THE DISCRETE OBJECTS IN THE EFFECTIVE TOPOS

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Introduction

The original aim of this paper was to give a rather quick and undemanding proof that the effective topos contains two non-trivial small (i.e. internal) full subcategories which are closed under all small limits in the topos (and hence in particular are internally complete). The interest in such subcategories arises from the fact that they provide a very natural notion of model for many of the strongly polymorphic type theories. The nature of these models is that types are interpreted as objects of the topos, and hence as 'sets' in an intuitionistic sense. They in fact provide 'set-theoretic' models in the sense of Reynolds [**16**], except that the set-theory involved is non-classical. At the end of the paper we indicate briefly how one (only) of the two full subcategories gives rise to a model of the theory of constructions. This sheds some light on how far we can travel with a simple set-theoretic picture of data types. However, our main concern is to present a clean mathematical characterization of the complete subcategories.

The basic idea, due to Peter Freyd, remains very simple—it is that the collection of objects orthogonal to any given object is automatically closed under all existing limits, and to look at the objects orthogonal to the subobject classifier in the effective topos. However, one of our original motivations for writing this paper was that we did not quite believe Freyd's identification of this category. During the course of the paper we shall show that the category of (families of) objects in $\mathcal{E}\!f\!f$ orthogonal to Ω is the category of (families of) subquotients of N. The chain of reasoning now proceeds by saying that this second category is equivalent to the externalisation of an internal category, and that the internal category is therefore small complete. Somewhat to our surprise, it was this second stage, trivial for an internal category in $\mathcal{S}ets$, that we found difficulty in making precise, and it is in the discussion of notions of equivalence and completeness for internal categories and fibrations that much of the content of the paper now lies. We should warn the reader that even so our treatment is not exhaustive. We discuss only two notions (weak and strong) of equivalence for internal or fibred categories, and only two notions of completeness. Our definitions are well-adapted for our present purposes, corresponding more or less to the distinction between being told for any instance of a given set of parameters that there is an object with certain properties, and being given a function that produces one.

Let $\mathbf{C}$ be a small full subcategory of a topos (see § 3 for details) which is closed under arbitrary products. Then $\mathbf{C}$ cannot contain Ω, the object of truth values (in fact $\mathbf{C}$ cannot contain any object into which Ω embeds)—for if A is the disjoint union of the sets in $\mathbf{C}$, then $\Omega^A \in \mathbf{C}$, and hence $\Omega^A \rightarrowtail A$, which contradicts one of the constructive versions of Cantor's theorem. This argument runs parallel

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to that used by Peter Freyd to show that there are no small complete categories in $\mathcal{S}ets$ other than posets. This result extends immediately to Grothendieck toposes, and so if we are to find a complete internal category, we must look in a non-Grothendieck topos, for example, the effective topos. If the internal category is to represent a category of sets, then we must ensure that it does not contain Ω. Thus it makes sense to look at the objects orthogonal to Ω, the objects X such that all maps from Ω into X are constant.

Now, the subobject classifier of the effective topos is a complicated object (see the note below), and one of our first tasks is to show that an object X is orthogonal to Ω if and only if it is orthogonal to $\Delta 2$, a much simpler object, and one with much better properties (in particular, it is internally projective). Our main result is that the families of objects orthogonal to $\Delta 2$ are the families of subobjects of N, the object of realisers, and hence that the fibred category of such families, which is of course complete, is (weakly) essentially small.

The paper begins with a section on fibrations. This is not a technical extravagance—we want to prove results that claim that a certain subcategory of a topos is essentially small. If we simply took, say, the external category of orthogonal objects (the orthogonal objects in the fibre over 1), then we would merely be saying that we had an internal category that had the right global sections. Global sections say so little about an object in a topos that such a result would be almost meaningless, and certainly would enable us to deduce nothing about the completeness of the internal category. Because of this we are compelled to look at the fibration consisting of the families of orthogonal objects and to compare that with the fibration which is the externalisation of our internal category (which does determine the internal category up to equivalence).

This theme is continued in later sections of the paper, where we discuss both for internal categories in a topos $\mathscr{E}$, and for fibrations over it, first notions of equivalence, and then notions of completeness.

The weak-strong distinction we make for notions of completeness and equivalence can be viewed as one aspect of the local versus global problem, since 'existence' in the internal logic of a topos corresponds to local existence. There is therefore a body of material emanating from the school of Grothendieck (for example, Giraud [**8**]), which is relevant to our present problems. Much of this is concerned with the linked notions of descent and of stack—a short section on which is included here. With a section on the general theory of categories of orthogonal objects, this concludes our preparatory work, and we can proceed to our main results in §§ 6 and 7. The paper concludes with a short section on the use of these categories in the model theory of various polymorphic lambda-calculi.

A note on the effective topos

Throughout the paper we shall refer to a topos called the effective topos, but as the reader familiar with the papers by Hyland, Johnstone, and Pitts [**12**] and by Hyland [**10**] will immediately realise, our results apply equally well to any realisability topos—a topos constructed from a partial applicative algebra in the same way that 'the' effective topos is constructed from the natural numbers with Kleene application. There is just one source of confusion that may arise. In the case of 'the' effective topos the object of realisers is isomorphic to the object of numerals (the natural numbers object of the topos). Throughout this paper by N we shall mean the object of realisers.

Our notation for the effective topos follows that of Hyland [10] but with one small difference. We recall from that paper that an object of $\mathscr{E}\!\mathit{ff}$ is given by a pair $(X, =)$ consisting of a set and a $\mathrm{P}\omega$-valued relation on $X \times X$ such that there are natural numbers h and k which for all x, y, z in X satisfy

$$h \Vdash x = y \to y = x,$$

$$k \Vdash x = y \wedge y = z \to x = z,$$

where $h \Vdash \ldots$ denotes the realisability satisfaction. In particular, the object of realisers N is given by $(\omega, =)$ where $[\![n = m]\!] = \{n\}$ if $n = m$, and $\varnothing$ otherwise.

Maps $[f]$: $(X, =) \to (Y, =)$ are (equivalence classes of) $\mathrm{P}\omega$-valued relations f: $X \times Y \to \mathrm{P}\omega$ single-valued and total. We modify this construction slightly by identifying two different equality structures, $=, ='$ on X if there is an n such that, for all x, x',

$$n \Vdash x = x' \leftrightarrow x =' x'.$$

In the framework of Hyland, Johnstone, and Pitts [12] this amounts to taking the partial order reflection on the fibres of the tripos defining the topos, not simply the original pre-order. It has the advantage that it gives us canonical subobjects and also canonical pullbacks. This is of interest in the section on fibrations, since it means that the fibration $\mathscr{E}\!\mathit{ff}^2 \to \mathscr{E}\!\mathit{ff}$ is cloven even without the use of choice, and then later in the section on complete internal categories, when we can use the canonical subobjects to establish strong completeness properties of the separated subquotients of N.

We take this opportunity to recall that the subobject classifier Ω consists of the set $\mathrm{P}\omega$ with equality defined by realisable equivalence:

$$p \leftrightarrow q = \{h \in \omega \mid \ h_0 \Vdash p \to q \,\&\, h_1 \Vdash q \to p\},$$

and that there is a geometric morphism $\Gamma \dashv \Delta$: $\mathscr{S} \to \mathscr{E}\!\mathit{ff}$ where Γ is the global section functor and Δ is defined by $\Delta S = (S, =)$ where

$$[\![s = t]\!] = \begin{cases} \omega & \text{if } s = t, \\ \varnothing & \text{otherwise.} \end{cases}$$

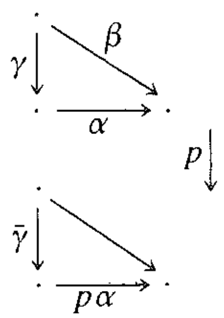
Finally, we refer the reader to Hyland [10] for a more complete account of the effective topos than it is possible to include here.

1. *Fibrations*

There is as yet no good reference for the theory of fibrations as it appears from the point of view of categorical logic: Grothendieck's development of the theory in, say, [9] was inspired by vastly different ends than those likely to concern readers of this paper, and the canonical reference on fibrations for the modern category theorist, Jean Bénabou's long-awaited book, has still to appear. However, it is no part of the purpose of the present paper to give a formal account of this theory, and although in the interests of being self-contained we include the basic definitions, readers are referred to Grothendieck [9], Giraud [8] and, especially, to Bénabou's JSL article [1].

Let $\mathscr{A}$ and $\mathscr{D}$ be categories, p: $\mathscr{A} \to \mathscr{D}$ a functor. A morphism α: $A_1 \to A_0$ of $\mathscr{A}$ is said to be *cartesian* with respect to p if for every β: $A_2 \to A_0$ and $\bar{\gamma}$: $pA_2 \to pA_1$ in $\mathscr{D}$ such that $p\alpha \circ \bar{\gamma} = p\beta$ there is a unique map γ: $A_2 \to A_1$ in $\mathscr{A}$ lying over $\bar{\gamma}$

(that is, $p\gamma = \bar{\gamma}$) such that $\alpha \circ \gamma = \beta$,



EXAMPLE. For any category $\mathscr{E}$, let $\mathscr{E}^2$ be the category of maps in $\mathscr{E}$: the objects are morphisms

$$A \xrightarrow{f} B$$

in $\mathscr{E}$ and the maps are commutative squares. Consider the natural codomain functor cod: $\mathscr{E}^2 \to \mathscr{E}$. Then a map

$$(h, k)\colon \left(C \xrightarrow{g} D\right) \to \left(A \xrightarrow{f} B\right)$$

is cartesian with respect to cod if and only if the square

$$\begin{array}{ccc} C & \xrightarrow{k} & A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ D & \xrightarrow{h} & B \end{array}$$

is a pullback.

For any I in $\mathscr{I}$, the morphisms $\alpha\colon A_1 \to A_0$ in $\mathscr{A}$ such that $p\alpha = \mathrm{id}_I$ form a subcategory $\mathscr{A}^I$ of $\mathscr{A}$ called the *fibre* over I.

DEFINITION. A functor $p\colon \mathscr{A} \to \mathscr{I}$ is a *fibration* if it satisfies either of the two following equivalent conditions:

(i) for any A_0 in $\mathscr{A}$ and any map $\bar{\alpha}\colon J \to pA_0$ in $\mathscr{I}$, there is a cartesian lifting α of $\bar{\alpha}$ with codomain A_0 (that is, a cartesian map $\alpha\colon A_1 \to A_0$ such that $p\alpha = \bar{\alpha}$);

(ii) any map $J \to pA$ in $\mathscr{I}$ lifts to $\mathscr{A}$, and any map $A_1 \to A_0$ in $\mathscr{A}$ factors as a map in the fibre over pA_1 followed by a cartesian map (note that such a factorisation is unique up to unique isomorphism if it exists).

If we have canonical cartesian liftings, that is, a function assigning a cartesian lifting to each pair ($\bar{\alpha}\colon D_1 \to pA_0$, A_0), then we say that the fibration is *cloven*. Of course the axiom of choice implies that every fibration is cloven—but in a sense the converse is also true, the assumption that we are given a cleavage of our fibration can be made to replace many important uses of choice.

To return for a moment to our example, we see that cod: $\mathscr{E}^2 \to \mathscr{E}$ is a fibration if and only if $\mathscr{E}$ has pullbacks, and that a cleavage amounts to being given for each pullback diagram

$$\begin{array}{ccc} & & \cdot \\ & & \downarrow \\ \cdot & \longrightarrow & \cdot \end{array}$$

a canonical pullback completion. In the sequel we shall be particularly concerned with subfibrations of cod: $\mathscr{E}^2 \to \mathscr{E}$ where $\mathscr{E}$ is the effective topos. We note that this fibration is cloven (see the note following the introduction).

1.1. Lemma. *Suppose $p: \mathscr{A} \to \mathscr{I}$ is a cloven fibration.*

(i) *Any map $A_1 \to A_0$ factors uniquely as a map in the fibre over pA_1 followed by a canonical cartesian.*

(ii) *For any $f: J \to I$ there is a 'pullback functor' $f^*: \mathscr{A}^I \to \mathscr{A}^J$.*

Proof (sketch). Part (i) is clear. In (ii) an important point is the way the use of a cleavage replaces choice. If $pA = I$, then for f^*A we take the domain of the canonical cartesian $f^*A \to A$ over f. If α maps A' to A in $\mathscr{A}^I$, then for $f^*\alpha$ we take the factorisation of

$$f^*A' \longrightarrow A' \xrightarrow{\alpha} A$$

through $f^*A \to A$.

Note however that in a cloven fibration canonical cartesian liftings do not necessarily compose (if they do, and if canonical liftings of identities are identities, then we say that the fibration is *split*), and so the composition functor f^*g^* is not necessarily equal to the functor $(gf)^*$, although there is a canonical isomorphism between them. This idea gives rise to another way of looking at fibrations—as indexed categories. Both notions attempt to capture the concept of one category varying continuously over another. In a fibration $p: \mathscr{A} \to \mathscr{I}$, the fibre $\mathscr{A}^I$ plays the role of the category as seen from I, and an object of $\mathscr{A}^I$ represents an I-indexed family of objects of the category. The cartesian maps capture the notion of relabelling. In a fibration, the variation of $\mathscr{A}$ over $\mathscr{I}$ is implicit; in an indexed category, on the other hand, it is explicit. Formally, an indexed category over $\mathscr{I}$ is a pseudo-functor $\mathscr{I}^{\mathrm{op}} \to \mathbf{Cat}$, where each I gets sent to the fibre $\mathscr{A}^I$, and each map to the corresponding pullback functor. The 'pseudo' refers to the fact that f^*g^* is not actually equal to $(gf)^*$, but only isomorphic to it, and that these isomorphisms satisfy certain coherence conditions.

