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Aspects of constructivity in mathematics

by J.M.E. Hyland

§1 The aim of this paper is to indicate points of contact between the following three topics which fall under the general heading of constructivity in mathematics:

(a) continuity of solutions in parameters;

(b) topological models for intuitionistic analysis;

(c) functional interpretations of analysis.

My interest in these matters arose from two sources.

1) In Kreisel 1959, a classical (or not specifically intuitionistic) notion of constructive result is discussed and specifically related to the notion of continuity in parameters (there thought of as typically real number generators). There was a clear need for a systematic treatment of the way in which constructive proofs give rise to continuity in parameters, though the main tool which is used in this paper to make the connection (viz. topic (b) above) did not begin to emerge until Scott 1968, 1970.

2) In a seminar on sheaves and logic organized by Scott in Oxford 1975-76, some time was spent proving results constructively and interpreting them in sheaf models over topological spaces. It became apparent to me that if one was interested only in truth in the topological models (as opposed to ones over arbitrary complete Heyting algebras), then one could dispense with the, at times, rather complex constructive proofs; the truth in topological models depended on simple considerations of continuity.

The main results of this paper are contained in §§7,8: those related to 2) above in §7 and those related to 1) in §8. §§2-6 contain a variety of preliminary material ideas; it seems likely that some of these may have a rôle to play in other areas, for example in the development of constructive analogues to classical model theory. The overall level of detail in this paper is very low. This will certainly frustrate some, but I did not want the ideas to become obscured by the presentation.

Of the three topics in constructivity mentioned above, the one which receives least explicit discussion is (c). Here there is contact with an issue alluded to by Kreisel in his contribution to this volume: the lack of significant returns for functional interpretations. The inconclusive remarks about Dialectica and modified realizability interpretations at the close of §7 and §8, indicate at least why it is that the classical mathematician has not used functional interpretations to formalize his constructive intuitions. Interesting results can be obtained using less subtle ideas: specifically without using constructive information from premises of implications. It would certainly be of interest to find branches of mathematics where applications of functional interpretations along the lines of §§7,8 were needed.

We apply notions of continuity in parameters to parameters of various types: natural numbers (N), reals (ℝ), continuous maps ℝ to ℝ (ℝ → ℝ), continuous maps from ℝ → ℝ to ℝ, and so on. All this takes place in a suitable cartesian closed category (FlL of Hyland 1977 say). We describe the topological models for these types. However sheaves on topological spaces model much more than finite types; there is a cumulative hierarchy of sheaves which models intuitionistic Zermelo-Fraenkel set theory (IZF) together with Zorn's Lemma (ZL). Then the topological models which we introduce can all be defined in the intuitionistic set theory; in particular ℝ is modelled as the Dedekind (not Cauchy) reals. Rather than introduce any particular constructive theory for the types which we do consider, we take as our basic notion of intuitionistic or constructive proof whatever may be proved in IZF + ZL.

The direct use of ℝ (as opposed to a zero-dimensional space of real number generators) is an innovation not only for topic (c), but also (for logicians) for topic (a). But it certainly coincides with the usual interest of mathematicians.

§2. Some important classes of formulae. The basic language which we are considering in this paper is that of finite type theory over two basic types N and R. Thus we have types for all products and mapping spaces over N and R, together with application, functional abstraction, pairing and unpairing of all appropraite types. In addition we may allow constant function and relation symbols for arbitrary elements of, continuous functions on and open relations on the basic types. Also in accordance with usual mathematical practice, we have maps to allow elements of ℕ to be taken as elements of ℝ. Particular classes of formulae from this language will play an important rôle in this paper.

First we describe two ways of defining new classes of formulae. Let Γ and Δ be classes of formulae. We define $PR(\Gamma;\Delta)$ to be the least class of formulae containing all of Δ, closed under $\vee, \wedge, \forall, \exists$, and such that if $\phi \in \Gamma$ and $\psi \in PR(\Gamma;\Delta)$ then $(\phi \rightarrow \psi)$ is in $PR(\Gamma;\Delta)$. Similarly we define $HPR(\Gamma;\Delta)$ to be the least class of formulae containing all of Δ, closed under $\wedge, \forall$, and such that if $\phi \in \Gamma$ and $\psi \in HPR(\Gamma;\Delta)$, then $(\phi \rightarrow \psi)$ is in $HPR(\Gamma;\Delta)$. In case Δ is the collection of atomic formulae, we write $(H)PR(\Gamma)$ for $(H)PR(\Gamma;\Delta)$. $PR(\Gamma;\Delta)$ is the collection of formulae built up from Δ with <u>p</u>remises <u>r</u>estricted to Γ and $HPR(\Gamma;\Delta)$ the collection of <u>H</u>arrop formulae so built up. These ways of defining collections of formulae have already been used in Troelstra 1973 (§3.6.3).

