APPLICATIONS OF CONSTRUCTIVITY

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This paper contains a sketch of some proof-theoretical results concerning constructive mathematics and indications how results may be used both to understand results in pure mathematics and (more optimistically) as a guide in discovering physically significant results. The proof-theoretic results are concerned with notions of (local) continuity in parameters. Interest in such questions goes back at least as far as *Hadamard's principle*: in order that differential equations be physically meaningful, they should have solutions uniquely determined by the initial and boundary conditions and should be stable (small changes in the data produce only small changes in solutions). Of course, as has been popularized by Catastrophe Theory, there are many instabilities in the real world; so no crude interpretation of Hadamard's principle is plausible (or useful).

The claims that underly this paper are

(i) that for all practical purposes questions about continuity in parameters are questions about the constructivity of arguments, and

(ii) that constructivity may operate as a useful heuristic principle in the application of mathematics.

As regards (i), what we give are results deriving continuity from constructivity, and simple examples of this connection. It is inevitably difficult to *show* that "natural" continuity results will always be obtainable by constructive arguments, though results from HYLAND (1977) can be used to give some plausibility to this. In Section 3, I suggest a test case for claim (i). As regards (ii), the hope is to make use of experience of what are good constructive definitions, in particular in the context of topos theory. Clearly considerable work will be needed to realize this hope.

The idea that there is some connection between constructivity and continuity is an old one (Brouwer's Theorem that all functions from reals to reals are continuous). But until recently, there was no presentation of useful formal results. HYLAND (1977) contained model-theoretic results giving connections both ways between constructivity and continuity. Independently, BEESON (1977) produced general proof-theoretic results connecting constructivity with his notion of stability. This paper merges Beeson's work with the fundamental distinction between systems with and without choice principles alluded to in HYLAND (1977). No attempt is made here to be comprehensive; the sole aim is to give the flavour of the area.

1. Formal systems

The proof-theoretic results with which we shall be concerned go through particularly smoothly for a basic system of intuitionistic type theory (with extensionality), or equivalently (see for example FOURMAN, 1977) in the context of topos theory (for which see JOHNSTONE, 1977). However, the reader may well prefer something more down to earth, so we consider a system with just two levels of higher types. Specifically our system is based on intuitionistic predicate logic, and has

(a) *types* closed under pairing, with basic type N for the natural numbers, and with two levels of power types (so e.g. $P(N)$ and $P(N\times P(N))$);

(b) *term forming operations* of application, pairing and unpairing among the types;

(c) *axioms*, full second order "Peano" axiomatization for N, extensionality and full comprehension.

(For ease of expression of ordinary mathematics, comprehension may be taken to include the introduction of definable subtypes; indeed, one would naturally make many conservative extensions of this system, allowing for example the direct formation of function spaces $A\to B$ and power types $P(A)$, restricting A to be of level 0 or 1.) In this system we will have definable types $\boldsymbol{R}$ for the Dedekind reals (as in FOURMAN and HYLAND, forthcoming, or JOHNSTONE, 1977), $\boldsymbol{R}\to\boldsymbol{R}$ for functions from reals to reals, $\mathrm{Cts}(\boldsymbol{R},\boldsymbol{R})$ for continuous such functions and so on. If we needed higher levels of types, we could add them.

For the purpose of formalizing mathematical practice, the basic system may be augmented in two distinct ways.

(1) We may add an axiom stating the compactness of Cantor space 2^N (i.e. the intuitionist's Fan Theorem). From this axiom we readily obtain

the compactness of the unit interval, the uniform continuity, and hence integrability of continuous functions from the unit interval to R, and also general ways of transforming local into global properties. It is important that we do not use the Cauchy reals in this system: they are not complete with respect to the usual uniformity. Monotonic bar induction could be added (though not to much effect), but *axioms of* (*dependent or countable*) *choice are excluded.* Let S denote this system, *the system without choice principles.*

(2) We may add to S the axiom DC of dependent choice. Then (see FOURMAN and HYLAND, forthcoming) the Cauchy and Dedekind reals may be identified. Approximation arguments involving choice become available and a theory of Lebesgue integration developed more or less along traditional lines (see BISHOP and CHENG, 1972). Let S^* denote this system, the *system with choice principles.*

The systems S and S^* are subsystems of systems formalizing classical mathematics. They are based on no philosophical analysis, and are simply designed to stay as close as possible to usual mathematical practice. But it seems worth commenting on the relation between S, S^* and other constructive approaches.

