# The Grunwald-Wang Theorem

## Abstract

Assume a global field $K$ has prescribed local cyclic extensions $L_\mathfrak{p}/K_\mathfrak{p}$. Is there a cyclic extension $L/K$ which realises the extensions $L_\mathfrak{p}/K_\mathfrak{p}$? The Grunwald-Wang theorem tells us that such an extension exists, except in a well-understood special case. Proving this is the aim of this essay.

After a review of the cohomology of finite and profinite groups we will develop class field theory – first abstractly, following [8], then in the local and global cases. For proofs we will refer to [3], [6], [7], [8], and [11]. We will prove local and global Tate duality for $\mathbb{Z}/n\mathbb{Z}$ and $\mu_n$. It will play a key role in our proof of Grunwald-Wang. This sets apart our approach from the that in [1] based on topological group theory. Our proof will not require the theory of $S$-idèles as used in [8].

Throughout we will emphasize local-global principles. We will encounter the Hasse Norm Theorem, the Hasse Principles for $n$-th powers and for $H^2$, as well as connections between the local and global reciprocity maps of class field theory.

# Contents

# Introduction

The Hasse principle, also called the local-global principle, is a broad-ranging philosophy in number theory. We say that the Hasse principle holds when a problem can be solved globally by solving it locally everywhere.

An example is given by the famous Hasse-Minkowski Theorem: let $q(x_1, ..., x_n)$ be a quadratic form. Then $q(x_1, ..., x_n) = 0$ has a solution over $\mathbb{Q}$ if and only if $q(x_1, ..., x_n) = 0$ has a solution over $\mathbb{R}$ as well as over $\mathbb{Q}_p$ for every prime $p$.

The Grunwald-Wang theorem answers a question about the interplay between local and global Galois extensions. As we will see, it has an equivalent cohomological formulation which makes the Hasse principle explicit. Before we outline our proof let us mention the recommended prerequisites for this essay.

It is essential the reader be familiar with the basic theory of number fields, as usually taught in a first course in algebraic number theory. We will also make heavy use of the theory of local fields: valuations, completions, and ramification theory. Possible references are chapters I and II of [3] or of [7]. Furthermore, we require some knowledge of the cohomology theory of finite groups.

Group cohomology is the most important technical tool for the modern development of class field theory. In Chapter 1 we review some of the main theorems. We define the Tate groups and extend the theory to profinite groups. This requires some knowledge of direct and inverse limits. Absolute Galois groups, which are the main object of study of class field theory, are examples of profinite groups. Our approach follows [6] and [8].

In Chapter 2 we will develop class field theory for abstract profinite groups, without referring to Galois theory. We will show that we can construct a powerful homomorphism between a profinite group $G$ and a so-called formation module $C$, which has very particular cohomological properties with respect to $G$. This homomorphism will allow us to prove more theorems about $G$ and $C$. Through this approach the structural similarities between local and global class field theory will become clear. For most proofs we refer to [8].

Chapters 3 and 4 show that there exist formation modules for the absolute Galois group of a local and a global field, respectively. Much of class field theory then follows straight away by appealing to the results of Chapter 2. We also prove local Tate duality for $\mathbb{Z}/n\mathbb{Z}$ and $\mu_n$ which provides us with an important perfect pairing. We mainly follow [6], [7], and [8].

In Chapter 5 we use the full strength of the results of the preceding chapters to prove a special case of global Tate duality. Together with a result from [1] about the Hasse principle for $n$-th powers, this gives a proof of a cohomological statement equivalent to the Grunwald-Wang theorem. We characterise the case for which the Hasse principle fails and give examples as well as generalisations of the theory developed.

# 1 Group Cohomology

Group cohomology is the most important technical tool for the development of class field theory. We assume that the reader is already familiar with the basic results of the cohomology of finite groups. We mention some of them for ease of reference. Later in this section we will extend the theory to profinite groups.

## 1.1 Review

Some of the standard results will be given without proof. Possible references are [3], Chapter IV and [6], Chapter II.

In this section let $G$ be a finite group.

Let $M, N$ be $G$-modules. Denote by $\mathrm{Hom}_G(M, N)$ the group of all $G$-module homomorphisms from $M$ to $N$.

Consider a resolution of $\mathbb{Z}$ by free $G$-modules $P_r$,

$$\cdots \to P_2 \to P_1 \to P_0 \to \mathbb{Z} \to 0.$$

Apply the left exact functor $\mathrm{Hom}_G(-, M)$ to get

$$0 \to \mathrm{Hom}_G(P_0, M) \xrightarrow{d_0} \mathrm{Hom}_G(P_1, M) \xrightarrow{d_1} \mathrm{Hom}_G(P_2, M) \xrightarrow{d_2} \cdots .$$

**Definition 1.1.1.** The $r$-th cohomology group $H^r(G, M)$ is defined as

$$H^r(G, M) = \mathrm{Ker}(d^r)/\,\mathrm{Im}(d^{r-1}).$$

We cite some basic properties of these cohomology groups.

**Proposition 1.1.2.** $H^0(G, M) = M^G$.

**Proposition 1.1.3.** For $I$ an injective $G$-module, $H^r(G, I) = 0$ for all $r > 0$.

**Proposition 1.1.4.** Let $0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$ be a short exact sequence of $G$-modules. Then there exists a long exact sequence

$$0 \to H^0(G, M) \to \cdots \to H^r(G, N) \to H^r(G, P) \xrightarrow{\delta^r} H^{r+1}(G, M) \to \cdots$$

This relationship is functorial: a morphism of short exact sequences induces a morphism of long exact sequences.

Let $N$ be a $H$-module where $H \leq G$. Define $\mathrm{Ind}_H^G(N)$ to be the set of all functions $\phi : G \to M$ such that $\phi(hg) = h\phi(g)$ for all $h \in H$. With an action of $G$ given by $(g\phi)(x) = \phi(xg)$, $\mathrm{Ind}_H^G(N)$ becomes a $G$-module.

**Definition 1.1.5.** When $H = \{1\}$ we simply write $\mathrm{Ind}^G(N)$. A $G$-module of this form is said to be *induced*.

$\mathrm{Ind}^G(N) \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}} N$, where the $G$-action is given by $g(s \otimes n) = gs \otimes n$.

**Theorem 1.1.6.** (Shapiro's Lemma)
Let $H \leq G$ and let $N$ be a $H$-module. Then for all $r \geq 0$,

$$H^r(G, \mathrm{Ind}_H^G(N)) \cong H^r(H, N).$$

**Corollary 1.1.7.** An induced module $\mathrm{Ind}^G(N)$ has trivial cohomology for $r > 0$.

Take a projective resolution

$$P_2 \to P_1 \to P_0 \to M \to 0$$

of a $G$-module $M$ and apply $(-)_G$ to it. This gives a complex

$$\cdots \xrightarrow{d_3} (P_2)^G \xrightarrow{d_2} (P_1)^G \xrightarrow{d_1} (P_0)^G \to 0$$

**Definition 1.1.8.** The $r$-th *homology group* of the $G$-module $M$ is

$$H_r(G, M) = \frac{\mathrm{Ker}(d_r)}{\mathrm{Im}(d_{r+1})}.$$

We list some basic properties of the homology groups.

**Proposition 1.1.9.** $H_0(G, M) = M_G$.

**Proposition 1.1.10.** For $P$ a projective $G$-module, $H_r(G, P) = 0$ for all $r > 0$.

**Proposition 1.1.11.** A short exact sequence $0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$ gives rise to a long exact sequence

$$\cdots \to H_r(G, N) \to H_r(G, P) \xrightarrow{\delta_r} H_{r-1}(G, M) \to \cdots \to H_0(G, P) \to 0$$

This assignment is functorial just as with cohomology groups.

**Theorem 1.1.12.** $H_1(G, \mathbb{Z}) \cong G^{ab}$, where $G^{ab} = G/[G, G]$ is the abelianisation of $G$, i.e. its largest abelian quotient. This theorem will play a crucial role in class field theory.

*Proof.* Let $I_G$ be the kernel of the map $\mathbb{Z}[G] \to \mathbb{Z}$ which sends $\sum_i n_i g_i$ to $\sum_i n_i$. $I_G$ is a free $\mathbb{Z}$-submodule of $\mathbb{Z}[G]$ with basis $\{g - 1 \mid g \in G\}$. Therefore $M/I_G M = M_G \cong H_0(G, M)$.
Consider $\mathbb{Z}$ as a $G$-module with trivial action. From the exact sequence $0 \to I_G \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$ we get a long exact sequence of homology

$$0 \to H_1(G, \mathbb{Z}) \to I_G/I_G^2 \to \mathbb{Z}[G]/I_G \to \mathbb{Z} \to 0,$$

noticing that $H_1(G, \mathbb{Z}[G]) = 0$ because $\mathbb{Z}[G]$ is projective.
The map $I_G/I_G^2 \to \mathbb{Z}[G]/I_G$ is induced by $I_G \hookrightarrow \mathbb{Z}[G]$, so it is zero. Therefore we have an isomorphism $H_1(G, \mathbb{Z}) \cong I_G/I_G^2$. Lastly, the map determined by $g \mapsto g - 1$ gives an isomorphism from $G^{ab}$ to $I_G/I_G^2$. □

Let $P_r$ be the free $\mathbb{Z}[G]$-module with basis $\{(1, g_1, ..., g_r) \mid g_i \in G\}$. Define homomorphisms $d_r : P_r \to P_{r-1}$ by

$$d_r(g_0, ..., g_r) = \sum_{i=0}^{r} (-1)^i (g_0, ..., g_{i-1}, g_{i+1}, ..., g_r)$$

Let $\varepsilon : P_0 \to \mathbb{Z}$ be the map that sends each generator $g_i$ to 1.