A morphism of fibrations (over the same base $\mathscr{I}$) is called a *cartesian* functor. A cartesian functor from $p: \mathscr{A} \to \mathscr{I}$ to $p': \mathscr{A}' \to \mathscr{I}$ is simply a functor $G: \mathscr{A} \to \mathscr{A}'$ over $\mathscr{I}$ which sends cartesian maps to cartesian maps. If $\mathscr{A}$ and $\mathscr{A}'$ are cloven, then we do *not* require that G preserves the canonical cartesians. Note that if $G: \mathscr{A} \to \mathscr{A}'$ is any functor over $\mathscr{I}$, then for any I in $\mathscr{I}$ it restricts to a functor $G^I: \mathscr{A}^I \to \mathscr{A}'^I$ between the fibres over I. If G is cartesian, then for any $f: J \to I$, there is a canonical natural isomorphism between the functors $G^J f^*$ and $f^* G^I: \mathscr{A}^I \to \mathscr{A}'^J$, though they are not in general equal.

The fibrations with which we shall be concerned will be almost exclusively subfibrations of

$$\begin{array}{c} \mathscr{E}^2 \\ \downarrow \\ \mathscr{E} \end{array}$$

(and in particular where $\mathscr{E}$ is the effective topos). In cod: $\mathscr{E}^2 \to \mathscr{E}$, the fibre over I is just the slice category $\mathscr{E}/I$, and for $f\colon J \to I$, the pullback functor $f^*\colon \mathscr{E}^I \to \mathscr{E}^J$ is given by pullback along f in $\mathscr{E}$. If $\mathscr{E}$ is a topos, then it is well known that each fibre $\mathscr{E}/I$ is also a topos, and that the functors f^* are logical, that is, they preserve finite limits, colimits, exponentials, and the subobject classifier. In particular, the subobject classifier of $\mathscr{E}/I$ is

$$\Omega_{\mathscr{E}} \times I \xrightarrow{\pi} I.$$

Even if $\mathscr{E}$ is a general finite limit category, cod: $\mathscr{E}^2 \to \mathscr{E}$ has strong properties. Not only is it a fibration, it is also a co-fibration; i.e. the opposite functor $\mathrm{cod}^{\mathrm{op}}\colon (\mathscr{E}^2)^{\mathrm{op}} \to \mathscr{E}^{\mathrm{op}}$ is a fibration. Put more concretely, this says that each f^* has a left adjoint $\Sigma_f\colon \mathscr{E}/J \to \mathscr{E}/I$ (given in this instance by composition with f), and that the Σ_f are preserved up to isomorphism by pullback functors, i.e. they satisfy the Beck–Chevalley condition: if

$$\begin{array}{ccc} \cdot & \xrightarrow{k} & \cdot \\ h\downarrow & & \downarrow f \\ \cdot & \xrightarrow{g} & \cdot \end{array}$$

is a pullback in $\mathscr{E}$, then the induced natural transformation $\Sigma_h k^* \to g^*\Sigma_f$ is an isomorphism. We express this by saying that cod: $\mathscr{E}^2 \to \mathscr{E}$ always has $\mathscr{E}$-indexed coproducts, for if we regard

$$\begin{pmatrix} Y \\ \downarrow \\ J \end{pmatrix} \in \mathscr{E}/J$$

as a family of sets Y_j indexed by J, or as a J-indexed variable type, then

$$\Sigma_f \begin{pmatrix} Y \\ \downarrow \\ J \end{pmatrix}$$

is the I-indexed variable type $\sum_{j \in J_i} Y_j$.

If $\mathscr{E}$ is a topos, then the functors f^* also have right adjoints Π_f corresponding to fibrewise products, and these adjoints automatically satisfy their corresponding version of the Beck–Chevalley conditions. In fact cod: $\mathscr{E}^2 \to \mathscr{E}$ is complete and cocomplete (for all $\mathscr{E}$-indexed limits and colimits).

2. *Categories of orthogonal objects*

The major result of this paper is a characterisation of the fibred category of objects in the effective topos $\mathscr{E}\!f\!f$ orthogonal to Ω or any codiscrete object ΔS for S a set with at least two elements. This section is devoted to a discussion of orthogonal categories in general.

We recall that an I-indexed family of objects $\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}$ is said to be orthogonal to a map $\theta\colon A \to A'$ if and only if the I-indexed family of maps

$$\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}^{I^*\theta} : \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}^{I^*A'} \to \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}^{I^*A}$$

is a family of isomorphisms (equal to an isomorphism in $\mathscr{E}/I$). Note that here we have taken the notion of orthogonality with respect to a single map, a 1-indexed family. The interested reader should however have no difficulty in formulating the definition of orthogonality with respect to a J-indexed family. Everything we have to say in this section holds also for orthogonality with respect to such a family, but this generality will not be needed in the sequel. The reader familiar with Freyd and Kelly [6] will immediately realise that several of the arguments carried out in this section are the internal versions of results there, although we shall not produce them in their greatest generality.

Let $\mathcal{O}rth(\theta)$ be the full subcategory of $\mathscr{E}^2$ with objects the families of orthogonal objects.

2.1. PROPOSITION. (i) *The composite* $\mathcal{O}rth(\theta) \to \mathscr{E}^2 \to \mathscr{E}$ *is a fibration and the inclusion* $\mathcal{O}rth(\theta) \to \mathscr{E}^2$ *a cartesian functor.*

(ii) *For each I, the fibre* $\mathcal{O}rth(\theta)^I$ *is closed under all external limits which exist in* $\mathscr{E}^I$. *In particular,* $\mathcal{O}rth(\theta)$ *is finitely complete.*

(iii) $\mathcal{O}rth(\theta)$ *is closed under internal limits in* $\mathscr{E}$.

Proof. (i) It suffices to show that if $J \xrightarrow{\alpha} I$ and

$$\begin{pmatrix} Y \\ \downarrow \\ J \end{pmatrix} \to \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}$$

is cartesian lying over α, then $\begin{pmatrix} Y \\ \downarrow \\ J \end{pmatrix}$ is an orthogonal family in case $\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}$ is such. However,

$$\begin{pmatrix} Y \\ \downarrow \\ J \end{pmatrix} \to \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}$$

cartesian implies $\begin{pmatrix} Y \\ \downarrow \\ J \end{pmatrix}$ isomorphic to $\alpha^*\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}$, and

$$\alpha^*\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}^{J^*\theta} \cong \alpha^*\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}^{\alpha^* I^*\theta} \cong \alpha^*\left(\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}^{I^*\theta}\right)$$

since α^* is logical. But this last is isomorphic to $\alpha^*\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}$, and the result follows.

(ii) This is left to the reader

(iii) We shall show that $\mathcal{O}rth(\theta)$ is closed under $\mathscr{E}$-indexed products, i.e. that

Π_α: $\mathscr{E}/J \to \mathscr{E}/I$ restricts to a functor $\mathcal{O}rth(\theta)^J \to \mathcal{O}rth(\theta)^I$, leaving the case of the equaliser to the reader. If $\begin{pmatrix} Y \\ \downarrow \\ J \end{pmatrix}$ is an arbitrary J-indexed family, and C is an arbitrary object of $\mathscr{E}$, then there is a canonical isomorphism

$$\Pi_\alpha(Y^{J^*C}) \cong (\Pi_\alpha Y)^{I^*C}.$$

If now $\begin{pmatrix} Y \\ \downarrow \\ J \end{pmatrix}$ is orthogonal, then this gives us

$$(\Pi_\alpha Y)^{I^*A'} \cong \Pi_\alpha(Y^{J^*A'}) \cong \Pi_\alpha(Y^{J^*A}) \cong \Pi_\alpha(Y^{J^*A}),$$

as required.

As our main concern in later sections will be with the notion of orthogonality with respect to θ: $A \to 1$ for A well-supported and internally projective, we concentrate our attention on this particular case.

Definition. If A is an object in $\mathscr{E}$, then $\mathcal{O}rth(A)$, the category of objects *orthogonal to* A, is the fibred category of objects orthogonal to $A \to 1$.

2.2. Lemma. *If there is an epimorphism from* A *onto* B *(a condition we shall henceforth write in shorthand as* $A \twoheadrightarrow B$*), then* $\mathcal{O}rth(A)$ *is a fibred subcategory of* $\mathcal{O}rth(B)$.

Proof. Suppose $\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}$ is orthogonal to A. Then the composite

$$\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix} \to \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}^{I^*B} \to \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}^{I^*A} \xrightarrow{\sim} \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix} \to \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}^{I^*A}$$

is iso. But since $A \twoheadrightarrow B$, the second component is monic, and hence is iso.

For the next group of results we shall assume that θ takes the special form $A \twoheadrightarrow 1$, so that A is well-supported (but we note that the apparently more general case of θ: $A \twoheadrightarrow A'$ can be obtained by localising to $\mathscr{E}/A'$).

2.3. Lemma. *Suppose* $A \twoheadrightarrow 1$. *Then* $\mathcal{O}rth(A)$ *is closed under subobjects.*

Proof. Without loss of generality we can assume that $I = 1$ as the proof rests only on elementary properties of toposes. Suppose that X is orthogonal to A and $X' \rightarrowtail X$. Since $A \twoheadrightarrow 1$, the induced map κ: $X \to X^A$ is monic. Moreover, the commutative square

$$\begin{array}{ccc} X' & \rightarrowtail & X \\ \downarrow & & \downarrow \\ X'^{A} & \rightarrowtail & X^{A} \end{array}$$

is a pullback. Now pullbacks of isos are iso, and hence X' is in $\mathcal{O}rth(A)$.

A corollary is that $\mathcal{O}rth(A)$ is a reflective subcategory of $\mathscr{E}^2 \to \mathscr{E}$. Since $\mathcal{O}rth(A)$ is closed under subobjects, any map

$$\begin{pmatrix} Z \\ \downarrow \\ I \end{pmatrix} \to \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}$$

factors through a quotient of $\begin{pmatrix} Z \\ \downarrow \\ I \end{pmatrix}$ $\left(\text{its image in } \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}\right)$ which is in $\mathcal{O}rth(A)$. This, together with the definability of $\mathcal{O}rth(A)$, gives the 'solution set condition', and we can apply the indexed adjoint functor theorem. The following characterisations of orthogonal objects and maps into orthogonal objects will be useful in the sequel.

2.4. Proposition. *Suppose $A \twoheadrightarrow 1$.*

(i) *The family $\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}$ is orthogonal to A if and only if there is a map λ such that the diagram*

$$\begin{array}{ccc} \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}^{I^*A} \times_I I^*A & \xrightarrow{\pi} & \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}^{I^*A} \\ {\scriptstyle \mathrm{ev}}\downarrow & & \downarrow{\scriptstyle \lambda} \\ \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix} & \xrightarrow{\mathrm{id}} & \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix} \end{array}$$

commutes. Note that when this is the case, the diagram is a pushout.

(ii) *Any map $\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix} \to \begin{pmatrix} Y \\ \downarrow \\ I \end{pmatrix}$ from an arbitrary I-indexed family into an orthogonal family factors through the pushout*

$$\begin{array}{ccc} \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}^{I^*A} \times_I I^*A & \overset{\pi}{\twoheadrightarrow} & \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}^{I^*A} \\ {\scriptstyle \mathrm{ev}}\downarrow & & \downarrow \\ \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix} & \longrightarrow & \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}^{\downarrow} \end{array}$$

Proof. Again assume that $I = 1$.

(i) A map $\lambda\colon X^A \to X$ satisfies $\mathrm{ev} = \lambda\pi = \pi(\lambda \times \mathrm{id})$ if and only if $\mathrm{id} = \kappa\lambda\colon X^A \to X^A$ where κ is the constant functions map, since κ is the exponential adjoint of π. As κ is monic, it has a right inverse if and only if it is an iso.

(ii) This is immediate from (i).

It will also be useful sometimes to rewrite the pushout condition of 2.4(i) as the equality of a pair of maps.

2.5. LEMMA. *Suppose $A \twoheadrightarrow 1$. Then, in each fibre,* $\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}$ *is orthogonal to A if and only if the two composites*

$$\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}^{I^*A} \times_I I^*A^{I^*2} \Rightarrow \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}^{I^*A} \times_I I^*A \times_I I^*A \underset{\pi_{02}}{\overset{\pi_{01}}{\rightrightarrows}} \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}^{I^*A} \times_I I^*A \xrightarrow{\mathrm{ev}} \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}$$

are equal.

2.6. REMARK. The meaning of this lemma is perhaps more intelligible if we make use of the internal logic of the topos. The family $(X_i \mid i \in I)$ is orthogonal to A if the formula

$$\text{‘}\forall i \in I \,.\, \forall \phi \in A \to X_i \,.\, \exists x \in X_i \,.\, \forall a \in A \,.\, \phi(a) = x\text{’}$$

holds in the internal logic. The lemma above says that if A is well-supported (that is, ‘$\exists a \in A \,.\, a = a$’ holds), then this condition for orthogonality is equivalent to

$$\text{‘}\forall i \in I \,.\, \forall \phi \in A \to X_i \,.\, \forall a, a' \in A \,.\, \phi(a) = \phi(a')\text{’}.$$

Proof. As the argument is the same for each fibre, assume $I = 1$. If X is orthogonal to A, then the two composites are obviously equal. To prove the converse note that since $A \twoheadrightarrow 1$, the diagram

$$X^A \times A \times A \rightrightarrows X^A \times A \to X^A$$

is a coequaliser, and that we thus have a commutative triangle

$$\begin{array}{ccc} X^A \times A & \xrightarrow{\pi} & X^A \\ {\scriptstyle \mathrm{ev}}\downarrow & \swarrow & \\ X & & \end{array}$$

The result now follows from Proposition 2.4.

We shall make use of this lemma immediately in proving a technical result which will be our main tool in relating different categories of orthogonal objects.