(A) Forget his remarks hinting at an intensional interpretation, and Bishop's mathematics (see BISHOP, 1967, or BRIDGES, 1979) can be formalized straightforwardly in S^*. So clearly there are connections with extensional systems designed for this (e.g. that of FRIEDMAN, 1977). Our set-theoretical apparatus has been restricted for simplicity.

(B) If from the system of KLEENE and VESLEY (1965), one drops the axiom of *continuous* choice (§ 7 of Chapter 1) which is at variance with classical mathematics, one obtains a subsystem of $S+$bar induction. Most of the development in KLEENE and VESLEY (1965) depends only on the fan theorem and numerical choices, and so can be formalized in S^*.

(C) KLEENE and VESLEY (1965) is similar in spirit to our systems in that there is no consideration of different kinds of sequences. There is naturally a much greater difference between our systems and the theory of choice sequences as exposed in KREISEL and TROELSTRA (1970). The difference can best be indicated by observing that the axiom of *bar continuity* may be analyzed as an amalgum of principles of *choice*, *continuity*, *effectivity* (using lawlike sequences) and *bar induction* (in that continuous functionals are in the inductively defined set K of KREISEL and TROELSTRA, 1970). Of this all we have is the weaker Fan Theorem in S and weaker choice principles in S^*.

(D) Without compactness, S^* would be a system consistent with Church's Thesis, and would formalize effective analysis (where all objects are effectively given), but we do not consider that here.

2. Continuous dependence and stability

The results given below extend to the general situation considered in BEESON (1977), namely dependence of values from a separable metric space on parameters from a complete separable metric space, and to other cases. But we restrict to the case of real values and parameters.

DEFINITION. Suppose $\forall x \in R.\, \exists y \in R.\, \Phi(x, y)$. We say that *locally y depends continuously on x* (i.e. can be chosen continuously) if and only if

$$\forall x \in R.\ \exists \text{ neighbourhood } N_x \text{ of } x.\ \exists f \in \mathrm{Cts}(N_x, R).\ \forall x' \in N_x.\ \Phi(x', f(x')).$$

We say (following BEESON, 1977) that *y is stable in x* (i.e. can be chosen stably) if and only if

$$\forall x \in R.\ \exists y \in R(\Phi(x, y) \wedge \forall \text{ neighbourhood } N_y \text{ of } y.\ \exists \text{ neighbourhood } N_x \text{ of } x.\ \forall x' \in N_x.\ \exists y' \in N_y.\ \Phi(x', y')).$$

Remarks. 1. Clearly, if globally y depends continuously on x, it does so locally, and if it does so locally, then y is stable in x. If y is uniquely determined by x, then the converse implications hold, but in general they do not (see Examples (1) and (2) below).

2. A solution locally continuous in parameters clearly satisfies the continuity intentions behind Hadamard's principle. It may be too strong, but stability appears to be too weak (see Example (4), though this is not a good physical example).

THEOREM. (i) (HAYASHI; HYLAND) *If* $\forall x \in R.\, \exists y \in R.\, \Phi(x, y)$ *is provable in S, then locally y is continuous in x.*

(ii) (BEESON) *If* $\forall x \in R.\ \exists y \in R.\ \Phi(x, y)$ *is provable in* S^*, *(and if* $\forall x \in R.\ \{y|\ \Phi(x, y)\}$ *is closed is provable in* S^**), then y is stable in x.*

A proof of (i) for the basic system (i.e. S without compactness of 2^N) is in HAYASHI (preprint). I hope to publish my independent (but later) proof based on category-theoretic ideas. (ii) is in BEESON (1977); I doubt whether the condition in brackets is really essential.