We define the class COH of <u>coherent formulae</u> to be the closure of the class of atomic formulae under $\wedge, \vee, \exists$. For some applications one can extend this definition: for example if all atomic formulae are decidable, then quantifier-free formulae may be treated as atomic in the definition of COH. There is a precise sense in which COH is the intuitionistic analogue of the existential formulae of classical model theory. We define the class POS of positive formulae to be the closure of the class of atomic formulae under $\wedge, \vee, \forall, \exists$. Again under some circumstances one can extend this definition.

We define STRICT the class of <u>strict formulae</u> to be PR(COH), and LOC the class of <u>local formulae</u> to be HPR(STRICT; COH). The intersection of STRICT and LOC is SLOC, the class of <u>strictly local formulae</u>; SLOC is HPR(COH; COH).

We now discuss an important way of extending the language which we introduced at the beginning of this section. This extension is needed for many of the applications which we give in §7 and §8. If $\phi(x)$ is strictly local with just x free then we can introduce a new <u>type</u> to our language, for $\{x|\phi(x)\}$. This means in particular that universal quantification over $\{x|\phi(x)\}$ is no longer regarded as involving an implication. This essentially expands the various classes of formulae defined above. In particular it expands SLOC; and that allows yet further expansion. It would be tedious to go into detail about all this. We simply stipulate that there be some convention for the <u>introduction of strictly local types</u>.

§3 Sheaf models for intuitionism.

Sheaves over topological spaces generalize both the Kripke and Beth models for intuitionism: they are themselves a special case of sheaves over a site. Let T be a topological space: we use its complete Heyting algebra $\mathcal{O}(T)$ of open sets as truth values. We have the usual definitions of the propositional operators (here $[\![\phi]\!]$ denotes the value of ϕ in $\mathcal{O}(T)$):

<u>Propositional logic</u>

$$[\![\phi \wedge \psi]\!] = [\![\phi]\!] \cap [\![\psi]\!]$$
$$[\![\phi \vee \psi]\!] = [\![\phi]\!] \cup [\![\psi]\!]$$
$$[\![\phi \rightarrow \psi]\!] = \text{Interior}\ ((T \setminus [\![\phi]\!]) \cup [\![\psi]\!])$$
$$[\![\bot]\!] = \text{the empty set.}$$

For ease of exposition we restrict attention to structures of the form

$$X_T = \{x \mid x \text{ maps } U \in \mathcal{O}(T) \text{ continuously into } X\}.$$

X_T is the sheaf of continuous X-valued functions on T. The case when X is $\mathbb{N}$ the natural numbers with the discrete topology is atypical. A more typical case is when X is $\mathbb{R}$ the reals with the usual topology. But X need not be topological: our higher types will be modelled using X's which are non-topological filter spaces (see §5).

Define an <u>e</u>xistence predicate (or predicate of <u>e</u>xtent) E by,

$$[\![Ex]\!] = \text{dom}(x) \quad \text{for all} \quad x \in X_T.$$

The quantifiers $\forall$ and $\exists$, respectively presuppose and imply existence, so we bring E into their definition:

<u>Predicate logic</u>

$$[\![(\forall x)\phi(x)]\!] = \text{Interior}\ (\bigcap\{[\![Ex \rightarrow \phi(x)]\!] \mid x \in X_T\})$$
$$[\![(\exists x)\phi(x)]\!] = \bigcup\{[\![Ex \wedge \phi(x)]\!] \mid x \in X_T\}.$$

The relations and functions on our structures are required to be extensional not just with respect to the ordinary equality = in the sheaf, but with respect to strong equivalence $\equiv$. For us, $[\![x = y]\!] = \text{Interior}\ \{t \mid x(t) = y(t)\}$, while $[\![x \equiv y]\!] = [\![Ex \vee Ey \rightarrow x = y]\!]$; thus $[\![x \equiv y]\!]$ includes the open sets where neither x nor y are defined. Then 1-ary relations $R : X_T \rightarrow \mathcal{O}(T)$ must satisfy

$$R(x) \cap [\![x \equiv y]\!] \subseteq R(y),$$

while a 1-ary function $f : X_T \rightarrow X_T$ must satisfy

$$[\![x \equiv y]\!] \subseteq [\![f(x) \equiv f(y)]\!].$$

Of particular interest for us are relations and functions arising in the following way. Let R be an open set in X^n; then we also denote by R the n-ary relation on X_T defined by,

$$R(x_1, \ldots, x_n) = \{t \mid (x_1(t), \ldots, x_n(t)) \in R\}.$$

Let f be a continuous function from X^n to X; then we also denote by f the n-ary function on X_T determined by the stipulation

$$(f(x_1,\dots,x_n))(t) = f(x_1(t),\dots,x_n(t)).$$

(In the above definitions, on the right hand side we have the original and on the left the defined meaning of the symbols R and f respectively).