2.7. PROPOSITION. *Suppose $B \twoheadrightarrow A \twoheadrightarrow 1$. Suppose moreover that there is a map $2 \to A$ inducing a surjection $B^A \twoheadrightarrow B^2$ by composition. Then the fibred categories $\mathcal{O}rth(B)$ and $\mathcal{O}rth(A)$ are equal.*

Proof. We already have that $\mathcal{O}rth(B)$ is contained in $\mathcal{O}rth(A)$. Suppose conversely that $\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}$ is orthogonal to A. Then the diagram

$$\begin{array}{ccc} \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}^{I^*B} \times_I I^*B^{I^*A} & \xrightarrow{\text{cmp}} & \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}^{I^*A} \\ \downarrow & & \downarrow {\scriptstyle K^{-1}} \\ \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}^{I^*B} \times_I I^*B^{I^*2} & \rightrightarrows & \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix} \end{array}$$

commutes. But the left-hand vertical is a surjection, and so the bottom arrows are equal. The result now follows from Lemma 2.5.

In the restricted case which we shall be considering in the rest of the paper, we have stronger closure properties for $\mathcal{O}rth(A)$ and a simpler description of the reflection into it. Assume that $A \twoheadrightarrow 1$ and that A is internally projective, that is, $(-)^A$ preserves surjections.

2.8. LEMMA. *Suppose $A \twoheadrightarrow 1$ and is internally projective. Then the fibred category $\mathcal{O}rth(A)$ is closed under quotients.*

Proof. Again we can assume that $I = 1$ and show that if $X \twoheadrightarrow Y$ and X is orthogonal to A, then so is Y. Now the diagram

$$\begin{array}{ccc} X & \twoheadrightarrow & Y \\ \downarrow & & \downarrow \\ X^A & \twoheadrightarrow & Y^A \end{array}$$

commutes, and the left-hand side is a surjection. On the other hand, it is also monic, since $A \twoheadrightarrow 1$, and hence is an isomorphism.

2.9. PROPOSITION. *Suppose that $A \twoheadrightarrow 1$ and that A is internally projective. Then in the pushout*

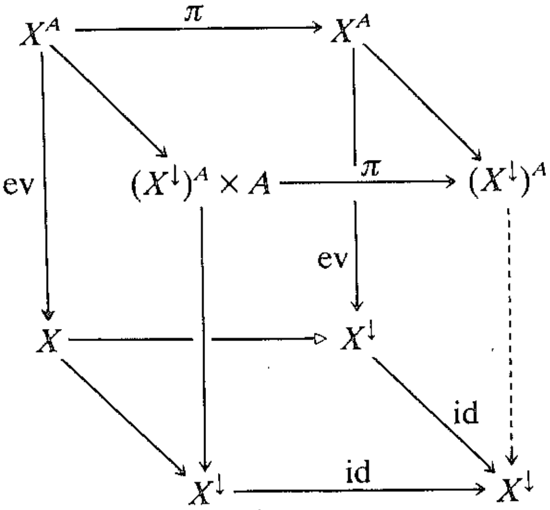
$$\begin{array}{ccc} \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}^{I^*A} \times_I I^*A & \xrightarrow{\pi} & \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}^{I^*A} \\ {\scriptstyle \text{ev}} \downarrow & & \downarrow \\ \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix} & \twoheadrightarrow & \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}^{\downarrow} \end{array}$$

$\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}^{\downarrow}$ *is the reflection of* $\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}$ *into* $\mathcal{O}rth(A)$.

Proof. As usual, we shall carry out the argument for $I=1$. Note first that for any $B \twoheadrightarrow C$ the diagram

$$\begin{array}{ccc} B\times A & \twoheadrightarrow & C\times A \\ \downarrow & & \downarrow \\ B & \twoheadrightarrow & C \end{array}$$

is always a pushout. As $(-)^A$ preserves epis, in the diagram



the top square is easily seen to be a pushout. The back square is a pushout by definition, and the bottom square is trivially a pushout. It follows that the fill-in for the right-hand vertical of the front face makes the front face a pushout too. The conclusion now follows from Proposition 2.4.

3. *Internal categories and equivalence*

Suppose that

$$\mathbf{C} = \left(C_1 \underset{\text{cod}}{\overset{\text{dom}}{\rightrightarrows}} C_0 \right)$$

is an internal category in an arbitrary left exact category $\mathscr{E}$. We recall that the *externalisation* of $\mathbf{C}$ is the fibred category $p\colon [\mathbf{C}] \to \mathscr{E}$ defined as follows: the objects of $[\mathbf{C}]$ in the fibre over some I in $\mathscr{E}$ are maps $I \to C_0$, to be thought of as I-indexed families of objects of $\mathbf{C}$. Given an I-indexed family $c\colon I \to C_0$ and a map $\alpha\colon J \to I$, we think of the J-indexed family $\alpha \circ c\colon J \to C_1$ as the relabelling of c along α. A map from $d\colon J \to C_0$ to $c\colon I \to C_0$ is then given by a pair ($\alpha\colon J \to I$, $f\colon J \to C_1$), such that the two diagrams

$$\begin{array}{ccc} J & \xrightarrow{f} & C_1 \\ & {\scriptstyle d}\searrow & \downarrow{\scriptstyle \text{dom}} \\ & & C_0 \end{array}$$

and

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & I \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle c} \\ C_1 & \xrightarrow[\text{cod}]{} & C_0 \end{array}$$

commute. In effect, a map lying above the relabelling α consists of a J-indexed family of maps in $\mathbb{C}$ from the J-indexed family of objects d to the relabelled family $\alpha \circ c = \alpha^* c$. Composition is defined using the internal composition of $\mathbb{C}$.

3.1. PROPOSITION. *The functor* $p\colon [\mathbb{C}] \to \mathscr{E}$ *is a (split) fibration (cf.* § 1).

Proof. The map (α, f) is cartesian if and only if f is a family of isomorphisms, that is, if and only if f factors through $\mathrm{Iso}(\mathbb{C}) \rightarrowtail C_1$, the object of isomorphisms of $\mathbb{C}$. Now (α, f) factors as (id_I, f) followed by $(\alpha, \mathrm{id}_{\mathbb{C}} \circ c \circ \alpha)$ where $\mathrm{id}_{\mathbb{C}}\colon C_0 \to C_1$ is the map giving the identities of $\mathbb{C}$. Notice that the canonical cartesian maps $(\alpha, \mathrm{id}_{\mathbb{C}} \circ c \circ \alpha)$ are closed under composition, and hence that the fibration is split.

As an example of this consider the case of a 'small full subcategory', where we now assume that $\mathscr{E}$ is a topos, or at least is locally cartesian closed. Suppose we are given a family $\begin{pmatrix} S \\ \downarrow \\ C_0 \end{pmatrix}$ of objects of $\mathscr{E}$. This gives an internal category $\mathbb{C}$ with objects C_0 and whose object of maps is the $C_0 \times C_0$-indexed family whose fibre over (x, y) is the collection of maps in the topos from S_x to S_y, in other words is the exponential $(\pi_1^* f)^{(\pi_0^* f)}$ in the slice $\mathscr{E}/C_0 \times C_0$. Write $(\mathbb{C})$ for the full subcategory of $\mathscr{E}^2$ on those maps obtained by pullback from the single map $\begin{pmatrix} S \\ \downarrow \\ C_0 \end{pmatrix}$. Of course $(\mathbb{C}) \xrightarrow{\text{cod}} \mathscr{E}$ is a full subfibration of $\mathscr{E}^2 \xrightarrow{\text{cod}} \mathscr{E}$, and our intuition tells us that $[\mathbb{C}]$ and $(\mathbb{C})$ should be equivalent.

3.2. LEMMA. *If the fibration* $\mathscr{E}^2 \to \mathscr{E}$ *is cloven (as is the case if we have enough choice in the meta-theory, or constructively in certain other cases including that of the effective topos), then there is a full and faithful cartesian functor from the fibration* $[\mathbb{C}] \to \mathscr{E}$ *to* $\mathscr{E}^2 \to \mathscr{E}$ *whose essential image is* $(\mathbb{C})$.

Proof. Of course we take the image of $\alpha\colon I \to C_0$ to be the domain of the canonical lifting of α to $\mathscr{E}^2$ with codomain f. It is routine to check that if α maps I to C_0 and β maps J to C_0, then the hom-set $[\mathbb{C}](\alpha, \beta)$ is one-to-one correspondence with the commutative squares

$$\begin{array}{ccc} \alpha^* S & \longrightarrow & \beta^* S \\ \downarrow & & \downarrow \\ I & \longrightarrow & J \end{array}$$

This establishes fullness and faithfulness. Finally, by definition, $(\mathbb{C})$ consists precisely of those objects of $\mathscr{E}^2$ isomorphic to some $\begin{pmatrix} \alpha^* X \\ \downarrow \\ I \end{pmatrix}$.

Note however that, except under special circumstances, in the absence of choice we cannot obtain a pseudo-inverse to the functor $[\mathbb{C}] \to (\mathbb{C})$. This is because for an arbitrary object $\begin{pmatrix} Y \\ \downarrow \\ I \end{pmatrix}$ of $(\mathbb{C})$, there may be many different

α: $I \to C_0$ expressing Y as a pullback of S. A functor $(\mathbb{C}) \to [\mathbb{C}]$ requires us to make a choice, but it is a use of choice at the level of meta-theory, and so is (relatively) harmless. Thus, in the sequel we shall assume that $(\mathbb{C})$ and $[\mathbb{C}]$ are equivalent categories over $\mathscr{E}$ (in the terminology introduced below we shall assume that they are strongly equivalent). Note however that we can still distinguish them: $[\mathbb{C}]$ is a split fibration, while $(\mathbb{C})$ is not.

One example of this full subcategory construction is the full subcategory of subsets of a given set X, obtained from the family $\varepsilon \to \mathrm{P}X$, where ε is the membership relation on $X \times \mathrm{P}X$. We however will be more interested in the internal category of subquotients of N. As in the normal category of sets we can take this category to have as objects the internal collection of symmetric and transitive relations on N:

$$Q_0 = \{R \in \mathrm{P}(N \times N) \mid \text{'}R \text{ is symmetric and transitive'}\}.$$

However, we can also describe Q_0 as a family of sets of equivalence classes of elements of N:

$$\{F \subset \mathrm{P}N \mid \forall X \in F\, \exists x \in N\,.\, x \in X \wedge \forall X, Y \in F[\exists x \in N\,.\, x \in X \wedge x \in Y \to X = Y]\}.$$

Living over Q_0 we have an obvious family $\begin{pmatrix} Q \\ \downarrow \\ Q_0 \end{pmatrix}$ of sets in $\mathscr{E}$, given by restricting the membership relation $\varepsilon \rightarrowtail \mathrm{P}N \times \mathrm{P}^2N$ to Q_0 (the fibre over F is thus F). We obtain the same result from the first presentation by observing that the disjoint union of the relations R determines a symmetric and transitive relation on $N \times Q_0$. If we take the domain of this relation, and then quotient by what is now an equivalence relation, we obtain a set Q which is of course still defined over Q_0, and is isomorphic to the family q: $Q \to Q_0$ given above. The internal category of subquotients of N is the full internal category on this family of objects.

An object over I of the externalisation $[\mathbb{Q}]$ is given by a map from I to Q_0, and the associated subfibration $(\mathbb{Q})$ of $\mathscr{E}^2 \to \mathscr{E}$ consists of those families $\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}$ such that there is

$$\begin{array}{ccc} \begin{pmatrix} A \\ \downarrow \\ I \end{pmatrix} & \rightarrowtail & I^*N \\ \downarrow & & \\ \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix} & & \end{array}$$

Externalisation is functorial in the sense that an internal functor F: $\mathbb{C} \to \mathbb{D}$ gives rise to a cartesian functor $[F]$: $[\mathbb{C}] \to [\mathbb{D}]$ over $\mathscr{E}$, and is 2-functorial since a similar property holds for internal natural transformations. However, extending this to $(\mathbb{C})$ and $(\mathbb{D})$ again involves some use of choice at the level of the metatheory. Nevertheless, we need to consider how (internal) properties of

F: $\mathbb{C} \to \mathbb{D}$ are related to corresponding properties of $[F]$: $[\mathbb{C}] \to [\mathbb{D}]$. While the internal statements that express the notion that 'F is full' or 'is faithful' are straightforward, the way to express 'F is an equivalence' is not. Suppose that F: $\mathbb{C} \to \mathbb{D}$ is an internal functor which is (internally) full faithful and essentially surjective on objects. Then in order to get a pseudo-inverse functor $\mathbb{D} \to \mathbb{C}$, we need some form of choice in $\mathscr{E}$ (what we have been calling choice in the metatheory will no longer do). This can come from a global choice principle in $\mathscr{E}$ (as is the case for small categories in $\mathscr{S}$) or it can arise more locally—given for example by the projectivity of D_0. It thus becomes useful to distinguish between two different internal notions of equivalence—we shall say that an internal functor F: $\mathbb{C} \to \mathbb{D}$ is an *equivalence* if and only if it has a global pseudo-inverse and a *weak equivalence* if and only if it is internally true in $\mathscr{E}$ that it is full, faithful and essentially surjective on objects:

$$\forall c, c' \in C_0\, \forall h \in D_1[\operatorname{dom} h = Fc \wedge \operatorname{cod} h = Fc' \to \exists f \in C_1 \,.\, Ff = h],$$

$$\forall f, g \in C_1[\operatorname{dom} f = \operatorname{dom} g \wedge \operatorname{cod} f = \operatorname{cod} g \wedge Ff = Fg \to f = g],$$

$$\forall d \in D_0\, \exists c \in C_0\, \exists i \in \operatorname{Iso}(\mathbb{C})[\operatorname{dom} i = Fc \wedge \operatorname{cod} i = d].$$

3.3. Remark. We do not consider here the intermediate condition when F has internally a pseudo-inverse (that is, when the object of pseudo-inverses is inhabited). This holds in the presence of the internal axiom of choice, and fails in general, unless we know, for example, that D_0 is internally projective.