**Proposition 1.1.13.** $\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ is exact, i.e. a projective resolution of $\mathbb{Z}$. This is called the *standard complex.*

*Proof.* See [6], II.1.15. □

Let us apply $\mathrm{Hom}_G(-, M)$ to the standard complex to find explicit descriptions of the cohomology groups.
An element of $\mathrm{Hom}_G(P_r, M)$ is just a function $\phi : G^{r+1} \to M$ such that $g\phi(g_0, ..., g_r) = \phi(gg_0, ..., gg_r)$. The induced coboundary maps $\mathrm{Hom}_G(P_r, M) \to \mathrm{Hom}_G(P_{r+1}, M)$ are given by

$$(d^r\phi)(g_0, ..., g_r) = \sum_{i_0}^{r} (-1)^i \phi(g_0, ...g_{i-1}, g_{i+1}, ..., g_r).$$

Notice that $\phi \in \mathrm{Hom}_G(P_r, M)$ is determined by its values on elements of the form $(1, g_1, g_1g_2, ..., g_1g_2...g_r)$. Setting

$$\phi(g_1, ..., g_r) = f(1, g_1, g_1g_2, ..., g_1g_2...g_r),$$

the coboundary maps become

$$(d^r\phi)(g_1, ..., g_{r+1}) = g_1\phi(g_2, ..., g_{r+1}) + \sum_{i=1}^{r} (-1)^i \phi(g_1, ..., g_{i-1}, g_{i+1}, ..., g_{r+1}) +$$

$$+ (-1)^{r+1}\phi(g_1, ..., g_r).$$

We can use this new characterisation of the boundary maps.

**Definition 1.1.14.** A function $\phi \in \mathrm{Ker}(d^1)$, i.e. a 1-cocycle, is called a *crossed homomorphism.* It satisfies $\phi(gh) = g\phi(h) + \phi(g)$ for all $g, h \in G$.
A function $\psi \in \mathrm{Im}(d^0)$, i.e. a 1-coboundary, is a called a *principal crossed homomorphism.* It is of the form $\psi(g) = gm - m$ for some $m \in M$.

This leads to the following characterisation of $H^1(G, M)$:

**Proposition 1.1.15.**

$$H^1(G, M) = \frac{\{\text{crossed homomorphisms}\}}{\{\text{principal crossed homomorphisms}\}}$$

If the action of $G$ on $M$ is trivial, then crossed homomorphisms are just homomorphisms, and principal crossed homomorphisms are all zero. In this case we have $H^1(G, M) = \mathrm{Hom}(G, M)$.

## 1.2 Functorial Properties

We can embed $M$ into $\mathrm{Ind}^G(M)$: map $m \in M$ to the function $g \mapsto gm$. This gives an exact sequence

$$0 \longrightarrow M \longrightarrow \mathrm{Ind}^G(M) \longrightarrow M_1 \to 0,$$

where $M_1$ is defined by the exactness of the above sequence.
The long exact sequence of cohomology gives

$$H^r(G, M_1) \xrightarrow{\delta} H^{r+1}(G, M)$$

is an isomorphism for all $r \geq 1$ and is surjective for $r = 0$.
Set $M_0 = M$ and $M_k = (M_{k-1})_1$. Iterating, we get the following proposition, called *downward dimension shifting*:

**Proposition 1.2.1.** For all $r, k \geq 0$ we have a canonical homomorphism

$$\delta^k : H^r(G, M_k) \to H^{r+k}(G, M).$$

This is a surjection for $r = 0$ and an isomorphism for $r \geq 1$.

Tensoring the exact sequence $0 \to I_G \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$ with $M$, we get

$$0 \to M_{-1} \to \mathrm{Ind}^G(M) \to M \to 0,$$

showing that each $G$-module $M$ is the quotient of an induced module.
$M_{-1}$ is defined by the exactness of the sequence. Define $M_k = (M_{k+1})_{-1}$ for $k < 0$.
We can use this get an analogous result called *upward dimension shifting*:

$$H^r(G, M) \cong H^{r-k}(G, M_k).$$

If $\alpha : G' \to G$ is a group homomorphism, it induces homomorphisms $\alpha_r : P'_r \to P_r$, which are compatible with the maps $d_r$ in the standard complex. Therefore $\alpha$ descends to a homomorphism $\alpha^* : H^r(G, M) \to H^r(G', M)$.
Now let $M$ and $M'$ be $G$- and $G'$-modules, respectively, and let $\beta : M \to M'$ be a group homomorphism. $\alpha$ and $\beta$ are said to be *compatible* when $\beta(\alpha(g)m) = g(\beta m)$. Then the pair $(\alpha, \beta)$ induces a homomorphism $H^r(G, M) \to H^r(G', M')$.

**Definition 1.2.2.** Let $H \leq G$. Let $\alpha$ be the inclusion map $H \hookrightarrow G$ and let $\beta$ be the identity on a $G$-module $M$. These homomorphisms are compatible, so we get the *restriction homomorphisms*

$$\mathrm{Res} : H^r(G, M) \to H^r(H, M).$$

**Definition 1.2.3.** Now let $H$ be a normal subgroup of $G$, let $\alpha$ be the quotient map $G \to G/H$ and let $\beta$ be the inclusion $M^H \hookrightarrow M$. This gives the *inflation homomorphisms*

$$\mathrm{Inf} : H^r(G/H, M^H) \to H^r(G, M)$$

.

**Definition 1.2.4.** Let $H$ be a finite index subgroup of $G$. We can find a set $S$ of coset representatives such that $G = \bigcup_{g \in S} gH$.
For any $G$-module $M$ we have a homomorphism $\mathrm{Ind}_H^G(M) \to M$ given by $\phi \mapsto \sum_{g \in S} g\phi(g^{-1})$. Together with the isomorphism from Shapiro's lemma we get the *corestriction homomorphisms*, given by the following composition.

$$\mathrm{Cor} : H^r(H, M) \xrightarrow{\sim} H^r(G, \mathrm{Ind}_H^G(M)) \longrightarrow H^r(G, M).$$

**Proposition 1.2.5.** Let $H \leq G$ be a finite index subgroup. Then

$$\mathrm{Cor} \circ \mathrm{Res} : H^r(G, M) \to H^r(G, M),$$

factoring through $H^r(H, M)$, is multiplication by $(G : H)$ for all $r \geq 0$.

*Proof.* Let $\phi_m : G \to M$ be given by $g \mapsto gm$. $\mathrm{Cor} \circ \mathrm{Res}$ is the map on cohomology induced by $M \to \mathrm{Ind}_H^G(M) \to M$; $m \mapsto \phi_m \mapsto \sum_s s\phi_m(s^{-1}) = \sum_s m = (G : H)m$. □

**Corollary 1.2.6.** If $|G| = m$ then $mH^r(G, M) = 0$. Therefore, if $G$ is finite and $M$ is a finitely generated $\mathbb{Z}$-module, $H^r(G, M)$ is finite.

*Proof.* Apply the proposition with $H = 1$. □

**Proposition 1.2.7.** (Inflation-Restriction Exact Sequence)
Let $H$ be a normal subgroup of $G$ and let $M$ be a $G$-module. Let $r \in \mathbb{Z}_{\geq 1}$. If $H^i(H, M) = 0$ for all $0 < i < r$, then the sequence

$$0 \longrightarrow H^r(G/H, M^H) \xrightarrow{\mathrm{Inf}} H^r(G, M) \xrightarrow{\mathrm{Res}} H^r(H, M)$$

is exact.

*Proof.* For $r = 1$ the condition on $H$ is vacuous and exactness can be checked using the explicit description of cocycles. For $r > 1$ one can use dimension shifting. □

**Definition 1.2.8.** Let $m \in H^r(G, M)$ and $n \in H^s(G, N)$. There is a unique family of bi-additive pairings

$$(m, n) \mapsto m \cup n : H^r(G, M) \times H^s(G, N) \to H^{r+s}(G, M \otimes N)$$

satisfying the following properties:

- the pairings are functorial in $M$ and in $N$,
- for $r = s = 0$, the pairing is $(m, n) \mapsto m \otimes n : M^G \otimes N^G \to (M \otimes N)^G$,
- if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of $G$-modules such that $0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$ is exact, then $(\delta m'') \cup n = \delta(m'' \cup n)$ for all $m'' \in H^r(G, M'')$ and $n \in H^s(G, N)$, and

- if $0 \to N' \to N \to N'' \to 0$ is an exact sequence of $G$-modules such that $0 \to M \otimes N' \to M \otimes N \to M \otimes N'' \to 0$ is exact, then $m \cup \delta n'' = (-1)^r \delta(m \cup n'')$ for all $m \in H^r(G, M)$ and $n'' \in H^s(G, N'')$.

We call this the *cup product*.

Let us list some of its properties:

**Proposition 1.2.9.**

- $(m \cup n) \cup p = m \cup (n \cup p)$,
- $m \cup n = (-1)^{rs} n \cup m$ where $m \in H^r(G, M), n \in H^s(G, M)$,
- $\mathrm{Res}(m \cup n) = \mathrm{Res}(m) \cup \mathrm{Res}(n)$,
- $\mathrm{Cor}(m \cup \mathrm{Res}\, n) = \mathrm{Cor}(m) \cup n$.

*Proof.* See [2], V.3. and [3], IV.7.4. □

## 1.3 Tate Groups

**Definition 1.3.1.** Let $H$ be a finite index subgroup of $G$ and let $S$ be a set of coset representatives for $G/H$. For a $G$-module $M$ we define the *norm map* $N_{G/H} : M \to M$ by $m \mapsto \sum_{s \in S} sm$.
When $G/H = \mathrm{Gal}(L/K)$ we might write $N_{L/K}$ instead.