We can now begin to describe the interpretation of the basic language of §2 over a topological space T. The basic types $\mathbb{N}$ and $\mathbb{R}$ are interpreted by the structures $\mathbb{N}_T$ and $\mathbb{R}_T$. Then we interpret the function and relation symbols in the way described above. Thus for example

$$[\![x < y]\!] = \{t \mid x(t) < y(t)\}.$$

We call the models we have just introduced the <u>models of sheaves over topological spaces</u>, or the <u>topological models</u>. The reader can now interpret any formula of the basic language which does not involve higher types, in the topological models. For a sentence ϕ, we shall say that ϕ <u>holds over</u> T iff $[\![\phi]\!] = T$.

<u>Remarks</u> 1) A full exposition of the theory of sheaf models sketched above is to appear in the eagerly awaited paper of Scott, Fourman 197?. The reader should also consult the pioneering papers Scott 1968, 1970, though the structures there are not quite sheaves but Ω-sets for $\Omega = \mathcal{O}(T)$.

2) There is a formulation of the above semantics (so-called Kripke-Joyal semantics - a misnomer as the notion of covering is used in the Beth models) closer to the familiar intuitionistic semantics. But the treatment we have sketched is far better for higher order logic.

3) Note that $\mathbb{N}_T$ is the standard model of the natural numbers in the topos of sheaves on T: an arithmetical sentence is true in the model iff it is true. But for analysis the standard structures differ from their classical counterparts.

§4 <u>Truth in fibres and truth in the model</u>

In this section we consider only the language of the intuitionistic first order predicate calculus for our base type $\mathbb{R}$; we allow constants for elements of $\mathbb{R}_T$. The discussion would be trivial for $\mathbb{N}$. For higher types even over $\mathbb{N}$, the discussion becomes interesting, and §5 is devoted to showing how the discussion can be made to go through unchanged for higher types.

Suppose $\phi(\vec{x})$ is closed with the constants $\vec{x} = x_1,\ldots,x_n$ exhibited. We say that $\phi(\vec{x})$ is true (in the fibre) at $t \in T$ iff $\phi(x_1(t),\ldots,x_n(t))$ (henceforth written $\phi(\vec{x}(t))$) is true. (Note the convention used without comment in §3, that if $\phi(\vec{x}(t))$ is true then $t \in [\![E\vec{x}]\!] = [\![Ex_1 \wedge \ldots \wedge Ex_n]\!]$). The relation between $[\![E\vec{x} \wedge \phi(\vec{x})]\!]$ and $\{t | \phi(\vec{x}(t))\}$ can be complex. But the following theorem is easy to establish (refer to §2 for definitions).

Theorem 4.1. Let $\phi(\vec{x})$ be closed with constants exhibited in the language for $\mathbb{R}$.

Then (a) if ϕ is coherent, $[\![E\vec{x} \wedge \phi(\vec{x})]\!] = \{t | \phi(\vec{x}(t))\}$;

(b) if ϕ is strict, $[\![E\vec{x} \wedge \phi(\vec{x})]\!] \subseteq \{t | \phi(\vec{x}(t))\}$;

(c) if ϕ is local, $[\![E\vec{x} \wedge \phi(\vec{x})]\!] \supseteq \text{Interior}(\{t | \phi(\vec{x}(t))\})$;

(d) if ϕ is strictly local, $[\![E\vec{x} \wedge \phi(\vec{x})]\!] = \text{Interior}(\{t | \phi(\vec{x}(t))\})$.

Proof:- By a routine induction on the definition of coherent, strict, local and strictly local formulae. (Needless to say, the same proof goes through for any X in place of $\mathbb{R}$ so long as the atomic formulae behave as the coherent ones do in (a) above.

(4.1)(d) shows that strictly local types $\{x | \phi(x)\}$ are interpreted as the sheaf of continuous maps from T to $\{x | \phi(x)\}$. Thus (4.1) goes through even with the convention for the introduction of strictly local types. This will again be true of results which use (4.1) and we will not always comment on this.

It follws from (4.1)(c) that if a local formula is locally true in the fibres, then it is locally true in the sheaf model. In particular if a local sentence is true, it is valid in all sheaf models over topological spaces. Not many interesting formulae are local so this result is rather uninspiring. (It should not be underestimated however: for example the principle that for any Dedekind real it is not the case that it is apart from every Cauchy real, is local when properly formulated in the higher types. However (as observed by Fourman) it is provable in the internal logic). Fortunately constructive interpretations of formulae are much more likely to be local than the formulae themselves, and this fact can be exploited to give interesting results.