We shall say that two (internal) categories $\mathbb{C}$ and $\mathbb{D}$ are weakly equivalent if there are an $\mathbb{E}$ and a pair of weak equivalence functors F: $\mathbb{E} \to \mathbb{C}$ and G: $\mathbb{E} \to \mathbb{D}$. It is easy to see that weak equivalence is symmetric reflexive and transitive, and that it is therefore the closure as an equivalence relation of the relation on categories generated by the weak equivalence functors. This notion has been extensively treated by Freyd [5] who notes that many proofs of equivalence in mathematics proceed by first establishing a weak equivalence in one direction and then using choice to obtain a pseudo-inverse.

As has been observed by, for example, Bunge and Paré in [3] (where the authors however use the language of indexed category theory), we can detect many interesting properties of an internal functor F at the level of its externalisation.

3.4. Proposition. *Suppose F: $\mathbb{C} \to \mathbb{D}$ is an internal functor. Then*

(i) *F is (internally) faithful if and only if $[F]$ is faithful;*

(ii) *F is (internally) full if and only if $[F]$ sastisfies the following condition:*

> *for every I in $\mathscr{E}$, C_1 and C_2 in $[\mathbb{C}]^I$ and map g: $[F](C_1) \to [F](C_2)$ in $[\mathbb{D}]^I$, there are a cover α: $J \twoheadrightarrow I$ and a map f: $\alpha^*(C_1) \to \alpha^*(C_2)$ in $[\mathbb{C}]^J$ such that $[F](f) = \alpha^*(g)$;*

(iii) *F is (internally) full and faithful if and only if $[F]$ is full and faithful;*

(iv) *F is a (strong internal) equivalence if and only if $[F]$ is an equivalence;*

(v) *F is (internally) essentially surjective on objects if and only if $[F]$ satisfies the following condition:*

> *for every I in $\mathscr{E}$, and every D in $[\mathbb{D}]^I$, there are a cover α: $J \twoheadrightarrow I$ and an object C in $[\mathbb{C}]^J$ such that $[F](C) \cong \alpha^*(D)$.*

Proof. The arguments for these are all essentially the same. It is straightforward to show that if the internal functor has one property, then the externalisation has the corresponding property. To proceed in the reverse direction consider the generic instance. Note that (iv) can be made to follow from the fact (cf. Bunge and Paré [3]) that

$$[\ ]\colon \mathrm{Cat}_{\mathscr{E}} \to \mathrm{Fib}_{\mathscr{E}}$$

is a local equivalence (i.e. for each C, C', the induced functor between the functor categories $\mathrm{Cat}_{\mathscr{E}}(C, C')$ and $\mathrm{Fib}_{\mathscr{E}}([\mathbf{C}], [\mathbf{C}'])$ is an equivalence)—a fact which itself follows from similar arguments to those given below. We sketch proofs for (ii) and (iv). A functor $F\colon \mathbf{C} \to \mathbf{D}$ is full if and only if

$$\forall A, B \in C_0\, \forall g\colon FA \to FB\; \exists f\colon A \to B\,.\, Fg = f.$$

Take the pullback

$$\begin{array}{ccc} P & \longrightarrow & D_1 \\ \downarrow & & \downarrow \\ C_0 \times C_0 & \xrightarrow{F_0 \times F_0} & D_0 \times D_0 \end{array}$$

Here $P \to D_1$ is a P-indexed family of maps from the P-indexed family

$$F^P\left(P \longrightarrow C_0 \times C_0 \xrightarrow{\pi_0} C_0\right) = P \longrightarrow C_0 \times C_0 \xrightarrow{\pi_0} C_0 \xrightarrow{F} D_0$$

of objects of $\mathbf{D}$ to

$$F^P\left(P \longrightarrow C_0 \times C_0 \xrightarrow{\pi_1} C_0\right) = P \longrightarrow C_0 \times C_0 \xrightarrow{\pi_1} C_0 \xrightarrow{F} D_0.$$

By hypothesis there is an $\alpha\colon J \twoheadrightarrow P$ and $J \to C_1$ such that

$$F^J(J \to C_1) = \alpha^*(P \to D_1),$$

but this implies that P (the set of triples A, B, g with $g\colon FA \to FB$) can be covered by J for which there is an assignment (A_j, B_j, f_j) such that $(A_j, B_j, Ff_j) = \alpha j$. For (iv) suppose we are given $F\colon [\mathbf{C}] \to [\mathbf{D}]$, $G\colon [\mathbf{D}] \to [\mathbf{C}]$ cartesian functors, together with natural isomorphisms $\eta\colon \mathrm{Id}_{[\mathbf{C}]} \to GF$, $\varepsilon\colon FG \to \mathrm{Id}_{[\mathbf{D}]}$. Look at the generic family of objects of $\mathbf{C}$, $C_0 \to C_0$ in $[\mathbf{C}]^{C_0}$. The transformation η gives a C_0-indexed family of isomorphisms from $C_0 \to C_0$ to $[GF](C_0 \to C_0)$, that is to say, a map $C_0 \xrightarrow{\eta} C_1$ factoring through $\mathrm{Iso}(\mathbf{C})$ making

$$\begin{array}{ccc} C_0 & \xrightarrow{\eta} & C_1 \\ & {\scriptstyle \mathrm{id}}\searrow & \downarrow{\scriptstyle \mathrm{dom}} \\ & & C_0 \end{array} \qquad \begin{array}{ccccc} C_0 & & \xrightarrow{\eta} & & C_1 \\ {\scriptstyle \mathrm{id}}\downarrow & & & & \downarrow{\scriptstyle \mathrm{cod}} \\ C_0 & \xrightarrow{F} & D_0 & \xrightarrow{G} & C_0 \end{array}$$

commute. We obtain $\varepsilon\colon D_0 \to D_1$ similarly, and it is now simple to verify that $C_0 \xrightarrow{\eta} C_1$ and $D_0 \xrightarrow{\varepsilon} D_1$ are (internal) natural isomorphisms $\mathrm{Id}_{[\mathbf{C}]} \to GF$ and $FG \to \mathrm{Id}_{[\mathbf{D}]}$.

Following this proposition we can define an equivalence of fibred categories to be a pseudo-inverse pair of cartesian functors over $\mathscr{E}$, and a weak equivalence

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{F} & \mathscr{D} \\ & \searrow \quad \swarrow & \\ & \mathscr{E} & \end{array}$$

of fibrations over $\mathscr{E}$ to be a cartesian functor which is full faithful and satisfies the condition:

for every I in $\mathscr{E}$, and every D in $\mathscr{D}^I$, there are a cover α: $J \twoheadrightarrow I$ and an object C in $\mathscr{C}^J$ such that $[F](C) \cong \alpha^*(D)$.

Finally, we note that Bunge and Paré prove

3.5. PROPOSITION. *If* $\mathbb{C}$ *and* $\mathbb{D}$ *are internal categories, then* $[\mathbb{C}]$ *and* $[\mathbb{D}]$ *are weakly equivalent fibrations if and only if* $\mathbb{C}$ *and* $\mathbb{D}$ *are weakly equivalent internal categories.*

4. *Strong and weak notions of completeness for internal categories*

The notion of equivalence is useful in naive category theory because it preserves and reflects all the categorical structure we are interested in. We, however, can no longer afford to be naive, and we must choose the notion of equivalence carefully so that it preserves the structure we are interested in—or more realistically we must look at various levels of structure and see how they are preserved and reflected by the two levels of equivalence which we introduced in the last section. In particular, we can no longer afford to be hazy about the distinction between a category having structure (in the sense of the internal logic) and a category actually coming equipped with the structure.

Consider even the simple case of an internal category $\mathbb{C}$ with a terminal object. Either $\mathbb{C}$ can have the terminal object in the strong sense of there being a designated map T: $1 \to C_0$ such that

$$\begin{array}{ccc} C_0 & \xrightarrow{\text{dom}} & C_1 \\ \downarrow & & \downarrow{\scriptstyle \text{cod}} \\ 1 & \xrightarrow{T} & C_0 \end{array}$$

is a pullback, or else it can have a terminal object in the weak sense that $\{c \in C_0 \mid \forall c' \in C_0 \, \exists! \, \alpha \in C_1 \, . \, \alpha \colon c' \to c\}$, the internal collection of terminal objects, is inhabited (i.e. the map from this into the terminal object of $\mathscr{E}$ is a surjection). Of course the strong sense implies the weak, but not conversely (the principle that every inhabited set should have a global element is equivalent to the projectivity of 1, and does not hold constructively). Suppose now that F: $\mathscr{C} \to \mathscr{D}$ is an internal functor. If F is a strong equivalence, then it preserves and reflects the property of having terminal objects—both in our weak and our strong sense. Similarly, if F is a weak equivalence, it preserves and reflects the weak structure, but does not reflect the strong. Moreover, although a weak equivalence preserves the property of having a strong terminal object, since $F \circ T$ will be a terminal object of $\mathscr{D}$ when T is a terminal object of $\mathscr{C}$, this is misleading (for example, weak equivalence does not preserve the property of having strong binary products). Of course if $\mathcal{D}$ has a strong terminal object, then $\mathcal{C}$ has a weak one. Note that the same problems exist at the level of fibrations: there is a distinction between there being a terminal object in each fibre of a fibration $p\colon \mathcal{A} \to \mathcal{E}$, and there being a cartesian functor from id: $\mathcal{E} \to \mathcal{E}$ that picks it out.

DEFINITION. An internal category $\mathbb{C}$ has *strong* (*binary*) *products* if there is an internal functor $\mathbb{C}^2 \to \mathbb{C}$ right adjoint (by a given internal adjunction) to the diagonal $\Delta\colon \mathbb{C} \to \mathbb{C}^2$. In addition, $\mathbb{C}$ has *weak* (*binary*) *products* if the following is internally true in $\mathcal{E}$:

$$\forall c, c' \in C_0\, \exists d \in C_0\, \exists \pi, \pi' \in C_1 \,.\, \pi\colon d \to c \wedge \pi'\colon d \to c'$$

$$\wedge \text{ `}d \text{ is a product of } c \text{ and } c' \text{ with projections } \pi \text{ and } \pi'\text{'}.$$

We define weak and strong pullbacks, equalisers, etc. similarly. $\mathbb{C}$ is *strong* (*respectively weak*) *finite complete* if it has strong (respectively weak) finite products and equalizers.

4.1. PROPOSITION. (i) *Strong equivalences preserve and reflect all strong* (*respectively weak*) *finite limits. In particular, if F: $\mathbb{C} \to \mathbb{D}$ is a strong equivalence, then $\mathbb{C}$ is strong* (*weak*) *complete if and only if $\mathbb{D}$ is.*

(ii) *Weak equivalences preserve and reflect weak finite structure. In general weak equivalences neither preserve nor reflect strong finite structure. Of course, since strong structure implies weak structure, if $\mathbb{C}$ and $\mathbb{D}$ are weakly equivalent categories, and $\mathbb{C}$ has some strong finite structure, then $\mathbb{D}$ has the corresponding weak structure.*

Suppose now that $\mathbb{D}$ is an internal category. Again we can say that $\mathbb{C}$ has strong limits if the diagonal $\mathbb{C} \to \mathbb{C}^{\mathbb{D}}$ has a right adjoint in the 2-category $\mathbf{Cat}(\mathcal{E})$, and that $\mathbb{C}$ has weak $\mathbb{D}$-limits if it is internally true that every functor $\mathbb{D} \to \mathbb{C}$ has a limit cone over it. Again, if F: $\mathbb{C} \to \mathbb{C}'$ is a strong equivalence with pseudo-inverse G say, then F preserves and reflects the property of having strong $\mathbb{D}$-limits (and also preserves and reflects the limits). Moreover, it still preserves and reflects the property of having weak $\mathbb{D}$-limits, since we can use G to lift $\mathbb{D}$-diagrams in $\mathbb{C}'$ to $\mathbb{D}$-diagrams in $\mathbb{C}$. As we would expect, if F is a weak equivalence, then it will in general neither preserve nor reflect the property of having strong $\mathbb{D}$-limits. Moreover, it will not preserve the property of having weak $\mathbb{D}$-limits, though it will reflect it. Consider the case for preservation. Suppose we have a $\mathbb{D}$-diagram in $\mathbb{C}'$; then in order to find a limit we have to lift the diagram to $\mathbb{C}$. However, although we know that we can lift the objects of the diagram to $\mathbb{C}$ individually, we have no constructive way of lifting them collectively (this is where the use of the pseudo-inverse G is crucial in the proof that strong equivalences preserve the existence of weak limits). More precisely, to prove that weak equivalence preserved the existence of weak limits we would like to know that a weak equivalence induced a weak equivalence of functor categories $F^{\mathbb{D}}\colon \mathbb{C}^{\mathbb{D}} \to \mathbb{C}'^{\mathbb{D}}$, but this is false. Suppose we regard objects C, C', and D as discrete internal categories. Then a weak equivalence functor from the discrete internal category corresponding to C to the internal category corresponding to C' is the same as a surjection $C \twoheadrightarrow C'$, and the category of D-diagrams in C is the discrete category on the exponential C^D. But the functor $(-)^D$ does not preserve surjections in general. If, on the other hand, we know that $\mathbb{C}'$ has weak $\mathbb{D}$-limits, then we can use F itself to transfer diagrams in $\mathbb{C}$ to diagrams in $\mathbb{C}'$, and then have to lift only the apex of the limit cone back to $\mathbb{C}$.

This problem remains quite serious, in that even if we know that $\mathbb{C}$ has weak limits for all internal diagrams, we can not deduce the same for $\mathbb{C}'$. The solution seems to be, instead of considering individual diagrams, to consider families of diagrams.

DEFINITION. An internal category $\mathbb{C}$ is *weakly complete* if, for all internal families of diagrams $\begin{pmatrix} \mathbb{D} \\ \downarrow \\ I \end{pmatrix} \to I^*\mathbb{C}$ in $\mathbb{C}$, it is internally true that there is a limit cone over each diagram. We say that $\mathbb{C}$ is *strongly complete* if, for each family of diagrams in $\mathbb{C}$ as above, there is a function on I which assigns limit cones.