3. Applications

To facilitate appreciation of the result above, I first give some

Examples from elementary mathematics.

(1) *Distinction between global and local continuity.* Consider the solution of the cubic equation

$$x^3 - 3x = a\,,$$

in reals for real parameter a. A diagram convinces one that there is no globally continuous solution, but that there are locally continuous ones. The equation can be proved to have a solution in S (split initially into cases: $a > -1/2 \vee a < 1/2$), so local continuity is a consequence of (i) of our Theorem.

(2) *Distinction between local continuity and stability.* Consider the solution of $z^2 = c$ in complex numbers for complex parameter c. There is no continuous solution of this equation in any neighbourhood of 0 (look at arg z—there is a homotopy obstruction)! Hence by part (i) of our Theorem, we cannot show the existence of a solution in S. *A fortiori*, the fundamental theorem of algebra is not provable in S. However, a constructive version is provable in S^* (as in BISHOP and CHENG, 1972), so in accordance with part (ii) of the Theorem, z can be chosen stably in c.

Warning: (1) and (2) should not be confused. (1) is frequently quoted to show that arbitrary choice principles cannot be used constructively with extensionality, and for this (2) would do just as well. But there is an important difference. Analogous to (2) is the solution of $x^3 + ax + b = 0$ in reals for real parameters a and b. The solutions form the fold and a solution cannot be chosen continuously in any neighbourhood of $a = b = 0$.

OPEN PROBLEM. One can express in a good constructive way (i.e. by a coherent formula) a condition $\mathrm{Sep}(f)$ on the degree n polynomial f, classically (and indeed in every Grothendieck topos) equivalent to the existence of at least one simple root of f. The schema

$$\mathrm{Sep}(f) \to \exists z.\ f(z) = 0\,,$$

expresses the separable closure of the complex numbers. There is no known proof of this in our basic system or in S. Since the simple root is certainly locally continuous in the coefficients, this seems a good *test case* for the claim that natural continuity results can always be obtained by considerations of constructivity.

(3) Using a simple extension of our main theorem, the difference between S and S^* can be detected in the proof in S^* that every Dedekind real is Cauchy! This gives for each element of R a sequence of rationals (element of $N \to Q$) converging to it. But there are no non-trivial continuous maps from open sets in R to $N \to Q$. Here $N \to Q$ may have the product topology with Q given *either* the subspace topology induced from R, *or* the discrete topology (so $N \to Q$ is isomorphic to Baire space). In either case, we will have stability on the basis of (ii) of the Theorem.

(4) *Computational but non-physical significance of stability.* Cantor's theorem on the uncountability of R formulated constructively states

$$\forall f \in N \to R.\ \exists x \in R.\ \forall n.\ x \neq f(n),$$

where $\neq$ is the intuitionistic apartness on R. x cannot be chosen (locally) continuously in f (an amusing exercise), so Cantor's theorem cannot be proved in S. But it can easily be proved in S^*, and indeed the stability of x is a triviality. It is hard (for me) to imagine stability of this kind being physically significant.

Next still at the very simplest level, I give the obvious

Application to differential equations. If a differential equation

$$\frac{dy}{dx} = f(x, y)$$

is such that f satisfies a Lipschitz condition on y, then for any a and b, there is an interval I containing a and a (continuous and) differentiable function $g: I \to R$ with $g(a) = b$ and satisfying the differential equation throughout I.

The standard argument for this may readily be formalized in S. So by a suitable generalization of our Theorem, I and g may be found continuously in f, a Lipschitz constant K, a and b. The only non-global feature is the interval I depending on K, so fix K and we have a quite general continuity of the solution g in the initial condition and the function f determining the equation. (Naturally, continuity here is with respect to the topology of continuous convergence or the compact open topology.)

Finally I give a brief discussion of

Applications to variational problems. This is a large area. There are many classical problems, for example Steiner's problem and Plateau's problem (Beeson has announced a paper devoted to the latter, BEESON, forthcoming).