§5 Models for the higher types

In this section we give a uniform definition of sheaves at all higher types over our basic types $\mathbb{N}$ and $\mathbb{R}$. We do this in the first place because we wish to consider applications to higher type parameters. Of secondary importance is the fact that the Dialectica and

and modified realizability interpretations involve higher types even for simple formulae; we wish to make sense of these interpretations in the sheaf models even though we do not seem to be able to make much use of them.

The main problem which we solve here is that of ensuring that the considerations of §4 go over to the higher types. In the first place this means that we must take as sheaves at higher types, sheaves of maps from (open sets in) T to suitable spaces of higher types. To see what these should in general be, consider a simple example. Let f be in our (still to be defined) sheaf of type $\mathbb{R} \to \mathbb{R}$, x and y of type $\mathbb{R}$ (i.e. members of $\mathbb{R}_T$) and suppose that all of f, x and y have their full extent. Now we know that

$$[\![f(x) < y]\!] = \{t | (f(x))(t) < y(t)\},$$

and we want that to be equal to

$$\{t | (f(t))(x(t)) < y(t)\}.$$

Thus f must map T to $\mathbb{R}^{\mathbb{R}}$, and for x in $\mathbb{R}_T$, f(x) is defined by

$$(f(x))(t) = (f(t))(x(t)).$$

But f cannot be an arbitrary map from T to $\mathbb{R}^{\mathbb{R}}$, as f(x) is required to be in $\mathbb{R}_T$, that is to say f(x) must be continuous. The obvious way to ensure this is to insist that $\lambda t.f(t)$ and application (or the evaluation map) be continuous. In general, though not in this particular example, this will take us outside the category TOP of topological spaces; we need to consider a cartesian closed category in which TOP embeds full and faithfully. Many such categories are known; the convergence spaces of Choquet, limit spaces or more generally filter spaces FIL (Hyland 1977). It makes no difference here which the reader chooses to consider.

Our sheaves at higher types over $\mathbb{N}$ and $\mathbb{R}$ are defined as follows: we take sheaf of continuous (in the sense of FIL) maps from T to be the space of appropriate type over $\mathbb{N}$ and $\mathbb{R}$ in the category FIL. This ensures that all the considerations of §4 go through unchanged when higher types are introduced.

Remarks 1) We are interpreting the finite types over $\mathbb{N}$ by sheaves of continuous (in the sense of FIL) maps into the continuous functionals of Kleene 1959 and Kreisel 1959. For an account of the relation between these original treatments and FIL see Hyland 1977. Occasionally hereafter we use the phrase continuous functionals to refer to the higher types in FIL over both $\mathbb{N}$ and $\mathbb{R}$.

2) Though we don't need this fact, it is amusing to note that the spaces we are defining externally at higher types are also the spaces internally defined using FIL, using the intuitionistic set theory (IZP) valid in our topological models.

§6 Functional Interpretations

Both the modified realizability and Dialectica interpretations (ϕ^{MR} and ϕ^{D} respectively) of a formula ϕ can be regarded as being derived from a crude constructive interpretation (ϕ^{C}) by simply varying the treatment of implication in the inductive definition. All these interpretations can be given in the finite type structure (of total objects) over the basic types: for us the basic types are N and R, the types of natural and real numbers respectively. Let I be an arbitrary one of our interpretations C, MR, D; and assume conventionally that ϕ^I is $(\exists\vec{x})(\forall\vec{y})\phi_I(\vec{x},\vec{y})$, and ψ^I is $(\exists\vec{s})(\forall\vec{t})\psi_I(\vec{s},\vec{t})$, where $\vec{x},\vec{y},\vec{s},\vec{t}$ are strings of variables. Then the interpretations are defined by the following inductive clauses:

(i) if ϕ is atomic ϕ^I is ϕ,

(ii) $(\phi \wedge \psi)^I$ is $(\exists\vec{x})(\exists\vec{s})(\forall\vec{y})(\forall\vec{t})(\phi_I \wedge \psi_I)$,

(iii) $(\phi \vee \psi)^I$ is $(\exists n)(\exists\vec{x})(\exists\vec{s})(\forall\vec{y})(\forall\vec{t})(n{=}0 \to \phi_I \wedge n{\neq}0 \to \psi_I)$,
or $(\exists r)(\exists\vec{x})(\exists\vec{s})(\forall\vec{y})(\forall\vec{t})(r{>}0 \to \phi_I \wedge r{<}1 \to \psi_I)$,

