{S}_I^f$".

Theorem 1.3 (Closure under reindexing and coincidence)

- *Reindexing preserves replete objects, i.e.* $\alpha: J \to I$ *in* $\mathcal{B}$ *and* $X \in \mathcal{S}_I$ *imply* $\alpha^* X \in \mathcal{S}_J$*; so* S*-replete objects forms a full subfibration* $q: \mathcal{S} \to \mathcal{B}$ *of* p*.*
- *Fibrewise replete and replete objects coincide, i.e.* $\mathcal{S}_I^f = \mathcal{S}_I$*, provided the left adjoint to* $\alpha^*: \mathcal{C}_J \to \mathcal{C}_I$ *exists for each* $\alpha: J \to I$ *in* $\mathcal{B}$*.*

Theorem 1.4 (Stability of reflections) *Reindexing preserves reflections into* $\mathcal{S}_I^f$*, i.e.* $\alpha: J \to I$ *in* $\mathcal{B}$ *and* $r_X: X \to RX$ *is the reflection of* $X \in \mathcal{C}_I$ *in* $\mathcal{S}_I^f$ *imply* $\alpha^*(r_X): \alpha^* X \to \alpha^*(RX)$ *is the reflection of* $\alpha^* X$ *in* $\mathcal{S}_J^f$*, provided the left and right adjoints to* $\alpha^*: \mathcal{C}_J \to \mathcal{C}_I$ *exist for each* $\alpha: J \to I$ *in* $\mathcal{B}$*. Moreover, if all reflections exist in the fibres, then* q *is a full reflective subfibration of* p *(i.e. the reflection is a morphism of fibrations).*

Definition 1.5 ([FK72]) *A factorization system* $(\mathcal{E}, \mathcal{M})$ *is* **proper** *iff all* $e \in \mathcal{E}$ *are epi and all* $m \in \mathcal{M}$ *are mono.*

Theorem 1.6 (Characterization) *If* $\mathcal{C}_I$ *has a proper factorization system* $(\mathcal{E}, \mathcal{M})$ *and arbitrary intersections of* $\mathcal{M}$*-subobjects [or* $\mathcal{C}_I$ *is a quasi-topos], then* $\mathcal{S}_I^f$ *is the least full reflective subcategory of* $\mathcal{C}_I$ *with the following properties:*

- $(!^*S) \in \mathcal{S}_I^f$*;*
- *closure under isomorphism, i.e.* $X \in \mathcal{S}_I^f$ *and* $X \cong Y$ *imply* $Y \in \mathcal{S}_I^f$*;*
- *closure under exponentiation, i.e.* $Y \in \mathcal{S}_I^f$ *and* $X \in \mathcal{C}_I$ *imply* $Y^X \in \mathcal{S}_I^f$*.*

Moreover, under the assumptions of Theorem 1.3, $q: \mathcal{S} \to \mathcal{B}$ *is the least full reflective subfibration of* p *containing* S *and closed under isomorphism and exponentiation.*

Remark 1.7 Closure under exponentiation (or being an exponential ideal in Freyd's terminology) amounts to saying that the reflection R of $\mathcal{C}_I$ into $\mathcal{S}_I^f$ preserves binary products. This property of reflections is half way between an arbitrary reflection and a localization (i.e. a reflection R which preserves finite limits).

In a quasi-topos $\mathcal{C}$ there are at least two proper [and stable] factorization systems (which become the same when $\mathcal{C}$ is a topos): (strong epis, monos) and (epis, strong monos). However, a quasi-topos (e.g. the Effective topos) may not have arbitrary intersections of strong subobjects, in this case the construction of the reflection may be described using the *internal language*.

Definition 1.8 ([CLW93]) *A category $\mathcal{C}$ with a terminal object 1 is* **extensive** $\overset{\Delta}{\iff}$ *$\mathcal{C}$ has finite sums and the functor $+ \colon \mathcal{C}^2 \to \mathcal{C}/2$ s.t. $(X,Y) \mapsto (!+! \colon X+Y \to 2)$ is an equivalence.*

In an extensive category sums are well-behaved (in the same way that they are in toposes).

Definition 1.9 (Admissible class of monos) *A class of monos $\mathcal{M}$ in a category $\mathcal{C}$ is* **admissible** *iff*

- *for any $m \colon X' \hookrightarrow X$ in $\mathcal{M}$ and $f \colon Y \to X$ in $\mathcal{C}$ there is a pullback* $\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \uparrow & \lrcorner & \uparrow m \\ \cdot & \longrightarrow & X' \end{array}$
- *$m \in \mathcal{M}$ implies $m' \in \mathcal{M}$, whenever* $\begin{array}{ccc} Y & \xrightarrow{f} & X \\ m' \uparrow & \lrcorner & \uparrow m \\ Y' & \longrightarrow & X' \end{array}$.

A family $\langle \eta_X \colon X \to LX | X \in \mathcal{C} \rangle$ is a **$\mathcal{M}$-partial map classifier** $\overset{\Delta}{\iff}$ *$\eta_X \in \mathcal{M}$ and for any $m \colon Y' \hookrightarrow Y$ in $\mathcal{M}$ and $f \colon Y' \to X$ in $\mathcal{C}$ there is a unique $\bar{f} \colon Y \to LX$ s.t.* $\begin{array}{ccc} Y & \xrightarrow{\bar{f}} & LX \\ m \uparrow & \lrcorner & \uparrow \eta_X \\ Y' & \xrightarrow{f} & X \end{array}$.

Remark 1.10 An admissible class of monos is uniquely determined by $\eta_1 \colon 1 \to L1$. We say that $t \colon 1 \hookrightarrow \Sigma$ is a **classifier** iff it is η_1 for an admissible class of monos. We do not trouble here with the richer notion of a dominance (see [RR88]).

The previous definitions (of extensive category and admissible class of monos) can be adapted in the obvious way to fibrations.

Theorem 1.11 (Closure properties)

- *If p is an extensive fibration, then q is closed under finite sums (computed in p), i.e. $X_1, X_2 \in \mathcal{S}_I$ implies $0, (X_1 + X_2) \in \mathcal{S}_I$, provided that $2 \in \mathcal{S}_1$ and $1 \lhd S$.*
- *If $\langle \eta_X \colon X \to LX | X \in \mathcal{C} \rangle$ is a $\mathcal{M}$-partial map classifier for an admissible class $\mathcal{M}$ of monos in p, then q is closed under lifting, i.e. $X \in \mathcal{S}_I$ implies $LX \in \mathcal{S}_I$, provided $L1 \in \mathcal{S}_1$ and $S \lhd LS$.*

1.2 Examples

The scope and applicability of the results stated above is demonstrated by a variety of examples. We consider various cartesian closed fibrations (and choices of S), and for each of them we say whether the requirements stated in the theorems above are met.

Example 1.12 Given a quasi-topos $\mathcal{B}$, let $p = cod: \mathcal{B}^{\rightarrow} \rightarrow \mathcal{B}$ (and $\mathcal{C}_I = \mathcal{B}/I$), then:

- $\mathcal{C}_I$ is a quasi-topos, and α^* preserves the quasi-topos structure;
- α^* has left and right adjoints, which satisfy the Beck-Chevalley condition, so in particular Theorems 1.3 and 1.4 apply;
- if $2 \in \mathcal{S}_1$, then $\mathcal{S}_I$ is closed under finite sums;
- if $t: 1 \rightarrow \Sigma$ is a classifier in $\mathcal{C}_1$ s.t. $\Sigma \in \mathcal{S}_1$ and $S \lhd LS$, then $!^*t$ is a classifier in $\mathcal{C}_I$, the partial map classifiers exist and are preserved by reindexing, and q is closed under the partial map classifier.