4.2. PROPOSITION. (i) *Strong equivalences preserve and reflect both strong and weak completeness.*

(ii) *Weak equivalences preserve and reflect weak but not strong completeness.*

Proof. It remains to be seen how, given a weak equivalence $F\colon \mathbb{C} \to \mathbb{C}'$, we lift a family of diagrams

$$G'\colon \begin{pmatrix} \mathbb{D}' \\ \downarrow \\ I \end{pmatrix} \to I^*\mathbb{C}'$$

in $\mathbb{C}'$ to a family in $\mathbb{C}$. To do this we take the iso-comma square for G' along I^*F in the 2-category $\mathbf{Cat}(\mathscr{E}/I)$ (a construction which acts as the bipullback in the 2-category):

$$\begin{array}{ccc} \mathbb{D} & \longrightarrow & \mathbb{D}' \\ {\scriptstyle G}\downarrow & \cong & \downarrow{\scriptstyle G'} \\ I^*\mathbb{C} & \longrightarrow & I^*\mathbb{C}' \end{array}$$

That is to say, we take the family $\begin{pmatrix} \mathbb{D} \\ \downarrow \\ I \end{pmatrix}$ of diagrams with objects triples $(d, \alpha, (c, i))$, where $c \in C_0$, $d \in D_0'$ over i, and α is an isomorphism $G'd \to Fc$. The maps are the obvious pairs of maps. The family G is a family of diagrams in $\mathbb{C}$ indexed by I, and therefore has a family of limits, but applying F to this family of limits gives a family of limits in $\mathbb{C}'$ for the family G'.

We could make analogous definitions of the weak and strong completeness of a fibration $p\colon \mathscr{A} \to \mathscr{E}$ with respect to families of $\mathscr{E}$-diagrams. It is however easier simply to say that $\mathscr{A}$ is *strongly complete* if it has finite limits in each fibre and each pullback functor $\alpha^*\colon \mathscr{A}^I \to \mathscr{A}^J$ has a right adjoint Π_α. We must of course require also that the finite limits are preserved by the α^* and that the Π_α satisfy Beck–Chevalley conditions. Now $\mathscr{A}$ is *weakly complete* if, given a finite diagram in the fibre over I, there is a $\beta\colon K \twoheadrightarrow I$ such that the pullback along β of the diagram has a limit in the fibre over K, and if for each α: $J \to I$ and each $A \in \mathcal{A}^J$, there is β: $K \twoheadrightarrow I$ such that $\Pi_{\beta^*\alpha}\beta^*A \in \mathcal{A}^K$ exists. The compatibility conditions we require are again that the α^* preserve all finite limits and $\mathscr{E}$-indexed products, though this now has to become subject to their existence.

We leave the following as exercises for the reader:

4.3. PROPOSITION. (i) *Strong equivalences of fibrations preserve and reflect both strong and weak completeness.*

(ii) *Weak equivalences of fibrations preserve and reflect weak but not strong completeness.*

4.4. PROPOSITION. *An internal category* $\mathbb{C}$ *is weakly (respectively strongly) complete if and only if its externalisation* $[\mathbb{C}]$ *is weakly (respectively, strongly) complete as a fibration.*

5. *Stacks*

The need to distinguish between strong and weak concepts arises from the fact that the notion of existence given by the internal logic of a topos is local existence, while we have been wanting to extract global information. This is possible if instead of an arbitrary fibration, we have a *stack* (see Giraud [8], Bunge and Paré [3]). In the first reference we can find the definition of stack (champs in French) over an arbitrary site. Here, however, we shall be interested only in stacks for the regular topology on a topos or quasi-topos $\mathscr{E}$, and for this case the definitions can be made somewhat simpler.

Let p: $\mathcal{A} \to \mathscr{E}$ be a fibration, and suppose that α: $J \twoheadrightarrow I$ is a surjection. Form the complex

$$J \times_I J \times_I J \rightrightarrows J \times_I J \rightrightarrows J \twoheadrightarrow I.$$

By an object with descent data with respect to α, we mean an object $A \in \mathcal{A}^J$ together with an isomorphism θ: $\pi_0^*A \to \pi_1^*A$ in the fibre over $J \times_I J$. In addition, we require that $\Delta^*\theta = \mathrm{id}_A$ and that $\pi_{02}^*\theta = \pi_{12}^*\theta \circ \pi_{01}^*\theta$. A map of objects with descent data (A, θ) to (A', θ') is just a map $A \to A'$ in the fibre over J which commutes with the descent maps θ and θ'. This gives us a category $\mathcal{D}esc_\alpha$. Note that any object of $\mathcal{A}^I$ gives us an object with descent data by pullback, and so a functor ρ_α: $\mathcal{A}^I \to \mathcal{D}esc_\alpha$.

DEFINITION. The fibration p: $\mathcal{A} \to \mathscr{E}$ is a *stack* if for all I and all α: $J \twoheadrightarrow I$ the functor ρ_α defined above is an equivalence of categories.

5.1. REMARK. The notion of descent data is a finitary way of expressing a compatible family of objects generated by A on the crible over I generated by α.

EXAMPLE. The fibration $\mathscr{E}^2 \to \mathscr{E}$ for $\mathscr{E}$ a topos is a stack. Suppose we have α: $J \twoheadrightarrow I$ and $\begin{pmatrix} X \\ \downarrow \\ J \end{pmatrix}$ in the fibre over J with descent data θ with respect to α. What this amounts to is being given, for each j, j' in J such that $\alpha j = \alpha j'$, an isomorphism

$$\theta_{j,j'}\colon X_j \xrightarrow{\sim} X_{j'}.$$

Moreover, the $\theta_{j,j'}$ compose $(\theta_{j,j''} = \theta_{j',j''}\theta_{j,j'})$ and $\theta_{j,j} = \mathrm{id}$. We can descend $\begin{pmatrix} X \\ \downarrow \\ J \end{pmatrix}$ to I by considering sections for this structure: as the fibre over I we take maps $\sigma\colon J_i \to X$ such that $\sigma_j \in X_j$ and $\theta_{j,j'}\sigma_j = \sigma_{j'}$.

We can obtain many more examples of stacks from this single instance.

5.2. Lemma. *Suppose* $p\colon \mathscr{A} \to \mathscr{E}$ *is a stack and* $p'\colon \mathscr{A}' \to \mathscr{E}$ *is a subfibration of* p *closed under isomorphisms such that if* $\alpha\colon J \twoheadrightarrow I$, $x \in \mathscr{A}^I$ *and* $\alpha^* x \in \mathscr{A}'$, *then* $x \in \mathscr{A}'$. *Then* $\mathscr{A}'$ *is a stack.*

5.3. Corollary. *Any subfibration of* $\mathscr{E}^2 \to \mathscr{E}$ *definable in the sense of Bénabou* [1] *is a stack, and hence any subfibration definable in the internal logic of the topos is a stack. In particular, the following are stacks:*

(i) *separated objects for a topology* j *on* $\mathscr{E}$;

(ii) *sheaves for a topology* j *on* $\mathscr{E}$;

(iii) *families of subquotients of a particular object* N *of* $\mathscr{E}$;

(iv) *the fibration* $\mathcal{O}\mathit{rth}(\theta)$ *for any map* θ.

However, many 'naturally occurring' fibrations are not stacks. For example, externalisations of internal categories are not generally stacks. An example is the full subcategory in $\mathscr{S}h(S^1)$ on the single object the connected double-covering of the circle. This double-covering is locally isomorphic to $S^1 \amalg S^1$, but of course is not globally so.

Bunge [2] shows how to construct, given a locally small fibration $\mathscr{F}$ over the topos $\mathscr{E}$, a stack $\bar{\mathscr{F}}$ with a weak equivalence $F\colon \mathscr{F} \to \bar{\mathscr{F}}$ over $\mathscr{E}$ such that for any $G\colon \mathscr{F} \to \mathscr{G}$ into a stack there is $H\colon \bar{\mathscr{F}} \to \mathscr{G}$ such that

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{F} & \bar{\mathscr{F}} \\ & {\scriptstyle G}\searrow & \downarrow{\scriptstyle H} \\ & & \mathscr{G} \end{array}$$

commutes up to isomorphism, and H is uniquely determined up to a unique isomorphism. The weak equivalence $F\colon \mathscr{F} \to \bar{\mathscr{F}}$ is called the *stack completion* of $\mathscr{F}$. From the universal property it follows that any stack which is weakly equivalent to $\mathscr{F}$ is (strongly equivalent to) its stack completion.

We shall not discuss the general construction of the stack completion as we do not need such generality, but we shall want a description of the stack completion for a full subfibration $\mathscr{F}$ of $\mathscr{E}^2$. In this case, because $\mathscr{E}^2$ provides us with a stack which contains $\mathscr{F}$, the construction of $\bar{\mathscr{F}}$ becomes simple. All we have to do is to collect the necessary objects: take those families $\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}$ such that there is a pullback

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ J & \overset{\alpha}{\twoheadrightarrow} & I \end{array}$$

with $\begin{pmatrix} Y \\ \downarrow \\ J \end{pmatrix}$ in $\mathscr{F}$. Let $\bar{\mathscr{F}}$ be the full subcategory of $\mathscr{E}^2$ on these. It is immediate from Lemma 5.2 that this is a stack and by definition the inclusion $\mathcal{F} \to \bar{\mathcal{F}}$ is a weak equivalence.

Recall that in the case that the subfibration $\mathcal{F}$ is the 'externalisation' $(\mathbb{C})$ of a small full subcategory $\mathbb{C}$ on a family of objects $S \to C_0$, it consists of those $\begin{pmatrix} Y \\ \downarrow \\ J \end{pmatrix}$ such that there is a pullback

$$\begin{array}{ccc} Y & \longrightarrow & S \\ \downarrow & & \downarrow \\ J & \longrightarrow & C_0 \end{array}$$

Its stack completion, which we shall write $\{\mathbb{C}\}$, is therefore the full subcategory of $\mathscr{E}^2$ of those $\begin{pmatrix} Z \\ \downarrow \\ I \end{pmatrix}$ such that there is a pullback

$$\begin{array}{ccc} Y & \longrightarrow & Z \\ \downarrow & & \downarrow \\ J & \xrightarrow{\ \alpha\ } & I \end{array}$$

with $\begin{pmatrix} Y \\ \downarrow \\ J \end{pmatrix}$ in $\mathcal{F}$.

In other words, $(\mathbb{C})$ consists of the objects such that there is a global map $J \xrightarrow{\alpha} C_0$ satisfying

$$\forall j \in J \, . \, Y_j \cong a(j),$$

while $\{\mathbb{C}\}$ consists of those objects such that

$$\forall i \in I \, \exists c \in C_0 \, . \, Z_i \cong S_c.$$

Note that in general these two conditions are not equivalent. For such a family $Z \to I$ to be in $(\mathbb{C})$ we need a function $I \to C_0$ giving such a c for each i, and in the absence of choice in the topos this is impossible to guarantee.

If $Q \to Q_0$ is the family of subquotients of N considered before in § 3, then the stack completion of $[\mathbb{Q}]$ is the fibration $\{\mathbb{Q}\}$ of families of subquotients of N—those $\begin{pmatrix} Z \\ \downarrow \\ I \end{pmatrix}$ such that

$$\forall i \in I \, \exists A \subset N \, . \, A \twoheadrightarrow Z_i.$$

In § 6 we will show that when N is the object of realisers in the effective topos, these two fibrations are not strongly equivalent, though they become so when we restrict the indexing category to the separated subobjects in the topos. This will enable us to apply the following result.

5.4. LEMMA. *If the stack $p: \mathscr{A} \to \mathscr{E}$ is weakly complete, then it is strongly complete.*

Proof. The proof is in fact routine, and follows from the observation that families of limits for the same diagram naturally carry descent data.

6. *Discrete objects in the effective topos*

The title of this section needs to be explained: we define the *discrete* objects in the effective topos as those orthogonal to $\Delta 2$. Our first result shows that we could obtain the same category from many different objects.

6.1. PROPOSITION. (i) *If S is a set containing at least two elements, then* $\mathcal{O}rth(\Delta S) = \mathcal{O}rth(\Delta 2)$.

(ii) *If U is a uniform object with (at least) two distinct global sections, then* $\mathcal{O}rth(U) = \mathcal{O}rth(\Delta 2)$.

(iii) $\mathcal{O}rth(\Omega) = \mathcal{O}rth(\Delta 2)$.

Proof. (i) We recall that $\Delta\colon \mathcal{S} \to \mathcal{E}\!f\!f$ is equivalent to the inclusion of the $\neg\neg$-sheaves. Now in any topos $\mathcal{E}$ and for any topology j, if X is a j-sheaf then the sheafification $Y \to aY$ of an arbitrary object Y gives rise to an isomorphism $X^{aY} \to X^Y$. In our case we get $\Delta S^{\Delta 2} \cong \Delta(S^2) \cong (\Delta S)^2$. The conclusion now follows from Proposition 2.7.

(ii) Recall (Hyland [10]) that the uniform objects are those U which are uniform with respect to N:

$$\forall \phi[\forall u \in U\, \exists n \in N\,.\, \phi(u, n) \to \exists n \in N\, \forall u \in U\,.\, \phi(u, n)].$$

In the effective topos the uniform objects are precisely the quotients of $\neg\neg$-sheaves. Suppose U is as in the statement. Then the two distinct global sections give a monic $2 \rightarrowtail U$. If now $\Delta S \twoheadrightarrow U$, then $\Delta S^U \cong \Delta S^{aU} \twoheadrightarrow \Delta S^{\Delta 2} \cong \Delta S^2$ since this is essentially happening in $\mathcal{S}$, and we again apply Proposition 2.7 to obtain the result.