Notice that because $H_0(G, M) = M/I_G M$ and $H^0(G, M) = M^G$, we have an exact sequence

$$0 \to \mathrm{Ker}(N_G) \to H_0(G, M) \xrightarrow{N_G} H^0(G, M) \to M^G/N_G(M) \to 0.$$

For an exact sequence $0 \to M' \to M \to M'' \to 0$ the norm map lets us relate the long exact sequences of homology and cohomology as follows.

$$\begin{tikzcd}
H^1(G,M'') \arrow[r] & H_0(G,M') \arrow[r] \arrow[d, "N_G"] & H_0(G,M) \arrow[r] \arrow[d, "N_G"] & H_0(G,M'') \arrow[r] \arrow[d, "N_G"] & 0 \\
0 \arrow[r] & H^0(G,M') \arrow[r] & H^0(G,M) \arrow[r] & H^0(G,M'') \arrow[r] & H^1(G,M')
\end{tikzcd}$$

We can apply the Snake lemma to the middle part of the diagram to get a long exact sequence extending to infinity in both directions:

$$\cdots \to H^r_T(G, M') \to H^r_T(G, M) \to H^r_T(G, M'') \xrightarrow{\delta} H^{r+1}_T(G, M) \to \cdots,$$

where the groups $H^r_T$ are defined as follows.

**Definition 1.3.2.** The *Tate groups* are given by

$$H_T^r(G,M) = \begin{cases} H^r(G,M) & r>0 \\ M^G/N_G(M) & r=0 \\ \mathrm{Ker}(N_G)/I_G M & r=-1 \\ H_{-r-1}(G,M) & r<-1. \end{cases}$$

for $r \in \mathbb{Z}$.

For $r > 0$ we might omit the subscript because there is no ambiguity.

**Proposition 1.3.3.** If $I$ is an induced $G$-module, then $H_T^r(G,I) = 0$ for all $r \in \mathbb{Z}$.

*Proof.* See [3], IV.6.6. □

One can show, employing dimension shifting in both directions, that the properties of the cup product which we quoted extend to Tate groups for all $r \in \mathbb{Z}$, see [8], I.9. Similarly Shapiro's lemma, (1.1.6), as well as (1.2.6) hold for the Tate groups for all $r \in \mathbb{Z}$. There are canonical restriction and corestriction maps Res : $H_T^r(G,M) \to H_T^r(H,M)$ and Cor : $H_T^r(H,M) \to H_T(G,M)$ for all $r \in \mathbb{Z}$.
The inflation map however is only defined for $r \geq 1$.

Throughout this essay we will consider $\mathbb{Q}, \mathbb{Z}$, and $\mathbb{Q}/\mathbb{Z}$ as $G$-modules with trivial $G$-action.

**Proposition 1.3.4.** Let $G$ be finite. Then

- $H_T^r(G,\mathbb{Q}) = 0$ for all $r \in \mathbb{Z}$,
- $H_T^0(G,\mathbb{Z}) = \mathbb{Z}/|G|\mathbb{Z}$ and $H^1(G,\mathbb{Z}) = 0$, and
- there exists a canonical isomorphism $\mathrm{Hom}(G,\mathbb{Q}/\mathbb{Z}) \cong H^2(G,\mathbb{Z})$.

*Proof.* Multiplication by $m \in \mathbb{Z}$ is an isomorphism on $\mathbb{Q}$, so the induced map $H_T^r(G,\mathbb{Q}) \xrightarrow{\cdot m} H_T^r(G,\mathbb{Q})$ is an isomorphism too. But if we take $m = |G|$, then (1.2.6) shows that $H_T^r(G,\mathbb{Q}) = 0$ for all $r \in \mathbb{Z}$.
$\mathbb{Z}^G = \mathbb{Z}$ and the norm map is just multiplication by $|G|$, so $H_T^0(G,\mathbb{Z}) = \mathbb{Z}/|G|\mathbb{Z}$.
$H^1(G,\mathbb{Z})$ is just $\mathrm{Hom}(G,\mathbb{Z})$ which is trivial because $\mathbb{Z}$ is torsion-free.
The last claim follows from the first, using the long exact sequence on cohomology of $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. □

**Proposition 1.3.5.** Let $G$ be cyclic. We have

$$H_T^r(G,M) \cong H_T^{r+2}(G,M)$$

for all $G$-modules $M$ and all $r \in \mathbb{Z}$.

*Proof.* See [6], II.3.4. □

## 1.4 Profinite Groups

We will now explain how profinite groups arise from considering infinite Galois extensions and extend previous results and definitions. For a detailed account see [8], Chapter I.

**Definition 1.4.1.** A group $G$ with a topology is a *topological group* if the group operations $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are continuous.

We can consider any finite group as a topological group by giving it the discrete topology.

**Definition 1.4.2.** We say that a topological group $G$ is *topologically generated* by a subgroup $H$ if $H$ is dense in $G$.

**Proposition 1.4.3.** A field $E$ is Galois over $K$ if and only if it is the union of finite Galois extensions of $K$.

*Proof.* See [6], I.A.2. □

**Definition 1.4.4.** Let $G = \mathrm{Gal}(E/K)$ be the Galois group of a finite or infinite extension. $G$ has a natural topology called the *Krull topology*: for $\sigma \in G$, let the cosets $\sigma G(E/L)$ form a basis of neighbourhoods of $\sigma$, where $L$ ranges over all finite Galois subextensions $L/K$.

**Proposition 1.4.5.** $G$, taken as above, is a compact Hausdorff topological group with respect to the Krull topology.

*Proof.* See [7], IV.1.1. □

We have a fundamental theorem of Galois theory for infinite extensions.

**Theorem 1.4.6.** Let $E$ be a possibly infinite Galois extension of $K$ and let $G = \mathrm{Gal}(E/K)$. Then the assignments

$$L \mapsto \mathrm{Gal}(E/L) = H \text{ and } H \mapsto E^H$$

are mutually inverse, giving a bijection between subextensions $L/K$ of $E/K$ and closed subgroups of $\mathrm{Gal}(E/K)$. Furthermore, open subgroups of $\mathrm{Gal}(E/K)$ are in bijection with the finite subextensions of $E/K$.

*Proof.* See [6], I.A.4. □

These topological Galois groups $G$ have the property that $1 \in G$ has a fundamental system of neighbourhoods consisting of normal subgroups. This leads to the following definition.

**Definition 1.4.7.** A *profinite group* $G$ is a compact Hausdorff topological group which admits a basis of neighbourhoods of $1 \in G$ consisting of finite index normal subgroups. In particular any finite group with the discrete topology is profinite.

**Proposition 1.4.8.** A subgroup $H$ of a profinite group $G$ is open if and only if it is closed and finite index.

*Proof.* $G$ is compact and $G = \bigcup_g gH$. Now the result follows easily. □

We will make use of some facts about projective and inductive limits. For a reference see [7], IV.2.

**Proposition 1.4.9.** A profinite group $G$ can be characterised as $G = \varprojlim_N G/N$, the projective limit of the quotients of $G$ by its open normal subgroups.
Conversely, if $\{G_i, g_{ij}\}$ is a projective system of finite groups, then $G = \varprojlim_i G_i$ is a profinite group.

*Proof.* See [7], IV.2.8. □

**Corollary 1.4.10.** Let $E/K$ be an infinite Galois extension. Then $\mathrm{Gal}(E/K) = = \varprojlim_L \mathrm{Gal}(L/K)$, where L ranges over all the finite subextensions $K \subseteq L \subseteq E$.

*Proof.* This follows from (1.4.3). □

**Definition 1.4.11.** A discrete $G$-module $M$ is a topological group $M$ with the discrete topology such that the action of $G$ on $M$ is continuous.

**Definition 1.4.12.** Let $G$ be a profinite group and let $M$ be a discrete $G$-module. Let $C^r(G, M) = \{\text{Continuous functions } \phi : G^r \to M\}$.
Define maps $d^r : C^r(G, M) \to C^{r+1}(G, M)$ by

$$(d^r\phi)(g_1, ..., g_{r+1}) = g_1\phi(g_2, g_3, ..., g_{r+1}) + \sum_{i=1}^{r}(-1)^i\phi(g_1, ..., g_{i-1}, g_{i+1}, ..., g_{r+1}) +$$

$$+(-1)^{r+1}\phi(g_1, ..., g_r).$$

Then the $r$-th cohomology group is given by

$$H^r(G, M) = \frac{\mathrm{Ker}(d^r)}{\mathrm{Im}(d^{r-1})}.$$

**Definition 1.4.13.** A discrete $G$-module $M$ such that $H_T^r(H, M) = 0$ for all closed subgroups $H \leq G$ and all $r \geq 1$ is called *cohomologically trivial.*

**Proposition 1.4.14.** Let $G$ be profinite and $M$ a discrete $G$-module. There is an isomorphism

$$\varinjlim_U H^r(G/U, M^U) \xrightarrow{\sim} H^r(G, M),$$

where $U$ ranges over all normal open subgroups of $G$.

*Proof.* See [8], 1.2.5. □

This result allows us to use the cohomology of finite groups to compute the cohomology of a profinite group.

**Theorem 1.4.15.** (Hilbert's Theorem 90)
Let $L/K$ be a finite or infinite Galois extension with $G = \mathrm{Gal}(L/K)$. Then $H^1(G, L^\times) = 0$.