When $S = \Omega_j$ where j is a topology, then $\mathcal{S}_1^f$ is the quasi-topos of j-sheaves.

An interesting example is the quasi-topos of filter spaces (see [Hyl79]), where topological spaces embed as a full reflective subcategory (which is not closed under exponentiation). When S is (the filter space corresponding to) Sierpinski's space, SFP domains form a full sub-CCC of $\mathcal{S}_1^f$.

Example 1.13 Given an extensive cartesian closed category $\mathcal{D}$ with small products and sums, and a proper factorization system $(\mathcal{E}, \mathcal{M})$ over $\mathcal{D}$ s.t. $\mathcal{M}$ is closed under arbitrary intersections, let $\mathcal{B}$ be the category of sets and $\mathcal{C}_I = \mathcal{D}^I$, then

- $\mathcal{C}_I$ inherit the structure of $\mathcal{D}$ (by pointwise definition), and α^* preserves such structure;
- α^* has left and right adjoints, which satisfy the Beck-Chevalley condition, so in particular Theorems 1.3 and 1.4 apply;
- if $2 \in \mathcal{S}_1$, then q is closed under finite sums;
- if $t: 1 \rightarrow \Sigma$ is a classifier in $\mathcal{C}_1$ with a partial map classifier s.t. $\Sigma \in \mathcal{S}_1$ and $S \lhd LS$, then $!^*t$ is a classifier in $\mathcal{C}_I$, the partial map classifiers exist and are preserved by reindexing, and q is closed under the partial map classifier.

There are many choices for $\mathcal{D}$ (and $\mathcal{M}$-subobjects), we consider some order-theoretic examples:

- the category of posets and monotonic maps, where an $\mathcal{M}$-subobject of $\underline{X} = (X, \leq_X)$ corresponds to a subset Y of X (with the induced order);
- the category of posets with pullbacks and stable maps (i.e. monotonic and pullback preserving), where an $\mathcal{M}$-subobject of $\underline{X} = (X, \leq_X)$ corresponds to a subset Y of X s.t. $\forall x_1, x_2, x \in Y. x_1, x_2 \leq_X x \supset (x_1 \wedge_X x_2) \in Y$;
- the category of ω-cpos and ω-continuous maps, where an $\mathcal{M}$-subobject of $\underline{X} = (X, \leq_X)$ corresponds to an ω-inductive subset Y of X, i.e. $(\forall i \in \omega. x_i \in Y) \supset (\sqcup_i x_i) \in Y$ for any ω-chain $\langle x_i | i \in \omega \rangle$ in $\underline{X}$;

Example 1.14 Let $\mathcal{B}$ be the category of ω-sets (see [LM91, Pho92]):

- an ω-set $\underline{X} = (X, \Vdash_X)$ consists of a set X and a **realizability relation** $\Vdash_X \subseteq N \times X$, i.e. $\forall x: X. \exists n: N. n \Vdash_X x$;

- a morphism $f\colon \underline{X} \to \underline{Y}$ is a **realizable map** $f\colon X \to Y$, i.e. $\exists e\colon N.\forall x\colon X.\forall m\colon N.m \Vdash_X x \supset e \cdot m \Vdash_Y f(x)$, e is called a **realizer** of f ($e \Vdash f$ for short).

$\mathcal{B}$ is equivalent to the quasi-topos of $\neg\neg$-separated objects in the Effective topos (see [Hyl82]).

Let $\mathcal{P} \in \mathbf{Cat}(\mathcal{B})$ be the full and internally complete category of $\neg\neg$-closed partial equivalence relations over $\underline{N} = (N, =_N)$ (see [Hyl88]), and p the externalization of $\mathcal{P}$. $\mathcal{C}_{\underline{I}}$ is isomorphic to the following category:

- an object is a family $\langle \underline{X}_i | i \in I\rangle$ of equivalence classes for a partial equivalence relation, i.e. X_i is a set of nonempty disjoint subsets of N and $\forall n\colon N, x\colon X_i.n \Vdash_{X_i} x \Longleftrightarrow n \in x$;
- a morphism from $\underline{X}$ to $\underline{Y}$ is a **realizable family** of maps $\langle f_i\colon \underline{X}_i \to \underline{Y}_i | i \in I\rangle$, i.e. $\exists e\colon N.\forall i\colon I.\forall m\colon N.m \Vdash_I i \supset e \cdot m \Vdash f_i$.

One can show that p has the following properties:

- $\mathcal{C}_{\underline{I}}$ is an extensive locally cartesian closed category with a proper factorization system $(\mathcal{E}, \mathcal{M})$, where an $\mathcal{M}$-subobject of $\langle \underline{X}_i | i \in I\rangle$ corresponds to a I-indexed family of subsets $Y_i \subseteq X_i$ (with the induced realizability relation), and $\mathcal{M}$ is closed under arbitrary intersections;
- α^* preserves the structure above and has left and right adjoints, which satisfy the Beck-Chevalley condition, so in particular Theorems 1.3 and 1.4 apply;
- if $2 \in \mathcal{S}_1$, then q is closed under finite sums;
- if $t\colon 1 \to \Sigma$ is a classifier in $\mathcal{C}_1$ s.t. $\Sigma \in \mathcal{S}_1$ and $S \lhd LS$, then $!^*t$ is a classifier in $\mathcal{C}_I$, the partial map classifiers exist and are preserved by reindexing, and q is closed under the partial map classifier.

In fact, the full subfibration of p of the (fibrewise) replete objects is isomorphic to the externalization of a full $\neg\neg$-closed internal subcategory of $\mathcal{P}$.

Similar results continue to hold if the Effective topos is replaced by another realizability topos.

2 Basic definitions and results

In this section we fix a category $\mathcal{B}$ with binary products, a cartesian closed $\mathcal{B}$-fibration $p\colon \mathcal{C} \to \mathcal{B}$ (i.e. each fibre $\mathcal{C}_I$ is cartesian closed and the structure is preserved by reindexing), and an object $S \in \mathcal{C}_U$.

Definition 2.1 *We say that:*

- *$e\colon P \to Q$ in $\mathcal{C}_I$ is S-**iso*** $\overset{\Delta}{\Longleftrightarrow}$ *$(\pi_1^* S)^{\pi_2^* e}$ is an iso in $\mathcal{C}_{U \times I}$;*
- *$X \in \mathcal{C}_I$ is **fibrewise** S-**replete*** $\overset{\Delta}{\Longleftrightarrow}$ *X^e is an iso in $\mathcal{C}_I$, for every $e\colon P \to Q$ S-iso in $\mathcal{C}_I$;*
- *$X \in \mathcal{C}_I$ is S-**replete*** $\overset{\Delta}{\Longleftrightarrow}$ *$(\pi_1^* X)^{\pi_2^* e}$ is an iso in $\mathcal{C}_{I \times J}$, for every $J \in \mathcal{B}$ and every $e\colon P \to Q$ S-iso in $\mathcal{C}_J$,*

We write $\mathcal{S}_I^f$ for the full subcategory of fibrewise S-replete objects in $\mathcal{C}_I$, and $\mathcal{S}_I$ for that of S-replete objects.