(iii) We remind the reader that, in the effective topos, $\Omega = (\mathrm{P}\omega, \leftrightarrow)$. It is easily seen that the relation $G\colon \mathrm{P}\omega \times \mathrm{P}\omega \to \mathrm{P}\omega$, defined by

$$G(S, T) = \begin{cases} \omega & \text{if } S = T, \\ \varnothing & \text{otherwise,} \end{cases}$$

gives rise to a surjection $\Delta(\mathrm{P}\omega) \twoheadrightarrow \Omega$, and hence Ω is uniform. Moreover '$\top$' and '$\perp$' are distinct global sections.

It is easy to deduce properties of the discrete objects as $\Delta 2$ is internally projective in $\mathcal{E}\!f\!f$. In fact all ΔS are, and a simple way to see this is to recall the description of function space for a topos based on a tripos (cf. Hyland, Johnstone and Pitts [12]). For $(X, =)$ and $(Y, =)$ in $\mathcal{E}\!f\!f$ the function space is $(Z, =)$ where $Z = \mathcal{E}\!f\!f((X, =), (Y, =))$ and

$$[\![f = g]\!] = \{n \in \omega \mid n \Vdash \text{'}f \text{ is functional'} \wedge \forall x \in X\, \forall y \in Y[f(x, y) \leftrightarrow g(x, y)]\}.$$

In case $(X, =) = \Delta X$, a map f in Z is completely determined by an X-sequence $(y_x)_{x \in X}$ such that $\bigcap_{x \in X} [\![y_x = y_x]\!] \neq \varnothing$. This collection with the suitably restricted equality is (isomorphic to) the function space $(Y, =)^{\Delta X}$. It is easy to see now that ΔX is internally projective.

6.2. COROLLARY. $\mathcal{O}rth(\Delta 2)$ *is closed under subobjects, quotients, and all the limits in the topos.*

This however only begins to become significant when we have

6.3. Lemma. *The object N is discrete.*

Proof. This follows from the uniformity principle of Hyland [**10**]. It is however easy to see with the representation of the function space $(Y, =)^{\Delta X}$ given above that $N^{\Delta 2}$ is based on the set of diagonal pairs of $N \times N$. Thus $N \cong N^{\Delta 2}$.

So we know that $\mathcal{O}rth(\Delta 2)$ is a fairly large category—it contains at least all the subquotients of N. In the remainder of this section we shall see that $\mathcal{O}rth(\Delta 2)$ is in fact the stack completion of the fibred category of subquotients N—the stack of families of subquotients of N. We begin by working in the fibre over 1.

Recall that the objects ΔS can be characterised internally—they are the sheaves for the double negation topology; cf. Hyland [**10**]. In the fibre over 1 we can also give a good description of the separated objects for this topology: any separated object is isomorphic to one given as $(X, |-|)$ where

$$[\![x = x']\!] = \begin{cases} |x| & \text{if } x = x', \\ \varnothing & \text{otherwise,} \end{cases}$$

as in [**10**]. Conversely, any such object is separated. It is now easy to see that any object in $\mathcal{E}\!f\!f$ is covered by a separated object, for if we have an arbitrary $(Y, =)$, then

$$\begin{array}{ccc} (Y, |-|) & \rightarrowtail & \Delta Y \\ \big\downarrow & & \\ (Y, =) & & \end{array}$$

where $|y| = [\![y = y]\!]$. Note however, that this covering is not canonical as it depends on the presentation of $(Y, =)$ we are given.

6.4. Lemma. *If X is a separated object in the effective topos, then there is a subobject $E \rightarrowtail X \times N$ such that*

(a) π: $E \twoheadrightarrow X$,

(b) $\begin{pmatrix} E \\ \downarrow \\ N \end{pmatrix} \in \mathcal{S}h_{\neg\neg}(\mathcal{E}\!f\!f/N)$.

In general we shall say that an object X has Property (†) if there is an $E \rightarrowtail X \times N$ satisfying (a) and (b) above.

Proof. Assume that X is presented as $(X, |-|)$ as above, and define E by E: $X \times \omega \to \mathrm{P}\omega$,

$$E(x, n) = \begin{cases} \{n\} & \text{if } n \in |x|, \\ \varnothing & \text{otherwise.} \end{cases}$$

Then E is trivially relational for both X and N, and hence defines the subobject E. In order to show that E surjects onto X we must show that there is an m realising

$$\forall x \in X\, \exists e \in E\,.\, \pi e = x.$$

The essential part of the statement is that given an $n \in |x|$ we should be able to find an $\langle x, n\rangle$ such that $\{m\}(n)_0 \in E(x, n)$. But now any m coding a total recursive function $\omega \to \omega \times \omega$ whose first component is the identity will do. It remains to prove that $\begin{pmatrix} E \\ \downarrow \\ N \end{pmatrix}$ is a $\neg\neg$-sheaf in $\mathcal{E}ff/N$. We show that $\begin{pmatrix} E \\ \downarrow \\ N \end{pmatrix}$ is a closed subobject of a sheaf. Since the functor N^* is logical, $N^*\Delta X = \begin{pmatrix} \Delta X \times N \\ \downarrow \\ N \end{pmatrix}$ is certainly a sheaf, and we have

$$\begin{pmatrix} E \\ \downarrow \\ N \end{pmatrix} \rightarrowtail N^*X \rightarrowtail N^*\Delta X.$$

This inclusion is $\neg\neg$-closed in $\mathcal{E}ff/N$ if and only if

$$E \rightarrowtail X \times N \rightarrowtail \Delta X \times N$$

is closed in the fibre over 1. Now $E \rightarrowtail \Delta X \times N$ can also be given by the map $E\colon X \times \omega \to P\omega$. We must show that

$$\Vdash \forall x \in \Delta X\ \forall n \in N[\neg\neg E(x, n) \to E(x, n)].$$

So given x in X, n in ω, $m \in [\![n = n]\!]$, and information that $E(x, n)$ is non-empty, we must find recursively in m an $m' \in E(x, n)$. But of course $m = n$, and if $E(x, n)$ is non-empty then it is $\{n\}$, so the task is not too difficult.

We now turn our attention back to other fibres of the fibration. We shall need some notation. We have a sheafification functor for the $\neg\neg$-topology which we shall write as a cartesian functor

$$\begin{array}{ccc} \mathcal{E}ff^{\,2} & \xrightarrow{\ a\ } & \mathcal{E}ff^{\,2} \\ & \searrow \quad \swarrow & \\ & \mathcal{E}ff & \end{array}$$

On each fibre this restricts to a functor a_I: $\mathcal{E}ff/I \to \mathcal{E}ff/I$ giving the sheafification functor in $\mathcal{E}ff/I$ for $I^*(\neg\neg)$, the $\neg\neg$-topology there. Sometimes we shall abuse notation and drop the suffix 1 from a_1.

It is well known that a_I can be defined in terms of a_1. For any I-indexed family $\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}$, the sheaf $a_I\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}$ is the left-hand vertical in the pullback

$$\begin{array}{ccc} a_I X & \longrightarrow & a_1 X \\ \downarrow & & \downarrow \\ I & \longrightarrow & a_1 I \end{array}$$

Recall that a family $\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}$ is separated if and only if the canonical map

$$\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix} \to a_I\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}$$

is monic. It is easy to see from the above that if X is separated in the fibre over 1, then $\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}$ is separated in the fibre over I. Furthermore, if I is itself separated in the fibre over 1, the 'if' becomes 'if and only if'.

6.5. LEMMA. *If I is separated, and $\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}$ is separated in the fibre over I, then $\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}$ has Property* (†) *in the topos $\mathscr{E}\!\mathit{ff}/I$ (cf. Lemma* 6.4).

Proof. Since I is separated, so is X, and hence there is $E \rightarrowtail X \times N$ satisfying the requirements $E \twoheadrightarrow X$ and $\begin{pmatrix} E \\ \downarrow \\ N \end{pmatrix} \in \mathscr{S}\!h_{\neg\neg}(\mathscr{E}\!\mathit{ff}/N)$. As

$$\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix} \times_I I^*N = \begin{pmatrix} X \times N \\ \downarrow \\ I \end{pmatrix},$$

we can take

$$\begin{pmatrix} E \\ \downarrow \\ I \end{pmatrix} \rightarrowtail \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix} \times_I I^*N$$

by appropriate composition. We still of course have that the composite

$$\begin{pmatrix} E \\ \downarrow \\ I \end{pmatrix} \rightarrowtail \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix} \times_I I^*N \xrightarrow{\pi} \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}$$

is a surjection, but more remarkably,

$$\begin{pmatrix} E \\ \downarrow \\ I^*N \end{pmatrix} \in \mathscr{S}\!h_{\neg\neg}((\mathscr{E}\!\mathit{ff}/I)/I^*N) = \mathscr{S}\!h_{\neg\neg}(\mathscr{E}\!\mathit{ff}/I^*N).$$

Consider the diagram

$$\begin{array}{ccccccc} E & \rightarrowtail & X \times N & \rightarrowtail & a_I X \times N & \longrightarrow & aX \times N \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ I \times N & \longrightarrow & I \times N & \longrightarrow & I \times N & \longrightarrow & aI \times N \end{array}$$

Since I is separated, the final map in the top row is monic (it is a pullback of a monic). Furthermore, the composition of the top row gives a closed monic. It follows that $E \rightarrowtail a_I X \times N$ is also closed, as closed subobjects are stable under pullback. Hence $\begin{pmatrix} E \\ \downarrow \\ I^*N \end{pmatrix}$ is a sheaf, as required.

6.6. PROPOSITION. *Suppose $\begin{pmatrix} Y \\ \downarrow \\ I \end{pmatrix}$ is an I-indexed family orthogonal to $\Delta 2 = a2$.*

Suppose furthermore that $\begin{pmatrix} Y \\ \downarrow \\ I \end{pmatrix}$ *is covered by a family* $\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}$ *with Property* (†). *Then* $\begin{pmatrix} Y \\ \downarrow \\ I \end{pmatrix}$ *is a subquotient of* I^*N.

Proof. Since the proof is carried out relative to, but not entirely in, the topos $\mathscr{E}\!f\!f/I$, we shall abuse notation and work as if $I=1$. Let $E \rightarrowtail X \times N$ be a subobject satisfying the requirements of Property (†). Define E' to be the image of E in $Y \times N$:

$$\begin{array}{ccc} E & \rightarrowtail & X \times N \\ \downarrow & & \downarrow \\ E' & \rightarrowtail & Y \times N \end{array}$$

Plainly $E' \twoheadrightarrow Y$. We shall show that the projection on the second coordinate $E' \to N$ is monic. Now we have $E' \rightarrowtail N^*Y = Y \times N$. But Y is orthogonal to $a2$, and hence so is N^*Y. Consequently, E' is orthogonal to $a2$ by Lemma 2.3. However, since $\begin{pmatrix} E \\ \downarrow \\ N \end{pmatrix}$ is a sheaf over N,

$$\begin{pmatrix} E \\ \downarrow \\ N \end{pmatrix}^{a2} \xrightarrow{\sim} \begin{pmatrix} E \\ \downarrow \\ N \end{pmatrix} \times_N \begin{pmatrix} E \\ \downarrow \\ N \end{pmatrix}$$

and we have a commutative diagram

$$\begin{array}{ccccc} & & \begin{pmatrix} E \\ \downarrow \\ N \end{pmatrix}^{N^*a2} & \xrightarrow{\sim} & \begin{pmatrix} E \\ \downarrow \\ N \end{pmatrix} \times_N \begin{pmatrix} E \\ \downarrow \\ N \end{pmatrix} \\ & & \downarrow & & \downarrow \\ \begin{pmatrix} E' \\ \downarrow \\ N \end{pmatrix} & \xrightarrow{\sim} & \begin{pmatrix} E' \\ \downarrow \\ N \end{pmatrix}^{N^*a2} & \longrightarrow & \begin{pmatrix} E' \\ \downarrow \\ N \end{pmatrix} \times_N \begin{pmatrix} E' \\ \downarrow \\ N \end{pmatrix} \end{array}$$

Now the top-right path is a surjection, and hence so is the composite of the bottom row. This is the diagonal of the pullback of $E \to N$ against itself, and so implies that $E' \to N$ is monic, and that the diagram

$$\begin{array}{ccc} E' & \rightarrowtail & N \\ \downarrow & & \\ Y & & \end{array}$$

presents Y as a subquotient of N.

6.7. Proposition. *Let* I *be a separated object. Then the following three full* subcategories of $\mathcal{E}\!f\!f/I$ *are identical:*

(i) *families* $\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}$ *orthogonal to* $a2$ (*i.e.* $\mathcal{O}rth(a2)^I$);

(ii) *families locally isomorphic to a subquotient of* N (*i.e.* $\{\mathbf{Q}\}^I$);

(iii) *families covered by a subquotient of* N (*i.e.* $(\mathbf{Q})^I$).

Proof. Clearly (iii) implies (ii) whether I is separated or not. Moreover, (iii) implies (i) by Lemma 6.3, and this lifts to give (ii) implies (i) since $\mathcal{O}rth(a2)$ is a stack, and (hence) closed under local isomorphism. We really only need the hypothesis on I to show that (i) implies (iii). If $\begin{pmatrix} Y \\ \downarrow \\ I \end{pmatrix}$ is an I-indexed family of orthogonal objects, then in $\mathcal{E}\!f\!f$ we can cover Y by a separated object X. This gives a separated cover $\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix} \twoheadrightarrow \begin{pmatrix} Y \\ \downarrow \\ I \end{pmatrix}$ in $\mathcal{E}\!f\!f/I$ and we apply Lemma 6.5 and Proposition 6.7 to see that $\begin{pmatrix} Y \\ \downarrow \\ I \end{pmatrix}$ is in (iii)^I.

Since any object J can be covered by a separated object, we obtain as an immediate corollary:

6.8. THEOREM. *The two stacks* $\mathcal{O}rth(a2)$ *of families of discrete objects, and* $(\mathbf{Q})$ *of families of subquotients of* N *are identical as fibred subcategories of* $\mathcal{E}\!f\!f^2 \to \mathcal{E}\!f\!f$.