*Proof.* By (1.4.14) we can reduce to the case of a finite extension $L/K$. Let $\phi : G \to L^\times$ be a crossed homomorphism. Take $a \in L^\times$ and set $b = \sum_{\sigma\in G} \phi(\sigma)\sigma a$. Because the $\sigma$ are linearly independent (see [5], VI.4.1) there must exist some $a$ such that $b \neq 0$. Then, for $\tau \in G$,

$$\tau b = \sum_{\sigma} \tau\phi(\sigma)\tau\sigma a = \sum_{\sigma} \phi(\tau)^{-1}\phi(\tau\sigma)\tau\sigma a = \phi(\tau)^{-1}b,$$

and so $\phi(\tau) = \tau(b^{-1})/b^{-1}$, showing that $\phi$ is principal. □

We want to extend the definition of the Tate groups to profinite groups as well.
Let $G$ be profinite, $M$ a discrete $G$-module and $V \subseteq U$ be normal subgroups of $G$. For $r \leq 0$ we need maps that make the $H^r_T(G/U)$ into an inverse system of which we can take the limit. This is accomplished with the following.

**Definition 1.4.16.** For $G$ profinite, $M$ a discrete $G$-module, and $V \subseteq U$ both normal in $G$, there are canonical *deflation maps* $\mathrm{Def} : H^r_T(G/V, M^V) \to H^r_T(G/U, M^U)$ for $r \leq 0$. For a precise definition see [8], pp.85.

**Definition 1.4.17.** We define Tate groups of a profinite group $G$ and a discrete $G$-module $M$ are defined as follows. For $r > 0$, we have $H^r_T(G, M) = H^r(G, M)$. For $r \leq 0$ we set

$$H^r_T(G, M) = \varprojlim_U H^r_T(G/U, M^U),$$

where $U$ runs through the open normal subgroups of $G$ and the limit is taken over the inverse system given by the deflation maps.

**Proposition 1.4.18.** Let $G$ be profinite and let $M$ be a discrete $G$-module. If $M = \varinjlim M_i$, then $H^r(G, M) = \varinjlim H^r(G, M_i)$.

*Proof.* See [6], II.4.4. □

**Definition 1.4.19.** Let $M$ be a Hausdorff, abelian, and locally compact group. Define the *Pontryagin dual* of $M$ to be

$$M^\vee = \mathrm{Hom}_{cts}(M, \mathbb{R}/\mathbb{Z}).$$

where $\mathbb{R}/\mathbb{Z}$ is given the natural quotient topology..

**Proposition 1.4.20.** (Pontryagin duality)
If $M$ is a Hausdorff, abelian, and locally compact group, then so is $M^\vee$ with the compact-open topology. Furthermore, there exists a canonical homomorphism

$$M \longrightarrow (M^\vee)^\vee$$

given by $m \mapsto \tau_m : M^\vee \to \mathbb{R}/\mathbb{Z}, \ \ \phi \mapsto \phi(m)$. This is a topological group isomorphism.

*Proof.* See [9], Theorem 5.3. □

# 2 Abstract Reciprocity

When we consider a finite group $G$, we will implicitly assume that both $G$ and any $G$-module $M$ have the discrete topology unless otherwise specified. Profinite groups $G$ come with their own topology, see (1.4.7), and will also be considered as topological groups. $\mathrm{Hom}(A, B)$, for $G$-modules $A$ and $B$, will consist only of the continuous homomorphisms between $A$ and $B$. We will keep these conventions for the rest of this essay.

## 2.1 Class Formations

**Definition 2.1.1.** Let $G$ be a profinite group and let $A, B$, and $C$ be $G$-modules. We call a continuous $G$-bilinear map $A \times B \to C$ a *pairing.*
A pairing induces continuous homomorphisms

$$A \xrightarrow{\Phi} \mathrm{Hom}(B, C) \quad \text{and} \quad B \xrightarrow{\Psi} \mathrm{Hom}(A, C).$$

We say a pairing is *non-degenerate* if both $\Phi$ and $\Psi$ are injective.
We say a pairing is *perfect* if both $\Phi$ and $\Psi$ are isomorphisms.

Let $G$ be a finite group and let $A$ and $B$ be $G$-modules. The pairing

$$\mathrm{Hom}(A, B) \times A \to B$$

given by $(f, a) \mapsto f(a)$ yields the cup product pairing

$$H_T^r(G, \mathrm{Hom}(M_1, M_2)) \times H_T^{n-r}(G, M_1) \xrightarrow{\cup} H_T^n(G, M_2).$$

If $M_2 = \mathbb{Q}/\mathbb{Z}$, we set $M_1^* = \mathrm{Hom}(M_1, \mathbb{Q}/\mathbb{Z})$.

When we apply the results of this section in Chapters 3 to 5, $M$ will always be abelian and either finite or profinite. For such $M$ one can show that the image of any continuous homomorphism $\phi : M \to \mathbb{R}/\mathbb{Z}$ has finite image, see [10], Theorem 2.9.6. Therefore $\mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(M, \mathbb{R}/\mathbb{Z})$ and we have $M^* = M^\vee$ and $M \cong (M^*)^*$ by Pontryagin duality, see (1.4.20).

**Proposition 2.1.2.** Let $G$ be a finite group and let $M$ be a $G$-module. Then for all $r \in \mathbb{Z}$ the pairing

$$H_T^r(G, M^*) \times H_T^{-r-1}(G, M) \xrightarrow{\cup} H_T^{-1}(G, \mathbb{Q}/\mathbb{Z}) = \frac{1}{|G|}\mathbb{Z}/\mathbb{Z} \subseteq \mathbb{Q}/\mathbb{Z}$$

induces an isomorphism

$$H_T^r(G, M^*) \xrightarrow{\sim} H_T^{-r-1}(G, M)^*.$$

*Proof.* $H_T^{-1}(G, \mathbb{Q}/\mathbb{Z}) = \frac{1}{|G|}\mathbb{Z}/\mathbb{Z}$ follows from (1.3.4).
For $r = 0$ the proposition can be proved through elementary calculations. For the other dimensions we can use dimension shifting. □

**Definition 2.1.3.** Let $G$ be finite. A $G$-module $C$ is a *class module* for $G$ if for all subgroups $H \leq G$ it satisfies

- $H^1(H, C) = 0$ and
- $H^2(H, C)$ is cyclic of order $|H|$.

We call a generator $\gamma$ of $H^2(H, C)$ a *fundamental class.*

**Theorem 2.1.4.** (Nakayama-Tate)
Let $G$ be finite and $C$ be a $G$-module, $\gamma$ a generator for $H^2_T(G, C)$. For all $r \in \mathbb{Z}$ and each $H \leq G$ there exists a homomorphism

$$\delta^2 : H^r_T(H, \mathbb{Z}) \longrightarrow H^{r+2}_T(H, C)$$

given by the cup product $x \mapsto \gamma_H \cup x$, where $\gamma_H$ is the image of $\gamma$ under Res : $H^2_T(G, C) \to H^2_T(H, C)$. $C$ is a class module for $G$ with fundamental class $\gamma$ if and only if $\delta^2$ is an isomorphism for all $r \in \mathbb{Z}$ and all $H \leq G$.

*Proof.* See [8], III.1.4. □

**Corollary 2.1.5.** Let $C$ be a class module for the finite group $G$. There exists a canonical isomorphism

$$\rho : G^{ab} \xrightarrow{\sim} C^G / N_G C.$$

*Proof.* Apply Theorem (2.1.4) in the case $r = -2$ and $H = G$.
$H^{-2}_T(G, \mathbb{Z}) = H_1(G, \mathbb{Z}) \cong G^{ab}$ by (1.1.12) and $H^0_T(G, C) = C^G / N_G C$. □

**Definition 2.1.6.** The inverse of $\rho$, precomposed with the projection $C^G \to C^G / N_G C$, is called the *reciprocity map*:

$$\mathrm{rec} : C^G \longrightarrow G^{ab}.$$

Corollary (2.1.5) is an abstract form of the reciprocity homomorphism of class field theory, making no reference to Galois theory. It tells us that all we will need to do to prove the main theorem of local resp. global class field theory is to find class modules for the Galois groups of finite extensions of local resp. global fields. This endeavour will occupy much of the next two chapters.

**Corollary 2.1.7.** If $C$ is a class module for the finite group $G$, then for all subgroups $H \leq G$ there are isomorphisms

$$\mathrm{inv}_H : H^2(H, C) \to \frac{1}{|H|}\mathbb{Z}/\mathbb{Z}$$

mapping $\gamma_H$ to $\frac{1}{|H|}$. We call these the *invariant maps.*

*Proof.* This follows at once from Theorem (2.1.4) in the case $r = 0$. □

What we have done so far is enough to develop class field theory for finite extensions. Absolute Galois groups however are not finite but profinite. This motivates us to look for a generalisation of the previous theorems.
To this end we need an analogue of a class module for a profinite group.