Notation 2.2 Before embarking on an analysis of replete objects, we introduce some notation for exponentials and the monad of *continuations.* Given a CCC $\mathcal{C}$ and $X \in \mathcal{C}$, we write:

- $X(\_)\colon \mathcal{C}^{op} \to \mathcal{C}$ for the functor s.t. $X(P) = X^P$ and $X(e)\colon k \in X(Q) \mapsto \lambda p\colon P.k(ep)$;
- $\eta_P\colon P \to X^2(P)$ for the natural transformation $p \in P \mapsto \lambda k\colon X(P).kp$;
- $\mu_P\colon X^4(P) \to X^2(P)$ for the natural transformation $X(\eta_{X(P)})$.

(X^2, η, μ) is a monad, the monad of continuations. Moreover, there is a *natural isomorphism* between $\mathcal{C}(P, X^2(Q))$ and $\mathcal{C}^{op}(X(P), X(Q))$, i.e. there is a duality between Kleisli category and the category of continuation transformers:

$$\frac{P \xrightarrow{f} X^2(Q)}{X(P) \xleftarrow[f^\star]{} X(Q)} \qquad \frac{P \xrightarrow{g_\star} X^2(Q)}{X(P) \xleftarrow[g]{} X(Q)}$$

namely $f^\star\colon k \in X(Q) \mapsto \lambda p\colon P.\alpha x k$.

We are primarily interested in the replete objects, as they form a subfibration $q\colon \mathcal{S} \to \mathcal{B}$ of $p\colon \mathcal{C} \to \mathcal{B}$ containing S.

Proposition 2.3 *$S \in \mathcal{S}_U$. $X \in \mathcal{S}_I$ and $\alpha\colon J \to I$ imply $\alpha^* X \in \mathcal{S}_J$.*

Proof The first claim is immediate from the definition of $\mathcal{S}_U$.

For the second claim, we show that $\pi_2^* e$ is strong $\pi_1^*(\alpha^* X)$-iso in $\mathcal{C}_{I\times K}$ for any S-iso e in $\mathcal{C}_K$. In fact, $(\pi_1^*(\alpha^* X))(\pi_2^* e) = ((\alpha\times 1)^*(\pi_1^* X))((\alpha\times 1)^*(\pi_2^* e)) = (\alpha\times 1)^*((\pi_1^* X)(\pi_2^* e))$ is iso, since $(\pi_1^* X)(\pi_2^* e)$ is iso, by the assumption $X \in \mathcal{S}_I$. ∎

Of course the notion of being S-iso is also stable under reindexing.

Lemma 2.4 *$e\colon P \to Q$ S-iso in $\mathcal{C}_I$ and $\alpha\colon J \to I$ in $\mathcal{B}$ imply $\alpha^*(e)\colon \alpha^* P \to \alpha^* Q$ S-iso.*

Proof This follows at once from the preservation of the CCC structure under reindexing. ∎

The aim of this section is to prepare the ground for results about the subfibration $q\colon \mathcal{S} \to \mathcal{B}$. We do this indirectly by establishing properties of the categories $\mathcal{S}_I^f$ of fibrewise S-replete objects. The relation between the two notions is that X is replete if and only if X is stably fibrewise replete.

Proposition 2.5 *Given $X \in \mathcal{C}_I$ the following assertions are equivalent:*

- $X \in \mathcal{S}_I$
- $\pi_1^*(X) \in \mathcal{S}_{I\times J}^f$, *for each $J \in \mathcal{B}$.*

Proof $1 \supset 2$. By Proposition 2.3 $\pi_1^* X \in \mathcal{S}_{I\times J}$. So it's enough to show that $\mathcal{S}_I \subseteq \mathcal{S}_I^f$ and then replace I with $I\times J$. Given $X \in \mathcal{S}_I$ and e S-iso in $\mathcal{C}_I$, we show that $X(e)$ is iso. In fact, $X(e) = \Delta^*((\pi_1^* X)(\pi_2^* e))$ is iso, since $(\pi_1^* X)(\pi_2^* e)$ is iso by the assumption $X \in \mathcal{S}_I$.

$2 \supset 1$. We prove that $(\pi_1^* X)(\pi_2^* e)$ is iso in $\mathcal{C}_{I\times J}$ for any e S-iso in $\mathcal{C}_J$. In fact, $\pi_2^* e$ is S-iso by Lemma 2.4, therefore $(\pi_1^* X)(\pi_2^* e)$ is iso by the assumption $\pi_1^* X \in \mathcal{S}_{I\times J}^f$. ∎

Our first task is to give a more concrete characterization of fibrewise S-replete (see Section 2.2) and to establish a few general properties of the categories $\mathcal{S}_I$ and $\mathcal{S}_I^f$

2.1 Auxiliary notions

We introduce the notions of weak and strong X-iso and prove some of their properties. These will enable us to give the more concrete characterization of the fibrewise S-replete objects from which their basic properties can readily be deduced. (Note that these notions are fibrewise, and so are weaker than our main notion of S-iso, see Proposition 2.11.)

Definition 2.6 (Weak and strong X-iso) *Given X and $e: P \to Q$ in $\mathcal{C}_I$, we say that:*

- *e is a* **weak X-iso** $\overset{\Delta}{\iff} \forall f: P \to X. \exists! g: Q \to X. f = e; g$*;*
- *e is a* **strong X-iso** $\overset{\Delta}{\iff} X(e)$ *is an iso.*

Lemma 2.7 *Given $e: P \to Q$ in $\mathcal{C}_I$, e iso iff for all $X \in \mathcal{C}_I$ e weak X-iso.*

Lemma 2.8 *Given X, W and $e: P \to Q$ in $\mathcal{C}_I$, the following assertions are equivalent:*

- *$e \times W$ weak X-iso;*
- *$W \times e$ weak X-iso;*
- *e weak $X(W)$-iso.*

Lemma 2.9 *Given $X \in \mathcal{C}_I$:*

- *e strong X-iso and $\alpha: J \to I$ in $\mathcal{B}$ imply $\alpha^* e$ strong $\alpha^* X$-iso;*
- *e strong X-iso and $W \in \mathcal{C}_I$ imply $(e \times W)$ and $(W \times e)$ strong X-iso;*
- *e and e' strong X-iso imply $(e; e')$ [and $(e \times e')$] strong X-iso.*

Proof

- We have to show that $X(e)$ iso implies $(\alpha^* X)(\alpha^* e)$ iso. This is immediate, since α^* preserves the CCC structure (and isomorphisms).
- Consider the following isomorphisms natural in P:

$$\begin{array}{ccccc} X(W \times P) & \xrightarrow{\sim} & X(P \times W) & \xrightarrow{\sim} & X(P)^W \\ \uparrow & & \uparrow & & \uparrow \\ X(W \times e) & & X(e \times W) & & X(e)^W \\ | & & | & & | \\ X(W \times Q) & \xrightarrow{\sim} & X(Q \times W) & \xrightarrow{\sim} & X(Q)^W \end{array}$$

  When $X(e)$ iso, then $X(e)^W$ iso, so $X(e \times W)$ and $X(W \times e)$ must be iso, too.
- We have to show that $X(e; e')$ is an iso. $X(e; e') = X(e'); X(e)$, as $X(\_)$ is a contravariant functor, therefore it is an iso, since $X(e)$ and $X(e')$ are.