To get a feel for what goes on, consider the simple result in analysis which provides the conceptual background for variational problems: a continuous function on the unit interval has a maximum (as is provable in S) and attains it (which is essentially non-constructive). By an extension of our Theorem (i), the maximum depends continuously on the function. But it is not possible to choose a point at which the maximum is attained stably in the function (consider $e^{ax}\sin(x)$ in $[0, 4\pi]$ and vary the parameter a through 0); so attainment of bounds is not provable in S^*. (The problem is connected with non-uniqueness, but it has a different effect here than it did above.) Continuity of the extremal value, but instability of the position of attainment, is a persistant feature of variational problems. Naturally, one then turns to the physically significant question of continuity of relative (strict) extrema, where there these exist. Such questions are difficult, and answers vary from problem to problem. At this time, I cannot give a constructive account of all the examples that come to mind.

I conclude this paper by making some very tentative remarks about

Possible heuristic value in science. Attempts to model many physical problems give rise to differential equations whose dynamics depends sensitively on initial conditions: for example every trajectory in some bounded region may be Liapunov unstable. The term *chaos* has been applied to extreme situations of this sort, and they are the subject of much research at this time. Some results are known concerning the topological structure of attracting sets which occur in particular situations, and much more is plausibly indicated on the basis of computer simulation. But the detailed structure of the flow is too complicated for any useful description. This is because of the sensitivity of the dynamics, or as I wish to say, because of the impossibility of a constructive treatment of the flow (in the large).

A recent paper by Shaw (preprint) gives a view of this situation based on a sophisticated interpretation of Hadamard's principle. His idea is that for physical applications the detailed structure does not matter: for example, exactly what the flow pattern is, is not of importance in turbulence, it is changing all the time; but certain general features of it remain the same. Again the exact shape of the record of one's heartbeat on an oscilloscope does not matter, though certain general features may be of considerable significance. So what are important are characteristic properties of a dynamical system which are continuously dependent on parameters; or which we can handle constructively! One such for which Shaw gives an interesting discussion is the rate of creation or loss of information in the flow. There must be others. Certainly for particular dynamical systems, there would be interest in regions where there were "approximately periodic orbits" of "approximately the same period", if this qualitative situation is stable. And we could expect such qualitative properties to be established constructively. Of course, a physicist does not need to be told by logicians what properties of a system are of importance. But our physical intuition of stability is fallible, so illustrations and applications mentioned in this paper may be of some use.

References

BEESON, M. J.. 1977, *Principles of continuous choice and continuity of functions in formal systems for contructive mathematics*, Annals of Mathematical Logic, vol. 12, pp. 249–322

BEESON, M. J., *Plateau's problem and constructive mathematics* (forthcoming)

BISHOP, E. A., 1967, *Foundations of constructive analysis* (McGraw-Hill)

BISHOP, E. A., and H. CHENG, 1972, *Constructive measure theory*, Memoir of the American Mathematical Society, No. 116

BRIDGES, D. S., 1979, *Constructive functional analysis* (Pitman)

FOURMAN, M. P., 1977, *The logic of topoi*, in: Handbook of Mathematical Logic (North-Holland)

FOURMAN, M. P., and J. M. E. HYLAND, *Sheaf models for analysis*, Proceedings of the Durham Symposium on Sheaves and Logic (forthcoming)

FRIEDMAN, H., 1977, *Set theoretic foundations for constructive analysis*, Annals of Mathematics (2), vol. 105, pp. 1–28

HAYASHI, S., *Derived rules related to the topos theoretic analysis in the intuitionistic higher order arithmetic*, preprint

HYLAND, J. M. E., 1977, *Aspects of constructivity in mathematics*, in: Logic Colloquium '76 (North-Holland)

JOHNSTONE, P. T., 1977, *Topos theory* (Academic Press)

KLEENE, S. C., and R. E. VESLEY, 1965, *The foundations of intuitionistic mathematics* (North-Holland, Amsterdam)

KREISEL, G., and A. S. TROELSTRA, 1970, *Formal systems for some branches of intuitionistic analysis*, Annals of Mathematical Logic, vol. 1, pp. 229-387

SHAW, R., *Strange attractors, chaotic behaviour and information flow*, preprint