**Definition 2.1.8.** Let $G$ be a profinite group. A *formation module* for $G$ is a discrete $G$-module $C$ together with isomorphisms

$$\mathrm{inv}_{U/V} : H^2(U/V, C^V) \xrightarrow{\sim} \frac{1}{(U:V)}\mathbb{Z}/\mathbb{Z}$$

for every pair of open subgroups $V \triangleleft U$ of $G$ such that additionally we have $H^1(U/V, C^V) = 0$ and, for $W$ normal and open in $V$, the following diagram commutes.

$$\begin{tikzcd}
H^2(U/V, C^V) \arrow[r, "\mathrm{Inf}"] \arrow[d, "\mathrm{inv}"] & H^2(U/W, C^W) \arrow[r, "\mathrm{Res}"] \arrow[d, "\mathrm{inv}"] & H^2(V/W, C^W) \arrow[d, "\mathrm{inv}"] \\
\frac{1}{(U:V)}\mathbb{Z}/\mathbb{Z} \arrow[r, hook] & \frac{1}{(U:W)}\mathbb{Z}/\mathbb{Z} \arrow[r, "(U:V)"] & \frac{1}{(V:W)}\mathbb{Z}/\mathbb{Z}
\end{tikzcd}$$

We call the pair $(G, C)$ a *class formation.*

**Definition 2.1.9.** The isomorphisms

$$\mathrm{inv}_{U/V} : H^2(U/V, C^V) \xrightarrow{\sim} \frac{1}{(U:V)}\mathbb{Z}/\mathbb{Z}$$

form a direct system, and its direct limit

$$\mathrm{inv} : H^2(G, C) \to \mathbb{Q}/\mathbb{Z},$$

which is injective but not necessarily an isomorphism, is again called the *invariant map.*

**Theorem 2.1.10.** Let $C$ be a formation module for the profinite group $G$. There is a reciprocity homomorphism

$$\mathrm{rec}_G : C^G \to G^{ab}$$

with dense image and kernel $N_G C := \bigcap_U N_{G/U} C^U$ for $U \subseteq G$ normal and open.

*Proof.* For $U \subseteq G$ open, one can easily see from (2.1.8) that $C^U$ is a class module for $G/U$. The surjective reciprocity homomorphisms $\mathrm{rec}\, G/U : C^G \to (G/U)^{ab}$ are compatible in the sense that, for $V$ normal in $U$, the following diagram commutes.

$$\begin{tikzcd}
C^G \arrow[r, "\mathrm{rec}_{G/V}"] \arrow[d, "id"] & (G/V)^{ab} \arrow[d, two heads] \\
C^G \arrow[r, "\mathrm{rec}_{G/U}"] & (G/U)^{ab}
\end{tikzcd}$$

Passing to the projective limit, we obtain the reciprocity homomorphism $\mathrm{rec}_G : C^G \to G^{ab}$. It has dense image because the $\mathrm{rec}_{G/U}$ are all surjective. An element $c \in C^G$ is in the kernel of $\mathrm{rec}_G$ if and only if it lies in $N_{G/U}C^U$ for all normal and open subgroups $U \subseteq G$. □

# 3 Local Theory

A major goal of the number theory of local fields is to understand and classify the finite Galois extension of a local field $K$. This is equivalent to determining the structure of $G_K = \mathrm{Gal}(\overline{K}/K)$, where $\overline{K}$ is a fixed separable closure of $K$. We call this the absolute Galois group of $K$. This is a hard problem and still far from being solved completely. However, there are partial results, namely on quotients of $G_K$. It is relatively elementary to determine $G_K^{nr}$, the Galois group of the maximal unramified extension of $K$. The theory of ramification then lets us compute the Galois group of the maximal tamely ramified extension. The wildly ramified part, which corresponds to the $p$-Sylow subgroup of $I_K$, is much harder to deal with.
Therefore, as the next step on the way to determining $G_K$, one can consider yet another quotient group: its abelianisation, $G_K^{ab} := G_K/\overline{[G_K, G_K]}$. Local class field theory completely explains the structure of this group.

We will not treat the archimedean cases $K = \mathbb{R}$ or $K = \mathbb{C}$.

## 3.1 Elementary Results

**Proposition 3.1.1.** Let $K^{nr}$ be the maximal unramified extension of a local field $K$. Then $\mathrm{Gal}(K^{nr}/K) \cong \varprojlim \mathbb{Z}/n\mathbb{Z} =: \widehat{\mathbb{Z}}$.

*Proof.* Let $L/K$ be a finite unramified extension with $[L : K] = n$. Let $l/k$ be the corresponding extension of residue fields. We know that $[l : k] = n$. Therefore $\mathrm{Gal}(L/K) \cong \mathrm{Gal}(l/k) = \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) = \mathbb{Z}/n\mathbb{Z}$. Now the result follows from Proposition 1.4.10. □

**Proposition 3.1.2.** $\mathrm{Gal}(K^t/K^{nr}) \cong \prod_{\ell \neq p} \mathbb{Z}_\ell$. Furthermore, $\mathrm{Gal}(K^t/K) \cong$
$\cong \prod_{\ell \neq p} \mathbb{Z}_\ell \rtimes \widehat{\mathbb{Z}}$, where the action of $\widehat{\mathbb{Z}}$ on $\prod_{\ell \neq p} \mathbb{Z}_\ell$ is determined by the Frobenius acting by conjugation.

*Proof.* We have $K^t = \bigcup_{m \geq 1, p \nmid m} K^{nr}(\pi_K^{1/m})$, where $\pi_K$ is a prime element of $K$. It follows that $\mathrm{Gal}(K^t/K^{nr}) \cong \varprojlim_{p \nmid m} \mathbb{Z}/m\mathbb{Z} \cong \prod_{\ell \neq p} \mathbb{Z}_\ell$.
Choosing a lift of the Frobenius to $\mathrm{Gal}(K^t/K)$ gives the semi-direct product structure. For a full proof see [8], VII.5.2. □

## 3.2 Local Class Field Theory

In this section we will show that $\overline{K}^\times$, the multiplicative group of the separable closure of $K$, is a formation module for $G_K$. This will immediately tell us that $L^\times$ it is a class module for the Galois group $\mathrm{Gal}(L/K)$ for any finite extension $L/K$. In view of 2.1.4, this will show at once that $G(L/K)^{ab} \cong K^\times/N_{L/K}L^\times$.

We set $H_T^r(L/K) = H_T^r(\mathrm{Gal}(L/K), L^\times)$.

**Proposition 3.2.1.** Let $L/K$ be an unramified Galois extension of local fields. Then the group of units $U_L$ of $L$ is a cohomologically trivial $\mathrm{Gal}(L/K)$-module.

*Proof.* See [6], III.1.1. □

**Proposition 3.2.2.** Let $K^{nr}$ be the maximal unramified extension of $K$. Then $H^2(K^{nr}/K) \cong \mathbb{Q}/\mathbb{Z}$.

*Proof.* Let $\Gamma_K := \mathrm{Gal}(K^{nr}/K)$. $U_{K^{nr}}$ is a cohomologically trivial $\Gamma_K$-module by (3.2.1), as is $\mathbb{Q}$. Therefore the exact sequences

$$0 \to U_{K^{nr}} \to K^{nr\times} \xrightarrow{v_{K^{nr}}} \mathbb{Z} \to 0,$$

where $v_K^{nr}$ is the (normalised) valuation, and

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

give

$$H^2(K^{nr}/K) \xrightarrow{\sim} H^2(\Gamma_K, \mathbb{Z}) \xrightarrow{\sim} H^1(\Gamma_K, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}.$$

The last isomorphism holds because $\widehat{\mathbb{Z}}$ is topologically generated by $1 \in \mathbb{Z}$, so $H^1(\Gamma_K, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(\widehat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$. □

**Theorem 3.2.3.** There is a canonical isomorphism

$$\mathrm{inv}_K : H^2(\overline{K}/K) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}.$$

Moreover, for a finite Galois extension $L/K$ the following diagrams commute:

$$\begin{tikzcd}
H^2(\overline{K}/L) \arrow[r, "\mathrm{inv}"] \arrow[d, shift left, "\mathrm{Cor}"] & \mathbb{Q}/\mathbb{Z} \arrow[d, shift left, "id"] \\
H^2(\overline{K}/K) \arrow[r, "\mathrm{inv}"] \arrow[u, shift left, "\mathrm{Res}"] & \mathbb{Q}/\mathbb{Z} \arrow[u, shift left, "L:K"]
\end{tikzcd}$$

$\mathrm{inv}_K$ induces a family of isomorphisms

$$\mathrm{inv}_{L/K} : H^2(L/K) \xrightarrow{\sim} \frac{1}{[L:K]}\mathbb{Z}/\mathbb{Z}.$$

Let $\gamma_{L/K}$ be the element such that $\mathrm{inv}_{L/K}(\gamma_{L/K}) = \frac{1}{[L:K]}$. Let $K \leq E \leq L$ where $E/K$ is Galois too. Then it follows from the diagram that

$$\mathrm{Res}(\gamma_{L/K}) = \gamma_{L/E}, \quad \mathrm{Cor}(\gamma_{L/E}) = [E:K]\gamma_{L/K}, \quad \mathrm{Inf}(\gamma_{E/K}) = [L:E]\gamma_{L/K}.$$

*Proof.* See [8], VII.1.3. and [6], III.2.7. □

**Theorem 3.2.4.** $\overline{K}^\times$ is a formation module for $G_K$, i.e. $(G_K, \overline{K}^\times)$ is a class formation. There is a reciprocity homomorphism

$$\mathrm{rec}_K : K^\times \to G_K^{ab}.$$

*Proof.* $\overline{K}^\times$ is a formation module due to to (3.2.3) and due to (1.4.15). Now apply (2.1.10). □

**Corollary 3.2.5.** Let $L/K$ be a finite Galois extension. Then $L^\times$ is a class module for $\mathrm{Gal}(L/K)$. Let $\gamma_{L/K}$ be a fundamental class, i.e. a generator of $H_T^2(L/K)$. The cup product

$$H_T^{-2}(G, \mathbb{Z}) \times H_T^2(L/K) \xrightarrow{\cup} H_T^0(L/K)$$

induces an isomorphism

$$\mathrm{Gal}(L/K)^{ab} \xrightarrow{\sim} K^\times / N_{L/K} L^\times$$

given by $\alpha \mapsto \alpha \cup \gamma_{L/K}$. Its inverse,

$$\mathrm{rec}_{L/K} : K^\times / N_{L/K} L^\times \xrightarrow{\sim} \mathrm{Gal}(L/K)^{ab},$$

we also call the reciprocity map. Furthermore $\mathrm{rec}_K |_L = \mathrm{rec}_{L/K}$.