  We have to show that $X(e \times e')$ is an iso. This follows from the previous point and closure under composition, since $e \times e' = (e \times \mathrm{id}); (\mathrm{id} \times e')$.

∎

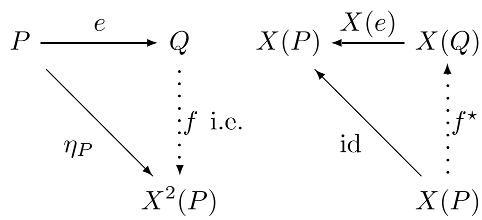
Lemma 2.10 *Given X and $e: P \to Q$ in $\mathcal{C}_I$, the following assertions are equivalent:*

- *e is a strong X-iso;*
- *e is a weak $X(W)$-iso, for every $W \in \mathcal{C}_I$.*

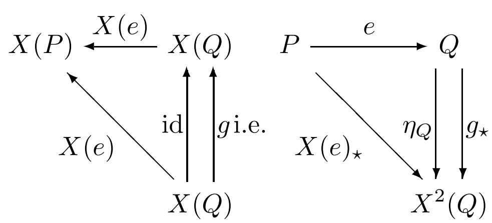
Proof $1 \supset 2$. Assume that e is strong X-iso, then:

- $e{\times}W$ strong X-iso, by Lemma 2.9;
- $e{\times}W$ weak X-iso, easy consequence;
- e a weak $X(W)$-iso, by Lemma 2.8.

$2 \supset 1$. Assume that e is weak $X(W)$-iso for every W, then



because e is weak $X^2(P)$-iso. Now we prove that $g \triangleq X(e); f^\star = \mathrm{id}$. Consider the two commuting triangles



since e is weak $X^2(Q)$-iso, then $\eta_Q = g_\star$. Therefore $X(e)$ is an iso (with $f^\star$ as inverse). ∎

2.2 Alternative characterization of fibrewise replete

First we give a few elementary properties of S-isos, and then we give the alternative concrete definition of fibrewise S-replete (see Proposition 2.13), which is easy to work with in practice.

Proposition 2.11 *e is S-iso in $\mathcal{C}_I$ iff $\pi_2^* e$ strong $\pi_1^* S$-iso in $\mathcal{C}_{U \times I}$. $X \in \mathcal{S}_I^f$ iff every S-iso $e\colon P \to Q$ in $\mathcal{C}_I$ is a strong X-iso.*

Lemma 2.12 *Given $X \in \mathcal{C}_I$:*

- *e S-iso and $W \in \mathcal{C}_I$ imply $e{\times}W$ and $W{\times}e$ S-iso;*
- *e and e' S-iso imply $e; e'$ and $e{\times}e'$ S-iso.*

Proof Because of Proposition 2.11, the claims follow easily from Lemma 2.9 and preservation of the CCC structure by reindexing functors. ∎

Proposition 2.13 *$X \in \mathcal{S}_I^f$ iff every S-iso $e\colon P \to Q$ in $\mathcal{C}_I$ is a weak X-iso.*

Proof

($\Rightarrow$) immediate by Proposition 2.11 and Lemma 2.10.

($\Leftarrow$) given $W \in \mathcal{C}_I$ and $e\colon P \to Q$ S-iso, by Lemma 2.12 $e{\times}W$ is S-iso. It follows by hypothesis that $e{\times}W$ is weak X-iso. Since this is true for any W, Lemma 2.10 tells us that e is strong X-iso and consequently $X \in \mathcal{S}_I^f$. ∎

2.3 Closure properties of fibrewise replete

It is simple to show that the fibrewise replete objects are closed under many universal constructions.

Theorem 2.14 *The full subcategories $\mathcal{S}_I^f$ of $\mathcal{C}_I$ satisfy the following properties:*

- *closure under isomorphism, i.e. $X \in \mathcal{S}_I^f$ and $X \cong Y$ imply $Y \in \mathcal{S}_I^f$;*
- *closure under exponentiation, i.e. $Y \in \mathcal{S}_I^f$ and $X \in \mathcal{C}_I$ imply $Y^X \in \mathcal{S}_I^f$;*
- *closure under fibrewise limits, i.e. $\lim F$ is the limit in $\mathcal{C}_I$ of $F: \mathcal{D} \to \mathcal{S}_I^f$ implies $\lim F \in \mathcal{S}_I^f$;*
- *closure under fibrewise internal products, i.e. $\Pi_\alpha X$ is the internal product in $\mathcal{C}_I$ of $X \in \mathcal{S}_J^f$ along $\alpha: J \to I$ implies $\Pi_\alpha X \in \mathcal{S}_I^f$.*

Proof We use the alternative characterization of $\mathcal{S}_I^f$ given in Proposition 2.13.

- Immediate by definition of $\mathcal{S}_I^f$.
- We show that every S-iso e in $\mathcal{C}_I$ is weak Y^X-iso. In fact, $e \times X$ is S-iso by Lemma 2.12, so $e \times X$ is weak Y-iso by $Y \in \mathcal{S}_I^f$, therefore e is weak Y^X-iso by Lemma 2.8.
- We show that every S-iso e in $\mathcal{C}_I$ is weak $\lim F$-iso. In fact, given a map $f: P \to \lim F$, by composing it with the limiting cone we get a cone from P into F. Applying the hypothesis on F we get a family of maps from Q into the vertices of F. It is easy to show (using $Fd \in \mathcal{S}_I^f$) that such family is a cone from Q into F, and so it determines a (unique) map $g: Q \to \lim F$, which satisfies the required properties.
- We show that every S-iso e in $\mathcal{C}_I$ is weak $\Pi_\alpha X$-iso. Lemma 2.12 tells us that $\alpha^*(e)$ is S-iso, and so weak X-iso by $X \in \mathcal{S}_J^f$. Therefore e is weak $\Pi_\alpha X$-iso, because

$$\begin{array}{ccc} P & \longrightarrow & \Pi_\alpha X \\ {\scriptstyle e}\downarrow & \nearrow & \\ Q & & \end{array} \quad \text{is equivalent (via the adjunction } \alpha^* \dashv \Pi_\alpha \text{) to} \quad \begin{array}{ccc} \alpha^* P & \longrightarrow & X \\ {\scriptstyle \alpha^* e}\downarrow & \nearrow & \\ \alpha^* Q & & \end{array}$$

∎

Corollary 2.15 *The full subfibration q of p satisfies the following properties:*

- *closure under isomorphism, i.e. $X \in \mathcal{S}_I$ and $X \cong Y$ imply $Y \in \mathcal{S}_I$;*
- *closure under exponentiation, i.e. $Y \in \mathcal{S}_I$ and $X \in \mathcal{C}_I$ imply $Y^X \in \mathcal{S}_I$.*

Note however that the other two closure properties of the fibrewise replete objects do not extend automatically to the replete objects, and are in any case inappropriate as properties of fibrations.