However we note that

6.9. PROPOSITION. *Not every family* $\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}$ *which is locally a quotient of* I^*N *is globally so. In particular,* $\mathcal{O}rth(a2)$ *and* $[\mathbf{Q}]$ *are not strongly equivalent fibrations.*

6.10. REMARK. At present we do not know whether there is an internal category $\mathbf{C}$ such that $\mathcal{O}rth(a2)$ is strongly equivalent to $[\mathbf{C}]$ or not.

Proof. We show that the fibrations differ in the fibre over the object $(2, \sim)$, where

$$[0 \sim 0] = [1 \sim 1] = \{0\},$$
$$[0 \sim 1] = \{1\},$$
$$[1 \sim 0] = \{2\}.$$

Note that this is not a separated object and that its sheaf reflection (in fact its separated reflection) is the terminal object 1. We have $\Delta 2 \twoheadrightarrow (2, \sim) \twoheadrightarrow 1$.

As our family over $(2, \sim)$ we take the object $A = (A, \simeq)$ where

$$A = \{0\} \times N \cup \{1\} \times N - \{0\}$$

with

$$[(i, n) \simeq (j, m)] = \begin{cases} \langle [i \sim j], n, m \rangle & \text{if } n - i = m - j, \\ \varnothing & \text{otherwise,} \end{cases}$$

labelled by $f\colon A \to (2, \sim)$ given by $f((i, n), j) = [i \sim j] \times \{n\}$.

The verification that this description does indeed give two objects and a map in $\mathscr{E}\!f\!f$ is routine, as is checking that the pullback of A over $\Delta 2$ is $C = (A, =)$ with $[(i, n) = (j, m)] = \{n \mid i = j \wedge n = m\}$. Thus C is a subobject over $\Delta 2$ of $\Delta 2^* N$. Finally, for future reference, we note that the map $A \to (2, \sim)$ is separated (the diagonal is closed), and so $A \to (2, \sim)$ is a family of separated subquotients of N.

We can now show that $A \to (2, \sim)$, despite being locally a subquotient of $(2, \sim)^* N$, can not be so globally. If it were, then we would have a map $(2, \sim) \to Q_0$ giving A by pullback from the generic subquotient of N. Since $A \to (2, \sim)$ is a family of separated subquotients, this would factor through the subobject P_0 of Q_0 which names the separated subquotients. This object P_0 is in fact a sheaf (see § 7), and so the map $(2, \sim) \to P_0$ would factor through 1. This would imply that the family $A \to (2, \sim)$ was a constant family, which it is not. It is perhaps easiest to see this by looking at the pullback C over $\Delta 2$. The fibre of C over 0 is N, and so C, if a constant family, would have to be isomorphic to $\Delta 2 \times N$ over $\Delta 2$. Suppose $\phi\colon C \to \Delta 2 \times N$ is this isomorphism; then $\pi \circ \phi$ maps C to N. Using the fact that both C and N are canonically separated, from the realiser for

$$\forall c \in C \,.\, \exists (i, n) \in \Delta 2 \times N \,.\, \phi(c) = (i, n),$$

we see that $\phi(0, n) = \phi(1, n)$, but then $\phi(1, -)$ is not surjective.

7. *Complete internal categories*

We summarise the results obtained at the end of the last section:

7.1. THEOREM. *Let* $\mathbf{Q}$ *be the full internal subcategory of the effective topos* $\mathscr{E}\!f\!f$ *on the subquotients of N. Then the externalization* $[\mathbf{Q}]_{\mathscr{E}\!f\!f}$, *which is equivalent to the full subfibration* $(\mathbf{Q})_{\mathscr{E}\!f\!f}$ *of global subquotients of N, is weakly (but not strongly) equivalent to* $\mathscr{O}rth(a2)$, *the stack of families orthogonal to a2, which may thus be taken to be its stack completion.*

Since $\mathscr{O}rth(a2)$ is (strongly) complete, we obtain as an immediate corollary:

7.2. COROLLARY. *The subcategory* $\mathbf{Q}$ *(respectively* $[\mathbf{Q}]_{\mathscr{E}\!f\!f}$*) is weakly complete.*

Dana Scott has observed that one small complete category in a topos gives rise to many more, for example any category of algebras for a triple over the small category. We however shall be more interested in another small category of sets—the internal category $\mathbf{P}$ of separated subquotients of N, which is the internal category corresponding to the subcategory of $\mathscr{E}\!f\!f$ called by Scott the category of *modest sets* (Scott [17]).

Let P_0 be the object of closed partial equivalence relations on N, that is, $P_0 = \{S \in \Omega_{\neg\neg}^{N} \mid S \in Q_0\}$. In the fibre over P_0 we have a generic closed partial equivalence relation on N,

$$\begin{array}{ccc} R & \rightarrowtail & P_0^*N \times_{P_0} P_0^*N \\ \downarrow & & \\ P_0 & & \end{array}$$

where $R = \{\langle S, n_1, n_2\rangle \mid S \in P_0 \wedge \langle n_1, n_2\rangle \in S\}$. We take the domain of R and quotient it with respect to what is now an equivalence relation to obtain $P_1 \to P_0$, an indexed collection of objects. Of course $\mathbb{P}$ is the full internal subcategory on this collection—the internal category with object of objects P_0 and object of maps given by the exponential

$$\left(P_1 \xrightarrow{\langle \mathrm{dom}, \mathrm{cod}\rangle} P_0 \times P_0\right) = \pi_1^*\begin{pmatrix} P_1 \\ \downarrow \\ P_0 \end{pmatrix}^{\pi_0^*\begin{pmatrix} P_1 \\ \downarrow \\ P_0 \end{pmatrix}}$$

in $\mathscr{Eff}/(P_0 \times P_0)$.

Note that P_0 is a retract of Q_0, and that R is obtained by pulling back the generic family $\begin{pmatrix} Q \\ \downarrow \\ Q_0 \end{pmatrix}$ for $\mathbb{Q}$. We thus obtain an inclusion functor $\mathbb{P} \to \mathbb{Q}$. This has a reflection $\mathbb{Q} \to \mathbb{P}$ obtained by internalising the separated reflection (a partial equivalence relation on N is mapped to its $\neg\neg$-closure).

In complete analogy with the results obtained previously for $\mathbb{Q}$, the externalisation $[\mathbb{P}]$ is equivalent to the full subfibration $(\mathbb{P})$ of $\mathscr{Eff}^2 \xrightarrow{\mathrm{cod}} \mathscr{Eff}$ consisting of the families $\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}$ of separated objects which are subquotients of I^*N. Moreover, the stack completion $\{\mathbb{P}\}$ may be taken to be separated families which are locally subquotients of N, and so to be the stack of separated families orthogonal to $a2$. Instead of $\{\mathbb{P}\}$ we shall sometimes write $\mathscr{Mod}$, as it is the stack corresponding to the category of modest sets (cf. Hyland [**11**]).

7.3. PROPOSITION. *The externalisation* $[\mathbb{P}]$ *is weakly but not strongly equivalent to* $\mathscr{Mod}$.

Proof. The counter-example given at the end of § 6 for Proposition 6.9 serves equally well in this case.

7.4. COROLLARY. *The category* $\mathbb{P}$ *(respectively* $[\mathbb{P}]_{\mathscr{Eff}}$*) is weakly complete.*

Proof. Both $\mathscr{Orth}(a2)$ and the fibration of families of separated objects are isomorphism-saturated subfibrations of $\mathscr{Eff}^2 \to \mathscr{Eff}$ closed under all limits.

Unfortunately, we cannot improve on this corollary to obtain strong completeness for $\mathbb{P}$ (or for that matter for $\mathbb{Q}$).

7.5. PROPOSITION. *The category* $\mathbb{P}$ *and hence (a fortiori)* $\mathbb{Q}$ *is not strongly complete as an internal category of* $\mathscr{Eff}$.

Proof. We show that ($\mathbb{P}$) (and at the same time ($\mathbb{Q}$)) is not strong complete. Consider the map $\top\colon 1 \to \Omega$, and take the indexed product $\Pi_\top 2$ of 2. We show that this cannot be globally a subquotient of Ω^*N, and hence is in neither ($\mathbb{P}$) nor ($\mathbb{Q}$). Now

$$\Pi_\top 2 \cong \begin{pmatrix} \tilde{2} \\ \downarrow \\ \Omega \end{pmatrix},$$

where dom: $\tilde{2} \to \Omega$ classifies $2 \rightarrowtail \tilde{2}$ (cf. Johnstone [**13**, Example 1.45]). If ($\mathbb{Q}$) were closed under indexed products, there would be a subset $A \rightarrowtail N \times \Omega$ and a surjection $\sigma\colon A \twoheadrightarrow \tilde{2}$ such that

$$\begin{array}{ccc} A & \rightarrowtail & N \times \Omega \\ \downarrow & & \downarrow \\ \tilde{2} & \xrightarrow{\text{dom}} & \Omega \end{array}$$

commutes. Thus

$$\forall \phi \in \tilde{2}\, \exists n \in N\,.\, \phi = \sigma(n, \operatorname{dom} \phi).$$

But $\tilde{2}$ is a retract of P2, and so is uniform with respect to N. It follows that

$$\exists n \in N\, \forall \phi \in \tilde{2}\,.\, \phi = \sigma(n, \operatorname{dom} \phi).$$

Now this implies that

$$\forall \phi, \phi' \in \tilde{2}\,.\, \operatorname{dom} \phi = \operatorname{dom} \phi' = \top \to \phi = \phi'.$$

On the other hand, $2 \cong \{\phi \in \tilde{2} \mid \operatorname{dom} \phi = \top\}$. This contradiction shows that ($\mathbb{P}$) is not closed under indexed products in $\mathscr{E\!f\!f}$ (although $\mathscr{M\!o\!d}$ of course is).

7.6. REMARK. This example also gives another proof that ($\mathbb{P}$) and $\mathscr{M\!o\!d}$ (respectively ($\mathbb{Q}$) and $\{\mathbb{Q}\}$) are not strongly equivalent.

At the time of writing we have no example of a (non-posetal) strongly complete internal category in a topos. However, $\mathbb{Q}$ and, in particular, $\mathbb{P}$ do have somewhat stronger completeness properties than has yet become apparent.

Let $\mathscr{S\!e\!p}$ be the full subcategory of $\mathscr{E\!f\!f}$ consisting of the separated objects. Then $\mathscr{S\!e\!p}$ is a locally cartesian closed left exact category, in fact a quasi-topos, and we shall consider the fibrations ($\mathbb{Q}$) and ($\mathbb{P}$) relativised to $\mathscr{S\!e\!p}$ (i.e. restricted to those fibres over separated objects).

7.7. PROPOSITION. *The fibrations* $(\mathbb{Q})_{\mathscr{S\!e\!p}}$ *and* $(\mathbb{P})_{\mathscr{S\!e\!p}}$ *are strongly complete.*

Proof. By Proposition 6.7, $(\mathbb{Q})_{\mathscr{S\!e\!p}}$ and $(\mathbb{P})_{\mathscr{S\!e\!p}}$ are the same as $\{\mathbb{Q}\}_{\mathscr{S\!e\!p}}$ and $\{P\}_{\mathscr{S\!e\!p}}$ respectively. They are thus closed under (finite limits in the fibres and) those families of internal products in the topos indexed by separated objects. Since Beck conditions are inherited from $\mathscr{E\!f\!f}$, this is rather more than is required. Alternatively, we can observe that since every object in $\mathscr{E\!f\!f}$ is covered by one from $\mathscr{S\!e\!p}$, a fibration over $\mathscr{E\!f\!f}$ is weakly complete if and only if its reduction over $\mathscr{S\!e\!p}$ is. Hence $(\mathbb{Q})_{\mathscr{S\!e\!p}}$ and $(\mathbb{P})_{\mathscr{S\!e\!p}}$ are weakly complete, but by Proposition 6.7 they are also stacks, and therefore strongly complete.

As far as $\mathbb{Q}$ is concerned, this is now the end of the road, but with $\mathbb{P}$ we can go a little further.

7.8. LEMMA. *The category $\mathbb{P}$ is an internal category in $\mathcal{S}ep$.*

Proof. We know that $\Omega_{\neg\neg}$ is a sheaf, and so also is any power of it. Now P_0 is a subobject of P^2N defined in the negative fragment of logic, and hence is also a sheaf (cf. Hyland [10]). Furthermore, R is separated over P_0, and hence the exponential which gives P_1 is separated over $P_0 \times P_0$. Since $P_0 \times P_0$ is separated, P_1 is separated in its own right.

As a corollary of the two results above we now have:

7.9. THEOREM. *The category $\mathbb{P}$ is weakly complete as an internal category of $\mathcal{E}ff$, and strongly complete as an internal category of $\mathcal{S}ep$.*

7.10. REMARK. This result can be extended slightly by using the fact that (any power of) P_0 is a sheaf, and so is projective.

Appendix: an awful warning on the subject of Beck conditions

Once again consider $\mathbb{P}$ as an internal category of $\mathcal{E}ff$. Since $\mathbb{P}$ is an internal category in $\mathcal{S}ep$, for any I the separated reflection $I \to \sigma I$ induces an isomorphism

$$[\mathbb{P}]^{\sigma I} \xrightarrow{\sim} [\mathbb{P}]^{I}.$$

It follows that, given any $\alpha\colon J \to I$, $\alpha^*\colon [\mathbb{P}]^I \to [\mathbb{P}]^J$ has a right adjoint $\Pi^{\mathbb{P}}_{\alpha}\colon [\mathbb{P}]^J \to [\mathbb{P}]^I$, since we can compute it from $\Pi_{\sigma\alpha}\colon [\mathbb{P}]^{\sigma J} \to [\mathbb{P}]^{\sigma I}$, which exists because of the strong completeness of $\mathbb{P}$ over $\mathcal{S}ep$. This however can *not* imply the strong completeness of $\mathbb{P}$ over $\mathcal{E}ff$ (cf. Proposition 7.5). In fact, the separated reflection σ is not left exact, and this induces a failure of the Beck–Chevalley conditions for the functors $\Pi^{\mathbb{P}}$. Specifically, the Beck condition fails for the pullback

$$\begin{array}{ccc} 2 & \rightarrowtail & \bar{2} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{dom}} \\ 1 & \xrightarrow{\top} & \Omega \end{array}$$

though at present our proof of this is indirect.