*Proof.* Since $\overline{K}^\times$ is a formation module for $G_K$ it follows from (2.1.8) that $L^\times$ is a class module for $\mathrm{Gal}(L/K)$. Thus we can use (2.1.5). The last statement follows from (2.1.10). □

The following theorem tells us that for every open subgroup of $K^\times$ there exists a finite abelian extension $L/K$ corresponding to it; hence the name.

**Theorem 3.2.6.** (Local Existence theorem)
There is a bijection between finite abelian extensions $L$ of $K$ and open subgroups of $K^\times$ which is given by $L \mapsto N_{L/K} L^\times$. In particular, every open subgroup of $K^\times$ is of the form $N_{L/K} L^\times$ for some finite abelian extension $L/K$. We call such groups *norm groups*.
Furthermore, setting $N_L = N_{L/K} L^\times$, we have

- $L_1 \subseteq L_2 \iff N_{L_1} \supseteq N_{L_2}$,
- $N_{L_1 L_2} = N_{L_1} \cap N_{L_2}$, and
- $N_{L_1 \cap L_2 / K} = N_{L_1} N_{L_2}$.

*Proof.* See [11], XIV.6.1 for the existence of the bijection and [7], IV.6.7 for remaining claims. □

The following theorem explains why $\mathrm{Gal}(\overline{K}/K)^{ab}$ is the largest quotient of $G_K$ the methods of local class field theory can tell us about: non-abelian structure cannot be detected by norm groups. We include it for completeness.

**Theorem 3.2.7.** (Norm Limitation Theorem)
Let $L/K$ be a finite extension and let $E/K$ be the maximal abelian subextension. Then

$$N_{L/K}L^\times = N_{E/K}E^\times.$$

*Proof.* See [6], III.3.5. □

## 3.3 Local Tate Duality

Before we can prove Tate duality we will need to characterise the local reciprocity map more explicitly.

**Proposition 3.3.1.** Let $L/K$ be a finite extension with $G = \mathrm{Gal}(L/K)$. Identify $\alpha \in K^\times$ with its image in $H_T^0(L/K)$ and let $\chi \in H^1(G, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$. Then

$$\mathrm{rec}_{L/K}(\alpha) \cup \chi = \chi(\mathrm{rec}_{L/K}(\alpha)).$$

*Proof.* See [11], Appendix to Chapter XI. □

**Proposition 3.3.2.** Given the above setting, we have

$$\chi(\mathrm{rec}_{L/K}(\alpha)) = \mathrm{inv}_K(\alpha \cup \delta\chi).$$

*Proof.* Let $[L : K] = n$ and set $\mathrm{rec} = \mathrm{rec}_{L/K}$, $\gamma = \gamma_{L/K}$. By (3.2.5), $\alpha = \gamma \cup \mathrm{rec}(\alpha)$. Using the properties of the cup product, we have $\gamma \cup rec(\alpha) \cup \delta\chi = \gamma \cup \delta(\mathrm{rec}(\alpha) \cup \chi)$. $\mathbb{Q}$ is cohomologically trivial by (1.3.4), so $\delta : H_T^{-1}(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} H_T^0(G, \mathbb{Z})$. $H_T^0(G, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$, also by (1.3.4). Therefore we have $\mathrm{rec}(\alpha) \cup \chi = r/n$ for some $r \in \mathbb{Z}$. Note that by (3.3.1) we have $\mathrm{rec}(\alpha) \cup \chi = \chi(\mathrm{rec}(\alpha)) = r/n$.
$\delta(\mathrm{rec}(\alpha) \cup \chi) = r \in H^0(G, \mathbb{Z})$, so $\gamma \cup (\mathrm{rec}(\alpha) \cup \delta\chi) = r \cup \gamma$. Applying the invariant map $\mathrm{inv}_K$ to $r \cup \gamma$ we get $r/n \in \frac{1}{n}\mathbb{Z}/\mathbb{Z} \subseteq \mathbb{Q}/\mathbb{Z}$. Therefore $\mathrm{inv}_K(\gamma \cup \mathrm{rec}(\alpha) \cup \delta\chi) = \mathrm{inv}_K(\alpha \cup \delta\chi) = \chi(\mathrm{rec}(\alpha))$. □

**Proposition 3.3.3.** Let $L/K$ be unramified with $G = \mathrm{Gal}(L/K)$ and let $\mathrm{Frob} \in G$ be the Frobenius element. Let $\alpha \in K^\times$ and let $v_K(\alpha) \in \mathbb{Z}$ be its (normalised) valuation. Then

$$\mathrm{rec}_{L/K}(\alpha) = \mathrm{Frob}^{v_K(\alpha)}.$$

*Proof.* The invariant map $\mathrm{inv}_{L/K} : H^2(L/K) \to \mathbb{Q}/\mathbb{Z}$ is a composition of isomorphisms

$$H^2(L/K) \xrightarrow{v} H^2(G, \mathbb{Z}) \xrightarrow{\delta^{-1}} H^1(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\phi} \mathbb{Q}/\mathbb{Z},$$

where $v$ is induced by the valuation of $L$. This follows from the proof of (3.2.2).
Identify again $\alpha \in K^\times$ with its image in $K^\times/N_{L/K}L^\times$. $v$ shall be the valuation or the map it induces on $H^2(L/K)$ as applicable. We have $v(\alpha \cup \delta\chi) = v_L(\alpha) \cup \delta\chi$. Therefore

$$\mathrm{inv}_K(\alpha \cup \delta\chi) = \phi \circ \delta^{-1} \circ v(\alpha \cup \delta\chi) = v(\alpha) \cup \phi(\chi) = v(\alpha)\chi(\mathrm{Frob}) = \chi(\mathrm{Frob}^{v(\alpha)}).$$

Now we apply (3.3.2) to see $\chi(\mathrm{rec}_{L/K}(\alpha)) = \chi(\mathrm{Frob}^{v(\alpha)})$ for any character $\chi$ of $G$. The valuations of $L$ and $K$ agree since $L/K$ is unramified, so we get $\mathrm{rec}_{L/K}(\alpha) = \mathrm{Frob}^{v_K(\alpha)}$. □

The following is the special case of local Tate duality we will eventually need for our proof of the Grunwald-Wang theorem. It holds in greater generality, namely for an arbitrary $\mathbb{Z}$-finitely generated $G_K$-module $M$ and its dual $M^\vee$, but we will only prove it for $M = \mathbb{Z}/n\mathbb{Z}$ and $M^\vee = \mu_n$, where $\mu_n$ denotes the group of $n$-th roots of unity.

**Theorem 3.3.4.** (Local Tate Duality)
Let $K$ be a local field. There exists a perfect pairing of finite groups

$$H^1(G_K, \mathbb{Z}/n\mathbb{Z}) \times H^1(G_K, \mu_n) \to H^2(G_K, \mu_n) \xrightarrow{\mathrm{inv}_K} \mathbb{Q}/\mathbb{Z},$$

given by $(\chi, \alpha) \mapsto \chi(\mathrm{rec}_K(\alpha))$. It induces isomorphisms

$$H^1(G_K, \mathbb{Z}/n\mathbb{Z}) \cong H^1(G_K, \mu_n)^\vee \quad \text{and} \quad H^1(G_K, \mu_n) \cong H^1(G_K, \mathbb{Z}/n\mathbb{Z})^\vee.$$

*Proof.* We begin with the cup product induced by $\mathbb{Z} \times \mathrm{Hom}(\mathbb{Z}, \overline{K}^\times) \to \overline{K}^\times$. It gives a pairing

$$H^2_T(G_K, \mathbb{Z}) \times H^0_T(\overline{K}/K) \xrightarrow{\cup} H^2(\overline{K}/K) \xrightarrow{\mathrm{inv}} \mathbb{Q}/\mathbb{Z}.$$

The subgroups of finite index in $K^\times$ intersect trivially and by (3.2.6) so do the norm groups. Therefore we have $H^0_T(\overline{K}/K) = K^\times$. By (1.3.4), $H^1(G_K, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G_K, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\delta} H^2(G_K, \mathbb{Z})$ is an isomorphism. Note that this group is locally compact because $G_K$ is. Now the pairing reads

$$\mathrm{Hom}(G_K, \mathbb{Q}/\mathbb{Z}) \times K^\times \to \mathbb{Q}/\mathbb{Z}, \quad (\chi, \alpha) \mapsto \mathrm{inv}(\alpha \cup \delta\chi).$$

Now let $\chi \in \mathrm{Hom}(G_K, \mathbb{Z}/n\mathbb{Z}) \subseteq \mathrm{Hom}(G_K, \mathbb{Q}/\mathbb{Z})$. Let $L$ be the field fixed by $\mathrm{Ker}(\chi)$. Then $L/K$ is a finite cyclic extension of degree dividing $n$. Obviously $\mathrm{rec}_{L/K}((K^\times)^n) \cong (K^\times)^n$ lies in the Kernel of $\chi$, and by applying (3.3.2) we get a pairing

$$\mathrm{Hom}(G_K, \mathbb{Z}/n\mathbb{Z}) \times K^\times/(K^\times)^n \to \mathbb{Z}/n\mathbb{Z}$$

given by $(\chi, \alpha) \mapsto \chi(\mathrm{rec}_K(\alpha))$.
From the cohomology sequences of $0 \to \mu_n \to \overline{K}^\times \xrightarrow{\cdot n} \overline{K}^\times \to 0$, we get that $K^\times/(K^\times)^n \cong H^1(G_K, \mu_n)$ and in particular that $H^1(G_K, \mu_n)$ is finite and abelian. Therefore by character theory we can construct a (non-canonical) isomorphism $H^1(G_K, \mu_n) \cong H^1(G_K, \mu_n)^\vee$. Once we have constructed the isomorphisms in the statement of the theorem, this will show that $H^1(G_K, \mathbb{Z}/n\mathbb{Z})$ is finite too.
Since $\mathrm{Hom}(G_K, \mathbb{Z}/n\mathbb{Z}) = H^1(G_K, \mathbb{Z}/n\mathbb{Z})$, we have constructed the required pairing. Let us now show that it is perfect.
Assume that $(\chi, \alpha) = 0$ for all $\alpha \in K^\times/(K^\times)^n$. By (3.3.2) and (3.2.4), $(\chi, \alpha) = \chi(\mathrm{rec}_K(\alpha)) = 0$.