Theorem 2.16 *If $d: \mathcal{D} \to \mathcal{B}$ is a full reflective subfibration of p closed under isomorphism and exponentiation s.t. $S \in \mathcal{D}_U$, then each $\mathcal{S}_I^f \subseteq \mathcal{D}_I$, and so a fortiori (the subfibration) $\mathcal{S}$ is included in $\mathcal{D}$.*

Proof We prove that the reflection $r_X: X \to RX$ is S-iso, then when $X \in \mathcal{S}_I^f$ it is easy to show that the unique g s.t. $r_X; g = \mathrm{id}_X$ is the inverse of r_X. So we want to show that $\pi_2^*(r_X)$ is strong $\pi_1^*(S)$-iso. By Lemma 2.10, it is enough to show that $\pi_2^*(r_X)$ is weak $(\pi_1^* S)^W$-iso for any W is $\mathcal{C}_{U \times I}$. $\pi_2^*(r_X) = r_{\pi_2^* X}$ (since the reflection is fibred) and $(\pi_1^* S)^W$ is in $\mathcal{D}_{U \times I}$ (since $\mathcal{D}$ is closed under exponentials). Therefore the universal property of the reflection implies that $\pi_2^*(r_X)$ is weakly $(\pi_1^* S)^W$-iso. ∎

3 Main results

In this section we assume for simplicity that $S \in \mathcal{C}_1$. In this way the definition of $\mathcal{S}_I^f$ is really fibrewise, i.e. it depends only on $\mathcal{C}_I$. In many proofs, we need look only at properties of $\mathcal{S}_1^f$; to deduce similar properties for $\mathcal{S}_I^f$ it suffices to replace the the fibration $p: \mathcal{C} \to \mathcal{B}$ with $p': \mathcal{C}' \to \mathcal{B}$ given by $\mathcal{C}'_J = \mathcal{C}_{I\times J}$. We investigate sufficient conditions on the fibration $p: \mathcal{C} \to \mathcal{B}$ and/or $S \in \mathcal{C}_1$ to ensure that:

- S-replete objects coincide with fibrewise S-replete objects;
- $q: \mathcal{S} \to \mathcal{B}$ is a reflective subfibration of p, and so (by Theorem 2.16) the least full reflective subfibration of p containing S and closed under isomorphism and exponentiation;
- q is closed under binary sums and $\mathcal{M}$-partial map classifiers (computed in p).

3.1 Stability under reindexing

In this section we consider the situation in which we are chiefly interested, that is when we can identify the replete and fibrewise replete objects. We have seen that the categories $\mathcal{S}_I^f$ have good closure properties, so in these circumstances we get good closure properties of the replete objects. We start with a basic lemma.

Lemma 3.1 *Suppose that the left adjoint $\Sigma_\alpha \dashv \alpha^*$ exists for $\alpha: J \to I$ in $\mathcal{B}$. Then given $X \in \mathcal{C}_I$ and $e: P \to Q$ in $\mathcal{C}_J$:*

- *e weak $\alpha^*(X)$-iso iff $\Sigma_\alpha e$ weak X-iso;*
- *e strong $\alpha^*(X)$-iso implies $\Sigma_\alpha e$ strong X-iso.*

Proof The first claim is immediate from the definition of weak iso, and the universal property of the adjunction $\Sigma_\alpha \dashv \alpha^*$. In fact,

$$\begin{array}{ccc} P & \longrightarrow & \alpha^* X \\ {\scriptstyle e}\downarrow & \nearrow & \\ Q & & \end{array} \quad \text{is equivalent (via the adjunction) to} \quad \begin{array}{ccc} \Sigma_\alpha P & \longrightarrow & X \\ {\scriptstyle \Sigma_\alpha e}\downarrow & \nearrow & \\ \Sigma_\alpha Q & & \end{array}$$

To prove the second claim, by Lemma 2.10, it is enough to show that $\Sigma_\alpha e$ weak $X(W)$-iso, when $W \in CC_I$ and $\forall U \in \mathcal{C}_J$ e weak $(\alpha^* X)(U)$-iso.

- e weak $(\alpha^* X)(\alpha^* W)$-iso (take $U = \alpha^* W$);
- e weak $\alpha^*(X(W))$-iso, as α^* preserves the CCC structure;
- $\Sigma_\alpha e$ weak $X(W)$-iso, by the first claim.

∎

Proposition 3.2 *If the left adjoint $\Sigma_J \dashv \pi_1^*$ (where $\pi_1: I\times J \to I$) exists for all $J \in \mathcal{B}$, then $\mathcal{S}_I^f = \mathcal{S}_I$.*

Proof By Proposition 2.5 we need to show that $X \in \mathcal{S}_I^f$ and $J \in \mathcal{B}$ imply $\pi_1^*(X) \in \mathcal{S}_{I\times J}^f$. Moreover, by Proposition 2.13 it suffices to prove that e S-iso in $\mathcal{C}_{I\times J}$ implies e weak $\pi_1^* X$-iso:

- $\Sigma_J e$ S-iso in $\mathcal{C}_I$, by Lemma 3.1, since S-iso in $\mathcal{C}_I$ iff strong $!_I^* S$-iso;
- $\Sigma_J e$ weak X-iso in $\mathcal{C}_I$, by Proposition 2.13;
- e weak $\pi_1^* X$-iso in $\mathcal{C}_{I\times J}$, by Lemma 3.1.

∎

We immediately deduce our first main result.

Theorem 3.3 *Suppose that the left adjoint $\Sigma_\pi \dashv \pi^*$ exists for every projection π in $\mathcal{B}$. Then the fibrewise S-replete objects form a subfibration of p identical with the subfibration of S-replete objects.*

Under the assumptions of Theorem 3.3, we get good closure properties of q as a subfibration of p. In the corollary below, when we refer to a limit of $F\colon \mathcal{D} \to \mathcal{S}_I$ in $\mathcal{C}_I$, we mean that the fibrewise limit is stable under reindexing. Similarly when we refer to an internal product along a map $\alpha\colon J \to I$ in $\mathcal{B}$, we mean that the Beck-Chevalley condition holds for all appropriate pullbacks in $\mathcal{B}$.

Corollary 3.4 *Suppose that the left adjoint $\Sigma_{\pi_1} \dashv \pi_1^*$ exists for every (first) projection π_1 in $\mathcal{B}$. Then the full subfibration q of p satisfies the following properties:*

- *closure under isomorphism, i.e. $X \in \mathcal{S}_I$ and $X \cong Y$ imply $Y \in \mathcal{S}_I$;*
- *closure under exponentiation, i.e. $Y \in \mathcal{S}_I$ and $X \in \mathcal{C}_I$ imply $Y^X \in \mathcal{S}_I$;*
- *closure under limits, i.e. $\lim F$ is the limit in $\mathcal{C}_I$ of $F\colon \mathcal{D} \to \mathcal{S}_I$ implies $\lim F \in \mathcal{S}_I$;*
- *closure under internal products, i.e. $\Pi_\alpha X$ is the internal product in $\mathcal{C}_I$ of $X \in \mathcal{S}_J$ along $\alpha\colon J \to I$ implies $\Pi_\alpha X \in \mathcal{S}_I$.*

Proof Immediate in view of Theorem 3.3 and Theorem 2.14. ∎

3.2 Existence of a reflection

We want to find sufficient conditions in the first instance to ensure that each $\mathcal{S}_I^f$ is a full reflective subcategory of $\mathcal{C}_I$, and so to derive sufficient conditions to ensure that the subfibration q of p is reflective. We will start by proving the following result:

> (external version) if $\mathcal{C}_1$ has a proper factorization system $(\mathcal{E}, \mathcal{M})$ and has intersections of $\mathcal{M}$-subobjects, then $\mathcal{S}_1^f$ is a full reflective subcategory of $\mathcal{C}_1$ and the reflection $R(X)$ of X is given by the following $\mathcal{M}$-subobject of $S^2(X)$
>
> $$R(X) = \cap\{X' \subseteq_{\mathcal{M}} S^2(X) | X' \text{ fibrewise } S\text{-replete and } \eta_X(X) \subseteq X'\}$$

There is also an internal variant of the result, which we will not prove in this paper:

> (internal version) if $\mathcal{C}_1$ is a quasi-topos, then $\mathcal{S}_1^f$ is a full reflective subcategory of $\mathcal{C}_1$ and the reflection $R(X)$ of X is given by the following regular subobject of $S^2(X)$ described in the internal language (where formulas are interpreted by regular subobjects)
>
> $$R(X) = \cap\{X' \subseteq S^2(X) | X' \ S\text{-replete and } \eta_X(X) \subseteq X'\}$$

Remark 3.5 When $(\mathcal{E}, \mathcal{M})$ is proper, then regular epis are in $\mathcal{E}$ and regular monos are in $\mathcal{M}$. In particular, this is true for split epis and split monos (since split implies regular).