(iv) $((\forall z)\phi)^I$ is $(\exists\vec{X})(\forall z)(\forall\vec{y})\phi_I(\vec{X}(z),\vec{y})$,

(v) $((\exists z)\phi)^I$ is $(\exists z)(\exists\vec{x})(\forall\vec{y})\phi_I(\vec{x},\vec{y})$,

(vi) (a) $I = C$:
$(\phi \to \psi)^{C}$ is $(\exists\vec{s})(\forall\vec{t})(\phi \to \psi_C)$,
(b) $I = MR$:
$(\phi \to \psi)^{MR}$ is $(\exists\vec{S})(\forall\vec{x})(\forall\vec{t})((\forall\vec{y})\phi_{MR} \to \psi_{MR}(\vec{S}(\vec{x}),\vec{t}))$,
(c) $I = D$:
$(\phi \to \psi)^{D}$ is $(\exists\vec{S})(\exists\vec{Y})(\forall\vec{x})(\forall\vec{t})(\phi_D(\vec{x},\vec{Y}(\vec{x},\vec{t})) \to \psi_D(\vec{S}(\vec{x}),\vec{t}))$.

Throughout the above, variables are supposed to be sensibly typed. In (iii), n is of type $\mathbb{N}$ and r of type $\mathbb{R}$; the interpretation simply uses the definability of $\vee$ with respect to elementary intuitionistic theories of natural or real numbers. $\vec{X}(z)$, $\vec{S}(\vec{x})$, $\vec{Y}(\vec{x},\vec{t})$ are interpreted in the obvious way; e.g. $\vec{S}(\vec{x})$ stands for some sequence $S_1(\vec{x}),\ldots,S_n(\vec{x})$. Finally (vi)(b) is an intuitionistically equivalent variant on the usual formulation of MR; it brings out the analogy with the other two interpretations.

Giving an inductive definition of ϕ^C is rather artificial; essentially it can be obtained using the notion of the strictly positive parts (s.p.p.'s) of a formula as follows:

(i) replace v's in the s.p.p.'s of the formula by a definition using $\forall$, $\wedge$, $\rightarrow$;

(ii) systematically move all quantifiers acting on the s.p.p.'s of the formula to the front;

(iii) bring to the required form by replacing "$(\forall x)(\exists y)(\ldots x,y\ldots)$" by "$(\exists Y)(\forall x)(\ldots x,Y(x)\ldots)$" (i.e. using Skolem functions). This brings out the fact that ϕ^C does not make much use of higher types; if ϕ^C is $(\exists\vec{x})(\forall\vec{y})\phi_C$ then the maximum level (in the usual sense) of the types of $\vec{x}$ is at most one greater than the maximum level of the types appearing in ϕ.

We close this section by giving some information about the relation between, ϕ, ϕ^C, ϕ^{MR} and ϕ^D. We define classes Γ and Δ of the formulae to be the least classes such that (i) $\Gamma \supseteq PR(\Delta)$ and (ii) $\Delta \supseteq HPR(\Gamma)$. In the following theorem $\vdash$ denotes derivation in a system of intuitionistic logic, which can be much weaker than the consequences in our basic language, of IZF.

Theorem 6.1. (a) For all ϕ, $\vdash \phi^C \rightarrow \phi$.
(b) If $\phi \in \Gamma$, then $\vdash \phi^{MR} \rightarrow \phi^C$ and $\vdash \phi^{MR} \rightarrow \phi$.
(c) If $\phi \in PR(HPR(COH))$, then $\vdash \phi^D \rightarrow \phi^C$ and $\vdash \phi^D \rightarrow \phi$.

Remark Some of (6.1) is a simple extension of Troelstra 1973 (see his §3.6.5).

§7 From continuity in parameters to truth in topological models

In this section we consider various notions of continuity in parameters, and use them to establish that certain propositions hold in all topological models. The crudest notion of continuity in parameters used, depends on the interpretation ϕ^C introduced in §6. We consider briefly why as yet we have not found a use for ϕ^{MR} (in a case where it differs from ϕ^C) to establish results about topological models.

The possibility of applying interpretations to the study of sheaf models arises out of the idea that the interpretation ϕ^I or $(\exists\vec{x})(\forall\vec{y})\phi_I$ expresses more explicitly the constructive content of ϕ. The result of this is that ϕ_I is much more likely to be local than ϕ is.

Lemma 7.1. (a) If $\phi \in PR(STRICT)$, then ϕ_C is local.
(b) If $\phi \in PR(PR(POS))$, then ϕ_{MR} and ϕ_D are intuitionistically equivalent to local formulae.