So $\chi$ is zero on $\operatorname{rec}_K(K^\times)$ which is dense in $\operatorname{Gal}(\overline{K}/K)^{ab}$. But $\chi$ is continuous, so it must be zero on all of $\operatorname{Gal}(\overline{K}/K)^{ab}$. Therefore $\operatorname{Hom}(G_K, \mathbb{Z}/n\mathbb{Z}) \to \operatorname{Hom}(K^\times/(K^\times)^n, \mathbb{Z}/n\mathbb{Z})$ is injective.

$(K^\times)^n$ is finite index in $K^\times$, so by the Existence Theorem (3.2.6) there is a finite extension $L/K$ such that $\operatorname{Gal}(L/K) \cong K^\times/(K^\times)^n$. Let $\pi$ be the quotient map $\pi : G_K \to \operatorname{Gal}(L/K)$. Let $\chi \in \operatorname{Hom}(K^\times/(K^\times)^n, \mathbb{Z}/n\mathbb{Z})$. Then $\chi \circ \operatorname{rec}_{L/K} \circ \pi$ lies in $\operatorname{Hom}(G_K, \mathbb{Z}/n\mathbb{Z})$. So $\operatorname{Hom}(G_K, \mathbb{Z}/n\mathbb{Z}) \to \operatorname{Hom}(K^\times/(K^\times)^n, \mathbb{Z}/n\mathbb{Z})$ is in fact an isomorphism, as required.

The second isomorphism follows from Pontryagin duality, (1.4.20). One can check that this is indeed the isomorphism induced by the pairing. □

**Proposition 3.3.5.** Let $\Gamma = \operatorname{Gal}(K^{nr}/K)$. We have $H^1(\Gamma, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$.

*Proof.* By (3.1.1), $\Gamma \cong \widehat{\mathbb{Z}}$, which has trivial $\mathbb{Z}/n\mathbb{Z}$-action. Therefore $H^1(\Gamma, \mathbb{Z}/n\mathbb{Z}) \cong \operatorname{Hom}(\widehat{\mathbb{Z}}, \mathbb{Z}/n\mathbb{Z})$. Because these homomorphisms are continuous, we see that they must be induced by homomorphisms $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ as $\mathbb{Z}$ lies densely in $\widehat{\mathbb{Z}}$. But $\operatorname{Hom}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$. □

**Proposition 3.3.6.** Assume that the residue characteristic of $K$ does not divide $n$. Then $H^1(\Gamma, \mu_n) \cong U_K(K^\times)^n/(K^\times)^n$.

*Proof.* $U_K(K^\times)^n/(K^\times)^n \cong U_K/(U_K \cap (K^\times)^n) = U_K/U_K^n$. We have $\mu_n \subseteq U_{K^{nr}}$. The long exact sequence of cohomology of $0 \to \mu_n \to U_{K^{nr}} \xrightarrow{\cdot n} U_{K^{nr}} \to 0$ yields the result since $U_{K^{nr}}^\Gamma = U_K$. □

**Proposition 3.3.7.** Assume again that the residue characteristic of $K$ does not divide $n$. With respect to the pairing in (3.3.4), $H^1_{nr}(K, \mu_n) := H^1(\Gamma, \mu_n) \subseteq H^1(G_K, \mu_n)$ is the orthogonal complement of $H^1_{nr}(K, \mathbb{Z}/n\mathbb{Z}) := H^1(\Gamma, \mathbb{Z}/n\mathbb{Z} \subseteq H^1(G_K, \mathbb{Z}/n\mathbb{Z})$.

*Proof.* $\chi \in H^1(\Gamma, \mathbb{Z}/n\mathbb{Z})$ tells us that $\chi$ factors as $\chi : G_K \to \Gamma \to \mathbb{Z}/n\mathbb{Z}$. $H^1(\Gamma, \mu_n) = U_K(K^\times)^n/(K^\times)^n$ by (3.3.6). Let $\alpha \in U_K(K^\times)^n/(K^\times)^n$ be represented by $uc^n$ for $u \in U_K$, $c \in K^\times$.

Let $L$ be the field fixed by $\operatorname{Ker}(\chi)$. Then by (3.3.2), $\operatorname{inv}_K(\alpha \cup \delta\chi) = \chi(\operatorname{rec}_{L/K}(\alpha)) = \chi(\operatorname{rec}_{L/K}(uc^n)) = \chi(\operatorname{rec}_{L/K}(u))\chi(\operatorname{rec}_{L/K}(c))^n$. Because $L/K$ is unramified, we can apply (3.3.3) to see that $\chi(\operatorname{rec}_{L/K}(\alpha)) \equiv 0 \mod n\mathbb{Z}$.

Conversely assume that for $\chi \in H^1(\Gamma, \mathbb{Z}/n\mathbb{Z})$, we have $(\chi, \alpha) = \chi(\operatorname{rec}_{L/K}(\alpha)) = \chi(\operatorname{Frob}^{v(\alpha)}) = 0 \mod n\mathbb{Z}$. This shows at once that $\alpha \in U_K(K^\times)^n/(K^\times)^n$.

Therefore $H^1(\Gamma, \mathbb{Z}/n\mathbb{Z})^\perp = H^1(\Gamma, \mu_n)$ as required. □

# 4 Global Theory

This chapter will show that the so-called idèle class group $C$ of a global field $K$ constitutes a formation module for the absolute Galois group $G_K$ of $K$, where $G_K = \mathrm{Gal}(\overline{K}/K)$ for a fixed separable closure $\overline{K}$ of $K$. The main theorem of global class field theory then follows from the abstract results of Chapter 2.

## 4.1 The Idèle Group

**Definition 4.1.1.** Define an *idèle* $\alpha = (\alpha_{\mathfrak{p}})$ to be a family of elements $\alpha_{\mathfrak{p}} \in K_{\mathfrak{p}}$ such that $\alpha_{\mathfrak{p}}$ lies in $\mathcal{O}_K^*$ for almost all $\mathfrak{p}$. Equivalently, $\alpha$ has non-trivial $\mathfrak{p}$-adic valuation for only finitely many $\mathfrak{p}$.

**Definition 4.1.2.** The *idèle group* $I_K$ is the restricted product

$$I_K = \prod_{\mathfrak{p}} K_{\mathfrak{p}}^*$$

with respect to the unit groups $\mathcal{O}_{\mathfrak{p}}^*$. This means that for almost all (i.e. all but finitely many) primes $\mathfrak{p}$ the values of an idèle $(\alpha_{\mathfrak{p}})$ lie in $\mathcal{O}_{\mathfrak{p}}^*$.

Since $K \subseteq K_{\mathfrak{p}}$ for all $\mathfrak{p}$, we can embed $K^*$ into $I_K$ diagonally. Unique prime factorisation in $\mathcal{O}_K$ ensures that this map is well-defined.

**Definition 4.1.3.** We can thus view $K^*$ as a subgroup of $I_K$ and call its elements the *principal idèles.*
The quotient

$$C_K = I_K/K^\times$$

is called the *idèle class group* of $K$.

**Proposition 4.1.4.** We can endow $C_K$ with a topology under which it is locally compact.

*Proof.* See [6], V.4. □

**Definition 4.1.5.** When $L/K$ is finite and separable, there is a natural injection $I_K \hookrightarrow I_L$. The direct limit

$$I = \varinjlim_{L/K} I_L$$

is the *idèle group of* $\overline{K}$. The *idèle class group of* $\overline{K}$ is

$$C := I/\overline{K}^\times = \varinjlim_{L/K} C_K.$$

**Proposition 4.1.6.** $I$ and $C$ have *Galois descent*, which means that $I_L^{\mathrm{Gal}(L/K)} = I_K$ and $C_L^{\mathrm{Gal}(L/K)} = C_K$. In particular, $C_K = C^{G_K}$.

*Proof.* See [7], VI.2.1 for the claim about the idèle group. The second claim can be deduced as follows. The exactness of $1 \to L^\times \to I_L \to C_L \to 1$ can be verified easily. In fact, the sequence $1 \to (L^\times)^G \to I_L^G \to C_L^G \to 1$ is exact as well: the functor $(-)^G$ is left exact, so we only need to prove exactness at $C_L^G$. This follows from an application of (1.4.15). □

## 4.2 Global Class Field Theory

In this section we set $H_T^r(L/K) = H_T^r(\mathrm{Gal}(L/K), C_L)$ for a Galois extension of global fields $L/K$.