3.2.1 Fibrewise reflection: external version

In this section we make the following additional assumptions: $(\mathcal{E}, \mathcal{M})$ is a proper factorization system for $\mathcal{C}_1$ and that intersections of $\mathcal{M}$-subobjects exist.

Lemma 3.6 *If $X \in \mathcal{S}_1^f$, then $\eta_X : X \to S^2 X$ is in $\mathcal{M}$.*

Proof Given $X \in \mathcal{S}_1^f$, let (e, m) be the factorization of $\eta_X : X \to S^2(X)$, We show that e is iso, and so $\eta_X \in \mathcal{M}$. First we show that e is S-iso, i.e. $\forall W \in \mathcal{C}_1 . \forall f : X \to S(W) . \exists ! g . f = e; g$.

- uniqueness of g follows from $e \in \mathcal{E}$ epi, in fact $(\mathcal{E}, \mathcal{M})$ is proper;
- for existence take $g = m; S^2(f); S(\eta_W)$, then $e; g = f$, because $\eta_{S(W)}; S(\eta_W) = \mathrm{id}_{S(W)}$.

Now we prove that e is iso. Since $e \in \mathcal{E}$, it is enough to prove that $e \in \mathcal{M}$.

- e is weak X-iso, because e is S-iso and $X \in \mathcal{S}_1^f$;
- e is split mono, because $\exists ! g . e; g = \mathrm{id}_X$;
- $e \in \mathcal{M}$, because $(\mathcal{E}, \mathcal{M})$ is proper.

∎

Remark 3.7 The above result can be reformulated as "every fibrewise S-replete object is an S-space", where X is an S-space $\overset{\Delta}{\iff} \eta_X \in \mathcal{M}$ (see [Pho90]). In proving the internal version referred to above, one has to rely on further properties of S-spaces.

Given $X \in \mathcal{C}_1$, define

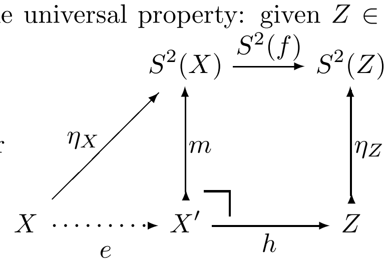
$$R(X) = \cap \{X' \subseteq_{\mathcal{M}} S^2(X) | X' \text{ fibrewise } S\text{-replete and } \eta_X(X) \subseteq X'\}$$

and let $r_X : X \to RX$ be the factorization of η_X through $RX \hookrightarrow S^2(X)$.

Theorem 3.8 *The reflection of X into $\mathcal{S}_1^f$ is $r_X : X \to R(X)$.*

Proof $RX \in \mathcal{S}_1^f$ by Theorem 2.14, being a limit of a diagram of fibrewise S-replete objects. We prove the universal property: given $Z \in \mathcal{S}_1^f$ and $f : X \to Z$, we seek $g : RX \to Z$ s.t. $f = r_X; g$.

Consider



- $\eta_Z \in \mathcal{M}$, by Lemma 3.6;
- $X' \in \mathcal{S}_1^f$ by Theorem 2.14
- $m \in \mathcal{M}$ as in any factorization system $\mathcal{M}$ is stable;
- $\eta_X \subseteq X'$ trivially;
- $RX \subseteq X'$, by definition of RX and the last three points;
- the sought g is now given by the inclusion of RX in X' followed by h.

To show uniqueness of g, suppose that $f = r_X ; g_i$, and consider the equalizer $m: X' \to R(X)$ of g_1 and g_2. We must show that m is an iso.

- $X' \in \mathcal{S}^f_1$, by Theorem 2.14;
- X' is an $\mathcal{M}$-subobject of $S^2(X)$, because X' is an $\mathcal{M}$-subobject of $R(X)$ (m is regular) and $R(X)$ is an $\mathcal{M}$-subobject of $S^2(X)$ (again, this is a property of every factorization system: if a diagram in $\mathcal{M}$ has a limit, it is in $\mathcal{M}$);
- η_X factors through X', because r_X equalizes g_1 and g_2;
- therefore $R(X) \subseteq X'$, by definition of $R(X)$, so they must be the same subobject $S^2(X)$.

∎

The argument above is in essence a standard one for the Special Adjoint Functor Theorem.

3.2.2 Fibred reflection

We start by deducing from the previous section a result in each fibre.

Proposition 3.9 *If each $\mathcal{C}_I$ has a proper factorization system $(\mathcal{E}, \mathcal{M})$ and has intersections of $\mathcal{M}$-subobjects, then each $\mathcal{S}^f_I$ is a full reflective subcategory of $\mathcal{C}_I$.*

Even when the replete and fibrewise replete objects coincide, and there are reflections $R_I: \mathcal{C}_I \to \mathcal{S}^f_I$ in each fibre, it is not enough to get a fibred reflection. We need to know that the reflections commute with reindexing.

Proposition 3.10 *Suppose that $\mathcal{S}_I = \mathcal{S}^f_I$ and the right adjoint $\alpha^* \dashv \Pi_\alpha$ exists for $\alpha: J \to I$ in $\mathcal{B}$. If $r_X: X \to RX$ is the reflection of $X \in \mathcal{C}_I$ in $\mathcal{S}^f_I$, then $\alpha^*(r_X): \alpha^* X \to \alpha^*(RX)$ is the reflection of $\alpha^* X$ in $\mathcal{S}^f_J$.*

Proof By Proposition 2.3 $\alpha^*(RX) \in \mathcal{S}_J$, and by Proposition 2.5 $\mathcal{S}_J \subseteq \mathcal{S}^f_J$. Therefore, we need to check only that for any $Y \in \mathcal{S}^f_J$

$$\begin{array}{ccc} \alpha^* X & \xrightarrow{f} & Y \\ {\scriptstyle \alpha^*(r_X)}\downarrow & \nearrow_{g} & \\ \alpha^*(RX) & & \end{array}.$$

Given $Y \in \mathcal{S}^f_J$ and $f: \alpha^* X \to Y$ we get g by the following chain of natural isomorphisms:

- $\mathcal{C}_J(\alpha^* X, Y) \dot{\cong}$, because $\alpha^* \dashv \Pi_\alpha$
- $\mathcal{C}_I(X, \Pi_\alpha Y) \dot{\cong}$, because $\Pi_\alpha Y \in \mathcal{S}^f_I$ by Theorem 2.14
- $\mathcal{S}^f_I(RX, \Pi_\alpha Y) \dot{\cong}$, because $\alpha^* \dashv \Pi_\alpha$
- $\mathcal{S}^f_J(\alpha^*(RX), Y)$.