Proof:- Straightforward. For (b), ϕ_{MR} and ϕ_D need not be local; but they can be made so by replacing some v's which have been defined away by the interpretation.

Given a sentence ϕ, where ϕ^C is $(\exists \vec{x})(\forall \vec{y})\phi_C$, the string $\vec{y}$ is the string of parameters (or more exactly positive parameters) of ϕ. These are the parameters arising in our various notions of continuity in parameters.

We discuss our crudest notion first. We say that ϕ holds with global continuity in parameters iff ϕ^C is true when the higher types are interpreted as spaces of appropriate type in FIL (see §5). This corresponds to saying not only that ϕ is true but that in transforming ϕ to ϕ^C (as described in §6), the Skolem functions needed can be chosen continuous.

Proposition 7.2. If a sentence ϕ holds with global continuity in parameters, and $\phi \in$ PR(STRICT) then ϕ holds in all sheaf models over topological spaces.

Proof:- By assumption $(\forall \vec{y})\phi_C(\vec{a},\vec{y})$ is true for a fixed choice of continuous $\vec{a}$, and so by (4.1) and (7.1), it holds over T (where now $\vec{a}$ represents the sequence of constant maps from T with values $\vec{a}$). Thus ϕ^C holds over T and since $\phi^C \to \phi$ (by (6.1)), ϕ holds over T.

It is an immediate result of (7.1) that suitable formulations of the following are valid in sheaves over any topological space:

(i) every continuous function has a least upper bound and is uniformly continuous on closed intervals;

(ii) the fan theorem (expressing compactness of Cantor space).

The above examples express pure compactness phenomena in analysis, and can be extended to many others (in general one will need to make heavy use of the convention described in §2 for the introduction of strictly local types). It appears that such propositions cannot be proved in IZF together with Zorn's Lemma. Realizability interpretations do not seem to have been extended to such strong systems, but I believe that I have a complete Heyting algebra over which the fan theorem fails.

One rather obvious defect of the notion of global continuity in parameters, is that (even with the use of the convention concerning the use of definable strictly local types), we are asking for continuity over an unnecessarily wide range of the parameters. Given a sentence ϕ, we can (up to trivial intuitionistic equivalence) take ϕ_C to be a conjunction of the form

$$(\psi_1 \to \chi_1)\wedge\dots\dots\dots\wedge(\psi_n \to \chi_n),$$

where the χ_i are atomic. Then we can give a definition of the parameter space of ϕ as

$$\{\vec{y}\mid \text{for some } i,\ 1\le i\le n,\ (\exists\vec{x})\psi_i(\vec{x},\vec{y})\},$$

where as usual ϕ^C is $(\exists\vec{x})(\forall\vec{y})\phi_C$. If ϕ is in PR(STRICT), then the ψ_i are strict. Hence using the property (4.1)(b) of strict, one can readily see that to make (7.1) go through, it is sufficient that the Skolem functions $\vec{x}$ be defined (and continuous) on enough of their domain to ensure that $\phi_C(\vec{x},\vec{y})$ is true (and so in particular has a truth value) for all $\vec{y}$ in the parameter space for ϕ.

What we have just sketched is a notion of global continuity in the parameter space. We do not pursue it further but turn to our weakest notion local continuity in parameters. To show the need for this notion consider the sentence expressing the existence of a root for the cubic x^3-3x-y,

$$(\forall y)(\exists x)(x^3-3x = y).$$

Obviously there ia no total continuous function giving x in terms of y. But for any y there will be an open neighbourhood U of y and a continuous function giving a root x(z) for each z in U. So the sentence does hold with local continuity but not global continuity in parameters. Another simple but illuminating example of a sentence of this kind is

$$(\forall x)(x > 0 \to (\exists n)[nx > 1]),$$

where x is of type $\mathbb{R}$ and n is of type $\mathbb{N}$.

First we define $\phi(\vec{x})$ is (locally) continuous in parameters from $A \subset \mathbb{R}^n$, where all free variables in ϕ are indicated, and the length of $\vec{x}$ is equal to n (for the induction to be smooth we must allow dummy free variables - the restriction to real parameters is one of convenience):

(i) if ϕ is atomic, $\phi(\vec{x})$ is continuous in parameters from A iff A is included in $\{\vec{x}\mid\phi(\vec{x})\}$;

(ii) $\phi\wedge\psi$ is continuous in parameters from A iff both ϕ and ψ are continuous in parameters from A;

(iii) $(\forall x)\phi(x,\vec{y})$ is continuous in parameters from A iff $\phi(x,\vec{y})$ is continuous in parameters from $\mathbb{R}\times A$;

(iv) $\phi(\vec{x}) \to \psi(\vec{x})$ is continuous in parameters from A iff ψ is continuous in parameters from $A\cap\{\vec{x}\mid\phi(\vec{x})\}$;

(v) $\phi\vee\psi$ is continuous in parameters from A iff there are relatively open B,C such that $A = B\cup C$ and ϕ,ψ are continuous in parameters from B, respectively C;

(vi) $(\exists x)\phi(x,\vec{y})$ is continuous in parameters from A iff there are continuous maps $f_i:A_i \to R$ on relatively open A_i covering A such that for each i $\phi(f_i(\vec{y}),\vec{y})$ is continuous in parameters in A_i.