8. *Models of polymorphic λ-calculi*

In this section we give a brief account of how the categories discussed in the previous section may be used to model various polymorphic extensions of the lambda-calculus. Our strategy is to take an internal category $\mathbf{C}$ (in our case it will be $\mathbb{P}$) as our collection of types. Thus, the interpretation of a closed type term will be a global element of C_0, that is, a map $1 \to C_0$. More generally, we have to interpret terms which depend on free (type and other higher-order) variables. We shall assume that these variables range over objects whose interpretation may be taken to be objects of the topos. Thus, if we take as the simplest real example

Girard's 'Système F' (Reynold's second-order lambda-calculus, Girard [7], Reynolds [16]), we have type expressions which may only depend on variables ranging over types, whereas in the extension to F_ω we have type expressions which depend on variables ranging over higher-order functionals on types. We interpret these type expressions in the natural way—as types varying over the interpretation of the domains of the free variables, and so as objects of $[\mathbf{C}]$ in the fibre over some I. For those familiar with the work of Seely in this area (Seely [18]), what we are essentially doing is using a restriction of the externalisation process to give a hyperdoctrine over some subcategory of the topos, and hence a version of Seely's notion of PL-category. More precisely, it is a version of the $2T\lambda C$-hyperdoctrines used by Pitts to prove a completeness result for internal category models with respect to the second-order lambda calculus. We refer the reader to Pitts [15] for a more detailed account of how to use internal categories than the sketch here.

We shall indicate how a sufficiently complete internal category $\mathbf{C}$ in $\mathscr{E}$ can be used to model Système F extended by notation for finite products (pairing and a terminal object). Consider $[\mathbf{C}]$ restricted to the fibres over the full subcategory $\mathscr{C}$ of $\mathscr{E}$ on objects $(C_0^n)_{n\in\omega}$. Note that in the case of $\mathbf{P}$ and $\mathscr{E\!f\!f}$, this subcategory is contained in $\mathscr{S\!e\!t}$, while in the case of $\mathbf{Q}$ it is not. We interpret a map $\phi\colon C_0^n \to C_0$ as a type-term in n free (type) variables $\phi(X_1, \ldots, X_n)$. We take these terms as constants, and generate the syntax for F over them. We wish to define recursively the interpretation of the type terms of the theory as follows:

interpret the substitution of one term into another by categorical composition (pullback in the fibration);

the term representing the product of two types which have, or previously have been coerced into having, the same free type variables, by taking their product in the relevant fibre: so

$$[\![\theta(X_1, \ldots, X_n) \times \theta'(X_1, \ldots, X_n)]\!] = [\![\theta]\!] \times [\![\theta']\!]$$

in the fibre $[\mathbf{C}]^{C_0^n}$;

the term representing the function space of two type expressions with the same free type variables similarly, by taking an exponential in the relevant fibre;

and finally the abstraction with respect to the variable X_1 of the term $\theta(X_1, \ldots, X_n)$ as $\Pi_{\pi_{2,\ldots,n}}$ where Π_π is the right adjoint to π^*, which we require to exist.

We regard the instantiation of a polymorphic type at a type as a syntactic operation, and thus interpret it using composition.

The individual terms in the calculus are now interpreted as maps in $[\mathbf{C}]$ in a more or less standard way. Thus a free variable of some type ϕ is interpreted as the identity on the interpretation $[\![\phi]\!]$ of ϕ. Of course ϕ may itself depend on some type variables, $[\![\phi]\!]\colon C_0^n \to C_0$, but this does not really alter things; we have

$$[\![x_\phi]\!]\colon\ C_0^n \longrightarrow C_0 \xrightarrow{\ \mathrm{id}\ } C_1.$$

In terms of the fibration, we need $[\mathbf{C}]_{\mathscr{C}}$ to be cartesian closed in each fibre and the $\alpha^*\colon [\mathbf{C}]^{C_0^m} \to [\mathbf{C}]^{C_0^n}$ to preserve this structure (so Frobenius reciprocity holds). Moreover, $[\mathbf{C}]_{\mathscr{C}}$ should have (strong) indexed products.

Let us consider, however, under what circumstances these necessary properties of the externalisation $[\mathbb{C}]$ imply the corresponding properties of the internal category $\mathbb{C}$. In general, completeness properties are forced to hold for $\mathbb{C}$ only if the syntax of the lambda calculus is strong enough to provide a generic instance, and then they are forced to be strong. For example, at the lowest possible level of the general schema given above, an internal category provides a model for the ordinary (non-polymorphic) typed λ-calculus if its global sections form a cartesian closed category. This of course says nothing about the category being internally a cartesian closed category, still less that the product and exponential structure should be strong. However, even in a weak polymorphic calculus, for example the second-order fragment, we have generic types $X \times Y$ and $Y \to X$ where X and Y are free type variables (both then interpreted as maps $C_0 \times C_0 \to C_0$). We also have the terms $\pi_1(u_{X\times Y})$ and $u_{Y\to X}(v_Y)$, both of type X, giving generic instances of the first projection and evaluation. It follows that $\mathbb{C}$ is forced to carry the strong structure of a cartesian closed category. However $\mathbb{C}$ is not forced to have strong C_0-indexed products, simply because the object $C_0^{C_0}$ is not forced to be in $\mathscr{C}$ (compare, however, the result of Pitts in [15]). Suppose now we wish to model F_ω rather than F. Syntactically F_ω has a generic polymorphic type-forming operator, and so any internal category $\mathbb{C}$ modelling F_ω will be forced to have strong C_0-indexed products (and more). More semantically, in order to model F_ω we consider the restriction of $[\mathbb{C}]$ to $\mathscr{D}$, the cartesian closed subcategory of $\mathscr{E}$ generated by C_0. As before we require that $[\mathbb{C}]_{\mathscr{D}}$ be cartesian closed in each fibre, that the re-indexing functors preserve this structure, and that $[\mathbb{C}]_{\mathscr{D}}$ have (strong) indexed products. There is a generic family of second-order type-forming operators which lives in the fibre of $[\mathbb{C}]$ over $C_0^{C_0} \times C_0$ and is given by the evaluation map ev: $C_0^{C_0} \times C_0 \to C_0$. If we now consider $\Pi_\pi \mathrm{ev}$, where

$$\pi\colon C_0^{C_0} \times C_0 \to C_0^{C_0}$$

is the projection, we obtain an object of $[\mathbb{C}]$ in the fibre over $C_0^{C_0}$, and so a map $C_0^{C_0} \to C_0$. This gives the object part of the internal product functor $\mathbb{C}^{C_0} \to \mathbb{C}$. More generally, by considering $\Pi_\pi \mathrm{ev}$, where now ev maps $C_0^D \times D$ to C_0 and π maps $C_0^D \times D$ to C_0^D, we can show that $\mathbb{C}$ has strong D-indexed products for any $D \in \mathscr{D}$.

The converse of the above is of course trivial. If $\mathbb{C}$ has strong structure, so has its externalisation. The category $\mathbb{P}$ is an example of this. It is strongly complete as an internal category in $\mathscr{S}\!e\!p$, and hence has all the structure needed to give a model for F_ω.

Note that a model arising from a strongly complete internal category in this way has more exactness properties than one might expect—Beck conditions (may be taken to) hold on the nose, rather than just up to isomorphism. For example, let the internal category $\mathbb{C}$ in $\mathscr{E}$ have strong products indexed by some object A. Thus we have an internal functor $\Pi_A\colon \mathbb{C}^A \to \mathbb{C}$ giving the A-indexed products. To check Beck conditions in $[\mathbb{C}]$ for A-indexed products we have to look at pullbacks in $\mathscr{E}$ of the form

$$\begin{array}{ccc} A \times B & \longrightarrow & A \times B' \\ \downarrow & & \downarrow \\ B & \xrightarrow{\beta} & B' \end{array}$$

The square giving the corresponding Beck condition is then the outer rectangle in

$$\begin{array}{ccc} \mathscr{E}(A\times B',\mathbb{C}) & \longrightarrow & \mathscr{E}(A\times B,\mathbb{C}) \\ \downarrow & & \downarrow \\ \mathscr{E}(B',\mathbb{C}^A) & \longrightarrow & \mathscr{E}(B,\mathbb{C}^A) \\ \downarrow & & \downarrow \\ \mathscr{E}(B',\mathbb{C}) & \longrightarrow & \mathscr{E}(B,\mathbb{C}) \end{array}$$

In this, the lower square commutes, since it is $\mathscr{E}(\beta, \Pi)$, and the top square commutes by naturality.

Perhaps it is worth commenting a little further on this. In Pitts's paper [15] the fibrations which model the polymorphic lambda calculus ('$2T\lambda C$-hyperdoctrines') are required to have Π-functors which satisfy Beck conditions on the nose. (We are however somewhat unclear about what precisely Seely [18] demands of his PL-categories in this respect.) Pitts observes that any theory can be modelled in a hyperdoctrine for which Beck conditions are strict, and comments that the condition is necessary for his completeness proof to work. He also remarks that it has the pleasant side-effect that the theory of $2T\lambda C$-hyperdoctrines is algebraic.

We however would like to enter a caveat about the strictness of Beck conditions in a general fibrational model. The requirement that Beck conditions should hold on the nose has been justified on the grounds that it is a manifest property of syntax. This manifest property perhaps becomes less so if examined closely with destructive intent. Intuitively, the Beck conditions say that the interpretation of $\Pi X \,.\, \phi$ is unchanged if we regard ϕ as having more free variables than it actually has. Since pretending that ϕ has extra free variables does not change the marks on the page, the Beck conditions should hold on the nose. This last point is debatable. If we decorate (in a syntax-directed way) expressions with the free variables they are deemed to have, then instead of identity we obtain a pair of combinators which should be inverses, and so the Beck conditions hold only up to isomorphism.

Finally, we note that we can use the fact that $\mathbb{P}$ is not just a complete internal category, but a complete internal *full sub*category of $\mathscr{E}\!\mathit{ff}$ (or at least $\mathscr{S}\!\mathit{ep}$) to obtain a model for the theory of constructions (cf. Coquand and Huet [4]). This theory can be seen as an extension of the full higher-order polymorphic lambda calculus more or less as follows:

1. the category of types (Coquand's and Huet's propositions) is indexed over itself, as well as over the category of orders (Coquand's and Huet's types), and has the corresponding products;
2. orders are not only closed under function space, but can also be indexed over both types and orders, and again are closed under these products.

In the model given by $\mathbb{P}$, types become special orders, and soundness follows from the fact that $\mathbb{P}$ is a locally cartesian closed full subcategory of a locally cartesian closed category, and is closed under all products from the ambient category.

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References

1. J. Bénabou, 'Fibred categories and the foundations of naive category theory', *J. Symbolic Logic* 50 (1985) 10–37.
2. M. Bunge, 'Stacks completion and Morita equivalence for categories in a topos', *Cahiers Topologie Géom. Différentielle* 20 (1979) 404–436.
3. M. Bunge and R. Paré, 'Stacks and equivalence of indexed categories', *Cahiers Topologie Géom. Différentielle* 20 (1979) 373–399.
4. T. Coquand and G. Huet, 'Constructions: a higher order proof system for mechanizing mathematics', *EUROCAL '85: European Conference on Computer Algebra* (ed. B. Buchberger), Lecture Notes in Computer Science 203 (Springer, Berlin, 1985), pp. 151–184.
5. P. J. Freyd, *On canonizing category theory, or on functorializing model theory, a pamphlet,* mimeographed notes, University of Pennsylvania, 1974.
6. P. J. Freyd and G. M. Kelly, 'Categories of continuous functors, I', *J. Pure Appl. Algebra* 2 (1972) 169–191.
7. J. Y. Girard, 'The system *F* of variable types, fifteen years later', *Theoret. Comput. Sci.* 45 (1986) 159–192.
8. J. Giraud, *Cohomologie non abélienne* (Springer, Berlin, 1971).
9. A. Grothendieck, 'Catégories fibrées et descente', Lecture Notes in Mathematics 224 (Springer, Berlin, 1971), pp. 145–194.
10. J. M. E. Hyland, 'The effective topos', *The L. E. J. Brouwer Centenary Symposium* (eds A. S. Troelstra and D. van Dalen, North-Holland, Amsterdam, 1982), pp. 165–216.
11. J. M. E. Hyland, 'A small complete category', *Ann. Pure Appl. Logic* 40 (1988) 135–165.
12. J. M. E. Hyland, P. T. Johnstone, and A. M. Pitts, 'Tripos theory', *Math. Proc. Cambridge Philos Soc.* 88 (1980) 205–232.
13. P. T. Johnstone, *Topos theory* (Academic Press, London, 1977).
14. R. Paré and D. Schumacher, 'Abstract families and the adjoint functor theorem', *Indexed categories and their applications* (eds. P. T. Johnstone and R. Paré), Lecture Notes in Mathematics 661 (Springer, Berlin, 1978), pp. 1–140.
15. A. M. Pitts, 'Polymorphism is set theoretic, constructively', *Proceedings of the conference on Category Theory and Computer Science, Edinburgh 1987* (ed. D. Pitt), Lecture Notes in Computer Science 283 (Springer, Berlin, 1987), pp. 12–39.
16. J. C. Reynolds, 'Polymorphism is not set-theoretic', *Semantics of data types,* Lecture Notes in Computer Science 173 (Springer, Berlin, 1984), pp. 145–156.
17. D. S. Scott, 'Church's thesis and a unification of types', paper presented at the conference 'Church's thesis after fifty years', Zeist, The Netherlands, June 1986.
18. R. A. G. Seely, 'Categorical semantics for higher order polymorphic lambda calculus', *J. Symbolic Logic* 52 (1987) 969–989.

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