∎

We can immediately deduce conditions sufficient to ensure that the replete objects form a reflective subfibration.

Theorem 3.11 *Suppose that*

1. *the left adjoint $\Sigma_\pi \vdash \pi^*$ exists for every projection π in $\mathcal{B}$;*
2. *each $\mathcal{C}_I$ has a proper factorization system $(\mathcal{E}, \mathcal{M})$ and has intersections of $\mathcal{M}$-subobjects;*
3. *the right adjoint $\alpha^* \dashv \Pi_\alpha$ exists for every morphism α in $\mathcal{B}$.*

Then q is a reflective subfibration of p; and in fact is the least reflective subfibration closed under exponentials in p and containing S.

3.3 Closure under sums

In this section we make the following additional assumptions:

1. p is an extensive fibration; that is, each $\mathcal{C}_I$ is extensive (see [CLW93]) and reindexing preserves finite coproducts;
2. $2 \in \mathcal{S}_1$;
3. S is inhabited (i.e. $1 \lhd S$).

3.3.1 Sums: fibrewise version

To start with we only use a weak fibrewise version of assumptions 1 and 2. We shall first show that $\mathcal{S}_1^f$ is closed under finite sums (computed in $\mathcal{C}_1$).

Proposition 3.12 *If*

$$\begin{array}{ccccc} 1 & \xrightarrow{in_1} & 2 & \xleftarrow{in_2} & 1 \\ \uparrow & & \uparrow f & & \uparrow \\ X_1 & \xrightarrow{in_1} & X & \xleftarrow{in_2} & X_2 \end{array}$$

and $X \in \mathcal{S}_1^f$, then $X_i \in \mathcal{S}_1^f$.

Corollary 3.13 $0 \in \mathcal{S}_1^f$.

Proof Use $1 \in \mathcal{S}_1^f$ and apply Proposition 3.12 to the coproduct $0 \longrightarrow 1 \longleftarrow 1$. ∎

Proposition 3.14 *If $X_1 \xrightarrow{in_1} X \xleftarrow{in_2} X_2$ is a coproduct and $X_1, X_2 \in \mathcal{S}_1^f$, then $X \in \mathcal{S}_1^f$.*

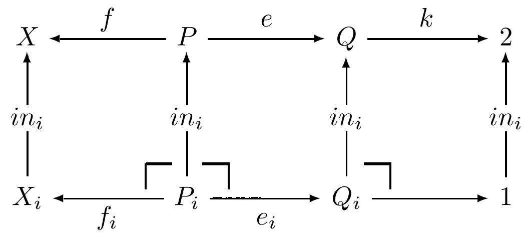
Proof Given $e\colon P \to Q$ S-iso and $f\colon P \to X$, we have to find $g\colon Q \to X$ s.t. $f = e; g$ (and show that it is unique):

- since $X_1 \xrightarrow{in_1} X \xleftarrow{in_2} X_2$ is a coproduct exists unique $h\colon X \to 2$ s.t.

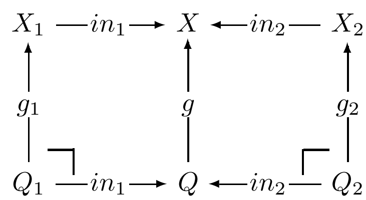
$$\begin{array}{ccccc} 1 & \xrightarrow{in_1} & 2 & \xleftarrow{in_2} & 1 \\ \uparrow & & \uparrow h & & \uparrow \\ X_1 & \xrightarrow{in_1} & X & \xleftarrow{in_2} & X_2 \end{array}$$

- since $2 \in \mathcal{S}_1^f$ and e is S-iso, exists unique $k\colon Q \to 2$ s.t. $f; h = e; k$

- therefore we have the following pullbacks of coproduct diagrams



- if the $e_i\colon P_i \to Q_i$ are S-iso, then exist unique $g_i\colon Q_i \to X_i$ s.t. $f_i = e_i; g_i$, and we can define $g\colon Q \to X$ as the unique map s.t.



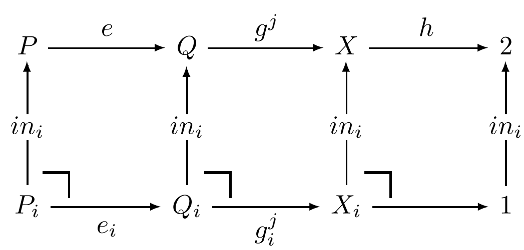
  clearly $f = e; g$, because $in_i; e; g = e_i; in_i; g = e_i; g_i; in_i = f_i; in_i = in_i; f$.

We prove that e_1 is S-iso (for e_2 the proof is similar), i.e. given $f_1\colon P_1 \to S^W$ exists unique $g_1\colon Q_1 \to S^W$ s.t. $f_1 = e_1; g_1$:

- since S is inhabited, exists a $f_2\colon P_2\colon \to S^W$;
- since e is S-iso, exists unique $g\colon Q \to S^W$ s.t. $[f_1, f_2] = e; g$;
- let $g_1 \stackrel{\Delta}{=} in_1; g$, then $e_1; g_1 = e_1; in_1; g = in_1; e; g = in_1; [f_1, f_2] = f_1$;
- moreover if g_1' had the same property of g_1, then $e; [g_1, g_2] = e; [g_1', g_2]$ (for a $g_2\colon Q_2 \to S^W$). But e is S-iso, so we must have $g_1 = g_1'$.

We prove that $g\colon Q \to X$ s.t. $f = e; g$ is unique. Suppose that $f = e; g^j$, then:

- $g^j; h = k$, because the $k\colon Q \to 2$ s.t. $f; h = e; k$ is unique;
- therefore we have the following pullbacks of coproduct diagrams



- $e_i; g_i^j = f_i$, because in_i are monic and $e_i; g_i^j; in_i = e_i; in_i; g^j = in_i; e; g^j = in_i; f = f_i; in_i$
- $g_i^1 = g_i^2$, because e_i are S-iso and $X_i \in \mathcal{S}_1^f$
- $g^1 = g^2$, because $g^j = g_1^j + g_2^j$.

∎

3.3.2 Sums: fibred version

We now deduce (under our additional assumptions) a result for the subfibration of replete objects. Of course we can immediately deduce (and so do not state separately) closure of each $\mathcal{S}_I^f$ under coproducts. But since we are assuming p is extensive, we in fact get closure of q under coproducts.

Theorem 3.15 *The subfibration q is closed under coproducts in p, and is itself an extensive fibration.*

Proof Suppose X_1 and X_2 are in $\mathcal{S}_I$. Then we have $X \cong X_1 + X_2$ in $\mathcal{S}_I^f$. But as p is extensive as a fibration we must have for any $\alpha\colon J \to I$ $\alpha^* X \cong \alpha^* X_1 + \alpha^* X_2$, and so also $\alpha^* X$ in $\mathcal{S}_J^f$. Hence by Proposition 2.5 we have $X \cong X_1 + X_2$ in $\mathcal{S}_I$. Now a similar stability argument using Proposition 3.12 shows that q must be an extensive fibration. ∎

3.4 Closure under lifting

In this section we make the following additional assumptions:

1. $\mathcal{M}$ is an admissible class of monos in p;
2. $\langle \eta_X\colon X \to LX | X \in \mathcal{C}\rangle$ is a $\mathcal{M}$-partial map classifier for $\mathcal{M}$ in p;
3. $L1 \in \mathcal{S}_1$;
4. $\eta_S\colon S \to LS$ is split monic (or equivalently $S \lhd LS$).