If n sentence ϕ is (locally) continuous in parameters from $\mathbb{R}^0$ (i.e. the one point space), then we say that ϕ is <u>continuous in parameters</u>. This is our notion of local continuity in parameters - but we drop the "local".

Unfortunately the above definition is cumbersome. Despite this I claim that it does represent the natural notion of (local) continuity in parameters. What is more, if a sentence is sufficiently simple for one to be able to read through and understand it, then with little further effort, one can read through and understand what it is for it to be continuous in parameters.

Clearly if ϕ holds with global continuity in parameters, then ϕ is continuous in parameters. Further, we can extend (7.2).

<u>Proposition 7.3</u>. If a sentence ϕ is continuous in parameters and $\phi \in$ PR(STRICT), then ϕ holds in all sheaf models over topological spaces.

Proof:- Use induction on the definition of continuity in parameters.

(7.3) allows us to improve the formulation of the compactness properties which follow from (7.1). One can use it to show many other things for example that an appropriate formulation of Dini's Theorem holds in all topological models. One can also use (7.3) to analyxe the general question of the existence of solutions to odd degree polynomials in one variable, in the topological models. One cannot find even a locally continuous solution in the neighbourhood of a point in the parameter space which gives rise to repeated roots; so by (8.2) in general such polynomials are not soluble in the topological models. But it is possible to write down a coherent fornula in terms of the coefficients of a polynomial, which expresses the fact that the polynomial has at least one non-repeated real root. Thus there is a non-trivial formula which expresses that separable polynomials of odd degree have roots; this does hold for all topological models in virtue of (7.3).

We close this section by considering the possibility of using more usual functional interpretations to establish facts about the sheaf models. The following proposition is an analogue of (7.2).

Proposition 7.4. (a) If the sentence ϕ is in PR(HPR(POS)) (which is the intersection of Γ of §6 and PR(PR(POS))) and ϕ^{MR} holds (i.e. classically over the continuous functionals) then ϕ holds in all sheaf models over topological spaces.

(b) If the sentence ϕ is in PR(HPR(COH)) (and so in PR(PR(POS))) and ϕ^{D} holds (i.e. classically over the continuous functionals) then ϕ holds in all sheaf models over topological spaces.

Proof:- As for (7.2) using (6.1) and (7.1).

Let me say first that I know of no example interesting or otherwise where (7.4) can be used to show that a proposition ϕ of analysis holds in the topological models, and where this could not be done by applying (7.2) or (7.3) to some ψ which trivially implies ϕ. In fact for the Dialectica interpretation no such example could exist using (7.4)(b) in its present form: one would need significantly to strengthen the result by weakening the hypotheses. However the present position is not so hopeless for modified realizability, and it seems reasonable to raise the following open problems.

1) If ϕ is in Γ, then by (6.1) if ϕ^{MR} holds so does ϕ^{C}; are there propositions ϕ such that ϕ^{MR} implies ϕ with intuitionistic logic, but ϕ^{MR} does not imply ϕ^{C} classically over the continuous functionals?

2) Even if the answer to 1) is negative, there is still a difference between the apparent range of applicability of (7.2) and (7.4)(a): PR(HPR(POS)) neither includes nor is included in PR(STRICT). Are there any interesting formulae in PR(HPR(POS)) not in PR(STRICT)?

§8 From truth in topological models to continuity in parameters

In this section we describe the most significant aspect of this work for classical mathematics: a simple way of obtaining continuity in parameters as a direct result of the constructivity of proofs. As we shall indicate, for strict formulae, truth over the parameter space (see §7) implies continuity in parameters; in fact by (7.3) it amounts to the same thing. Of course for different propositions the parameter space will be different. However, if a proposition has a constructive proof (see §1) then it is valid over any topological space, in particular over the parameter space; if in addition it is strict, it will thus be continuous in parameters. If, as I claim, what it means for a proposition to be continuous in parameters can be easily read of from the proposition, then the classical (or otherwise) mathematician has information immediately available to him arising out of the constructive nature of his proof. Particular features of a problem may enable him to improve this information to obtain some form of global continuity in parameters.