We spell 1 and 2 out in slightly more detail. First we are assuming that each $\mathcal{M}_I$ is an admissible class in $\mathcal{C}_I$, and that admissible monos are stable under reindexing. Secondly we are assuming that not only is there a partial map classifier at each $\mathcal{C}_I$, but also the family $\langle \eta_X\colon X \to LX | X \in \mathcal{C}\rangle$ is (at least up to isomorphism) stable under reindexing.

3.4.1 Lifting: fibrewise version

In this section, using only our assumptions in the fibre 1, we shall show that $\mathcal{S}_1^f$ is closed under lifting.

Lemma 3.16 *If*
$$\begin{array}{ccc} P' & \xrightarrow{\;e'\;} & Q' \\ {\scriptstyle m'}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle m} \\ P & \xrightarrow[\;e\;]{} & Q \end{array}$$
, m is a $\mathcal{M}$-subobject and e is S-iso, then e' is S-iso.

Proof We prove that $e' \times W\colon P' \times W \to Q' \times W$ is a weak S-iso (for every W), i.e. for every $f'\colon P' \times W \to S$ exists unique g' s.t. $(e' \times W); g' = f'$:

- consider the unique $\bar{f}'\colon P \times W \to LS$ s.t.
$$\begin{array}{ccc} P'\times W & \xrightarrow{\;f'\;} & S \\ {\scriptstyle m'\times W}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \eta_S} \\ P\times W & \xrightarrow[\;\bar{f}'\;]{} & LS \end{array}$$
and let $\tilde{f}' = \bar{f}'; \alpha\colon P \times W \to S$, where $\eta_S; \alpha = \mathrm{id}_S$. Clearly $f' = (m' \times W); \tilde{f}'$;

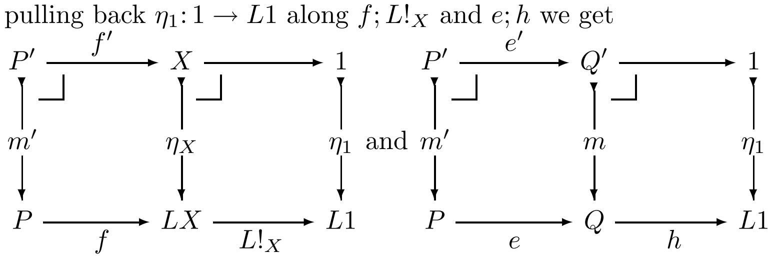
- exists unique $g\colon Q\times W \to S$ s.t. $\tilde{f'} = (e\times W); g$, because e is S-iso and $S \in \mathcal{S}^f_1$;
- let $g' = (m\times W); g$, then $(e'\times W); g' = f'$, because
  $(e'\times W); g' = (e'\times W); (m\times W); g = (m'\times W); (e\times W); g = (m'\times W); \tilde{f'} = f'$.

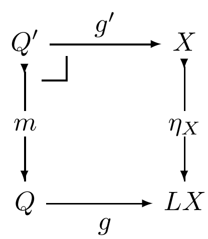
Uniqueness of g' s.t. $(e'\times W); g' = f'$ is easy, because $(e'\times W); g' = f'$ iff $(e\times W); \tilde{g'} = \tilde{f'}$. ∎

Proposition 3.17 *If $X \in \mathcal{S}^f_1$, then $LX \in \mathcal{S}^f_1$.*

Proof We prove that given a S-iso $e\colon P \to Q$ and $f\colon P \to LX$, exists unique g s.t. $f = e; g$.

- Exists a unique $h\colon Q \to L1$ s.t. $f; L!_X = e; h$, since $L1$ is S-iso;
- pulling back $\eta_1\colon 1 \to L1$ along $f; L!_X$ and $e; h$ we get



- e' is S-iso, by Lemma 3.16;
- exists unique $g'\colon Q' \to X$ s.t. $f' = e'; g'$, since $X \in \mathcal{S}^f_1$;
- consider the unique $g\colon Q \to LX$ s.t. 
- Clearly $f = e; g$.

Uniqueness of g follows from the one-one correspondence between total maps from Q to LX and $\mathcal{M}$-partial maps from Q to X. ∎

3.4.2 Lifting: fibred version

As always the results of the previous section extend to an arbitrary fibre: if $X \in \mathcal{S}^f_I$, then $LX \in \mathcal{S}^f_I$. But because because we have made strong stability assumptions about $\mathcal{M}$ and L, we get also a result for the subfibration q.

Theorem 3.18 *The subfibration q is closed under $\mathcal{M}$ monos, and under lifting.*

Proof The first claim follows from the second by the stability of $\mathcal{M}$ as each $\mathcal{S}^f_I$ is closed under pullbacks in $\mathcal{C}_I$. But each $\mathcal{S}^f_I$ is closed under L, and stably so in view of our assumptions. Hence q is closed under L by Proposition 2.5. ∎

Acknowledgements

We would like to thank Pino Rosolini and other members of the SDT mailing-list, who contributed to the discussion on the replete construction. We have used Paul Taylor's commutative diagrams.

References

[CLW93] A. Carboni, S. Lack, and R.F.C. Walters. Introduction to extensive and distributive categories. *Journal of Pure and Applied Algebra*, 84, 1993.

[FK72] P. Freyd and G.M. Kelly. Categories of continuous functors, i. *Journal of Pure and Applied Algebra*, 2, 1972.

[Fre91] P. Freyd. Algebraically complete categories. In A. Carboni, C. Pedicchio, and G. Rosolini, editors, *Conference on Category Theory '90*, volume 1488 of *LNM*. Springer Verlag, 1991.

[Hyl79] J.M.E. Hyland. Filter spaces and continuous functionals. *Annals of Mathematical Logic*, 16, 1979.

[Hyl82] J.M.E. Hyland. The effective topos. In A.S. Troelstra and D. van Dalen, editors, *The L.E.J. Brouwer Centenary Symposium*. North Holland, 1982.

[Hyl88] J.M.E. Hyland. A small complete category. *Annals of Pure and Applied Logic*, 40, 1988.

[Hyl91] J.M.E. Hyland. First steps in synthetic domain theory. In A. Carboni, C. Pedicchio, and G. Rosolini, editors, *Conference on Category Theory '90*, volume 1488 of *LNM*. Springer Verlag, 1991.

[LM91] G. Longo and E. Moggi. Constructive natural deduction and its modest interpretation. *Mathematical Structures in Computer Science*, 1, 1991.

[Mog94] E. Moggi. A general semantics for evaluation logic. In *9th LICS Conf.* IEEE, 1994.

[Pho90] W. Phoa. Effective domains and intrinsic structure. In *5th LICS Conf.* IEEE, 1990.

[Pho92] W. Phoa. An introduction to fibrations, topos theory, the effective topos and modest sets. Technical Report ECS-LFCS-92-208, Edinburgh Univ., Dept. of Comp. Sci., 1992.

[RR88] E. Robinson and G. Rosolini. Categories of partial maps. *Information and Computation*, 79(2), 1988.

[Sim92] A.K. Simpson. Recursive types in kleisli categories. available via FTP from theory.doc.ic.ac.uk, 1992.

[Tay91] P. Taylor. The fixed point property in synthetic domain theory. In *6th LICS Conf.* IEEE, 1991.