In order to formulate the basic proposition of this section we define the parameter space of a formula $\phi(\vec{z})$ with free variables $\vec{z}$ and with bound parameters $\vec{y}$ (i.e. where ϕ^C is $(\exists\vec{x})(\forall\vec{y})\phi_C(\vec{x},\vec{y},\vec{z})$). We already described this for sentences ϕ in §7, and in general we may take the parameter space of $\phi(\vec{z})$ to be that of the universal closure of $\phi(\vec{z})$. This parameter space can be defined inductively on the structure of ϕ, in an obvious way; but the details are messy and we omit them here.

Proposition 8.1. Let $\phi(\vec{z})$ be in STRICT, and let A be an open subset of the range of the parameters $\vec{z}$; suppose that over the parameter space of ϕ, $[\![\phi(\vec{z})]\!] \subseteq \{t|\vec{z}(t)\in A\}$; then ϕ is continuous with parameters in A.

Indication of proof:- By induction on the structure of ϕ. The only real interest is in the steps for $\vee$ and $\exists$ where use can be made of "generic" free parameters, and the homogeneity of the parameter space.

Our main interest is in the immediate corollary:

Proposition 8.2. If a sentence ϕ is strict and ϕ holds over its parameter space, then ϕ is continuous in parameters.

Remark. As the reader will realize in (8.1) and (8.2) we are treating the parameter space not just as a set but as a topological space. For higher type parameters of level two and above, this space will not immediately have the structure of a topological space; it will be a more general filter space. So to make sense of (8.1) and (8.2) one has to take the induced topology or what here amounts to the same thing, regard the continuous functionals as being in the category of sequential spaces (see Hyland 1977). As a result there are some subtle points in the proofs for higher type parameters, but we are not going into details about that here.

More interesting than a general proof of (8.1), is the consideration of a special case. Let ϕ and ψ be coherent and consider $(\forall x)(\exists y)(\phi(x,y) \to \psi(x,y))$. For this formula, the parameter space X is the space over which x ranges. The generic element $\bar{x}$ of X is the identity on X (considered as an element of X_X). If the formula considered holds over X, then so does $(\exists y)(\phi(\bar{x},y) \rightarrow \psi(\bar{x},y))$. Thus X is covered by sets U and elements y_U of Y_X (where the variable y ranges over Y) with extent U such that $\phi(\bar{x},y_U) \rightarrow \psi(\bar{x},y_U$ holds over U. But $y_U:U \rightarrow Y$ is then such that $\phi(x,y_U x)) \rightarrow \psi(x,y_U(x))$ is true for all x in U (since $\phi \rightarrow \psi$ is strict). If this information is unravelled it amounts to the fact that $(\forall x)(\exists y)(\phi(x,y) \rightarrow \psi(x,y))$ is continuous in parameters.

The way (8.2) is applied is expressed in the following proposition.

Proposition 8.3. If the sentence ϕ is strict and has a constructive proof (possibly using special axioms valid over the parameter space), then ϕ is continuous in parameters.

Proof:- By (8.2) and the remark in §1 that IZF+ZL holds over any topological space (and constitutes our notion of constructive proof).

Remark. Among the special axioms which can often be used are bar induction (which is valid over all complete metric and compact Hausdorff spaces) and the axiom of choice from numbers to sets (valid if parameters are restricted to the higher types over N).

(8.3) has immediate application to differential and integral equations. This is because

(i) such statements as "g is the derivative of f" can be expressed by strict formulae, and so statements that particular equations are soluble turn out to be strict, and

(ii) a large body of elementary work on differential and integral equations (the use of contraction mappings and Arzela's theorem for example) is constructive. So continuity of solutions to differential and integral equations in parameters, is an immediate consequence of the constructive way in which the existence of solutions is established. Of course where we have unique existence of a solution, we get global continuity.

One advantage of the method sketched above, over possible realizability methods, seems worth mentioning. In the constructive proofs considered, propositions (about sets of reals for example) for which we have no notion of continuity in parameters may occur. But this does not matter at all. In contrast with realizability, we do not need to establish continuity in parameters at each stage in a proof.

It is clearly not possible to use the modified realizability of Dialectica interpretations to establish continuity of real parameters. Too many propositions which we will meet in the course of proofs have only local continuity in parameters, and modified realizability and Dialectica interpretations use total not partial realizing objects. (This is not true for the higher types over ℕ, but we do not discuss the possibility for them here). An appropriate form of realizability using partial realizing objects needs to be developed over ℝ, before a proper general comparison between the traditional ideas of interpretations and the sheaf theoretic methods of this paper, can be made.

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