Let $L/K$ be a Galois extension of global fields and let $\mathfrak{P}, \mathfrak{P}'$ be primes of $L$ lying above the prime $\mathfrak{p} \in K$. Then the local extensions $L_\mathfrak{P}/K_\mathfrak{p}$ and $L_{\mathfrak{P}'}/K_\mathfrak{p}$ are isomorphic. We will leave the choice of prime above $\mathfrak{p}$ implicit and write instead $L_\mathfrak{p}/K_\mathfrak{p}$ for any such extension.

The following theorem is the cornerstone of global class field theory. Its proof is long and very intricate.

**Theorem 4.2.1.** (Class Field Axiom)
Let $L/K$ be a finite cyclic extension. Then

$$|H_T^r(L/K)| = \begin{cases} [L:K] & r = 0 \\ 0 & r = -1. \end{cases}$$

*Proof.* See [7], VI.3-4. □

An immediate consequence is the following local-global principle.

**Theorem 4.2.2.** (Hasse Norm Theorem)
Let $L/K$ be a finite cyclic extension. An element $x \in K^\times$ is a norm if and only if it is a norm locally everywhere, that is, a norm in every completion $L_\mathfrak{p}/K_\mathfrak{p}$.

*Proof.* Set $G = \mathrm{Gal}(L/K)$ and $G_\mathfrak{p} = \mathrm{Gal}(L_\mathfrak{p}/K_\mathfrak{p})$. The exact sequence $1 \to L^\times \to I_L \to C_L \to 1$ gives an exact sequence

$$0 \to H_T^0(G, L^\times) \to H_T^0(G, I_L).$$

By [7], VI.3.2. we have $H_T^0(G, I_L) = \bigoplus_\mathfrak{p} H_T^0(G_\mathfrak{p}, L_\mathfrak{p}^\times)$. Therefore the homomorphism

$$K^\times / N_{L/K} L^\times \to \bigoplus_\mathfrak{p} K_\mathfrak{p}^\times / N_{L_\mathfrak{p}/K_\mathfrak{p}} L_\mathfrak{p}^\times$$

is injective, which proves the theorem. □

This can be used to deduce the Hasse-Minkowski theorem; see [3] Exercise 3 or [7] VI.4, Exercises 3-5.

The Hasse Norm theorem, paired with (1.3.5), is an ingredient in the proof of the following more general local-global principle.

**Theorem 4.2.3.** (Hasse Principle for $H^2$)
There is an exact sequence

$$0 \to H^2(\overline{K}/K) \to \bigoplus_{\mathfrak{p}} H^2(G_{K_\mathfrak{p}}, \overline{K}_\mathfrak{p}^\times) \xrightarrow{\mathrm{inv}_K} \mathbb{Q}/\mathbb{Z} \to 0,$$

where $\mathrm{inv}_K$ is the sum of the local invariant maps $\mathrm{inv}_{K_\mathfrak{p}} : H^2(G_{K_\mathfrak{p}}, \overline{K}_\mathfrak{p}^\times) \to \mathbb{Q}/\mathbb{Z}$.

*Proof.* See [8], VIII.1.16. □

**Proposition 4.2.4.** The sequence

$$0 \to H^2(G_K, \overline{K}^\times) \to H^2(G_K, I) \to H^2(\overline{K}/K) \to 0$$

is exact, yielding a canonical isomorphism

$$H^2(\overline{K}/K) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z},$$

which in turn induces isomorphisms $\mathrm{inv}_{L/K} : H^2(L/K) \xrightarrow{\sim} \frac{1}{[L:K]}\mathbb{Z}/\mathbb{Z}$.

*Proof.* See [8], VIII.1.20. □

Another consequence of the Class Field Axiom (4.2.1) is the following.

**Proposition 4.2.5.** $H^1(\overline{K}/K) = 0$. Therefore $C = C_{\overline{K}}$ is a class module for the Galois group of any finite extension $L/K$.

*Proof.* See [8], VIII.1.12. □

**Theorem 4.2.6.** Furthermore, the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & H^2(L/K) \arrow[r] \arrow[d, "\mathrm{inv}_{L/K}"] & H^2(\overline{K}/K) \arrow[r, "\mathrm{Res}"] \arrow[d, "\mathrm{inv}_K"] & H^2(\mathrm{Gal}(\overline{K}/L), C) \arrow[r] \arrow[d, "\mathrm{inv}_L"] & 0 \\
0 \arrow[r] & \frac{1}{[L:K]}\mathbb{Z}/\mathbb{Z} \arrow[r] & \mathbb{Q}/\mathbb{Z} \arrow[r, "L:K"] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

This allows us to conclude that $(G, C)$ is a class formation in the sense of (2.1.8).

*Proof.* See [8], VIII.1.10. □

Now the main theorem of this section immediately follows from (2.1.4) and (2.1.10).

**Theorem 4.2.7.** (Main Theorem of Global Class Field Theory)
Let $L/K$ be a finite Galois extension of global fields with Galois group $G = \mathrm{Gal}(L/K)$. Then the cup product

$$H_T^{-2}(G,\mathbb{Z}) \times H_T^2(L/K) \xrightarrow{\cup} H_T^0(L/K)$$

induces a canonical isomorphism

$$\mathrm{rec}_{L/K} : C_K/N_{L/K}C_L \cong G^{ab}.$$

Furthermore we obtain a reciprocity homomorphism

$$\mathrm{rec}_K : C_K \to G_K^{ab}$$

with dense image and kernel $N_{G_K}C := \bigcap_L N_{L/K}C_L$. Lastly we have $\mathrm{rec}_K|_L = \mathrm{rec}_{L/K}$.

There is a strong link between the local and global reciprocity maps as the following three theorems will illustrate.

**Theorem 4.2.8.** Let $L/K$ be an abelian extension of global fields and let $\mathfrak{p}$ be a prime of $K$. Then the following diagram commutes, where the left and right hand side vertical maps are given by inclusion into the idèle group composed with the projection onto $C_K$ and by inclusion, respectively.

$$\begin{tikzcd}
K_\mathfrak{p}^\times \arrow[r, "\mathrm{rec}_{L_\mathfrak{p}/K_\mathfrak{p}}"] \arrow[d] & \mathrm{Gal}(L_\mathfrak{p}/K_\mathfrak{p}) \arrow[d] \\
C_K \arrow[r, "\mathrm{rec}_{L/K}"] & \mathrm{Gal}(L/K)
\end{tikzcd}$$

*Proof.* See [7], VI.5.6. □

**Theorem 4.2.9.** Let $L/K$ be an abelian extension of global fields and let $\alpha = (\alpha_\mathfrak{p}) \in I_K$. Then

$$\mathrm{rec}_{L/K}(\alpha) = \prod_\mathfrak{p} \mathrm{rec}_{L_\mathfrak{p}/K_\mathfrak{p}}(\alpha_\mathfrak{p}).$$

*Proof.* Almost all local extensions $L_\mathfrak{p}/K_\mathfrak{p}$ are unramified and an idèle $\alpha$ lies in $\mathcal{O}_{K_\mathfrak{p}}$ for almost all $\mathfrak{p}$. Therefore we can use (3.3.3) to conclude that the product is finite. Alternatively one could use the fact that for an unramified local extension the norm map restricted to $\mathcal{O}_{L_\mathfrak{p}}^\times$ surjects onto $\mathcal{O}_{K_\mathfrak{p}}^\times$, see [6], III.1.2.
$I_K$ is topologically generated by idèles of the form $\alpha = (a_\mathfrak{p})$ for $a_\mathfrak{p}$ in $K_\mathfrak{p}^\times$, so it is enough to prove the theorem for these idèles. But this is just the statement of (4.2.8). Now extend by continuity to all of $I_K$. □

**Theorem 4.2.10.** Let $L/K$ be a finite abelian extension of global fields. Let $N$ and $N_\mathfrak{p}$ denote the global and local norm maps, respectively. Then we have

$$NC_L \cap K_\mathfrak{p}^\times = N_\mathfrak{p}L_\mathfrak{p}^\times.$$

*Proof.* This can be deduced from (4.2.8) and (4.2.9). See [8], VI.5.8. □

**Theorem 4.2.11.** (Global Existence Theorem)
For $L/K$ abelian, the norm groups $N_{L/K}C_L$ are exactly the closed finite index subgroups of $C_K$.
Furthermore, setting $N_L = N_{L/K}L^\times$, we have

- $L_1 \subseteq L_2 \Longleftrightarrow N_{L_1} \supseteq N_{L_2}$,
- $N_{L_1L_2} = N_{L_1} \cap N_{L_2}$, and
- $N_{L_1 \cap L_2/K} = N_{L_1}N_{L_2}$,

in complete analogy with the local case.

*Proof.* See [7], VI.6.1. □

# 5 The Grunwald-Wang Theorem

We shall write $H^1(K, A)$ to mean $H^1(G_K, A)$ for $K$ a local or global field and $A$ a $G_K$-module. The aim of this chapter is to show that the Grunwald-Wang theorem follows from the surjectivity of

$$\mathrm{Res} : H^1(K, \mathbb{Z}/n\mathbb{Z}) \to \prod_{\mathfrak{p} \in S} H^1(K_\mathfrak{p}, \mathbb{Z}/n\mathbb{Z}).$$

We will use the global Tate pairing to relate the surjectivity of this map to the injectivity of

$$\mathrm{Res} : H^1(K, \mu_n) \to \prod_{\mathfrak{p} \notin S} H^1(K_\mathfrak{p}, \mathbb{Z}/n\mathbb{Z})^\vee$$

## 5.1 Global Tate Duality

**Proposition 5.1.1.** (Global Tate Pairing)
Let $K$ be a global field. There is a perfect pairing of locally compact groups

$$\prod_{\mathfrak{p}} H^1(K_\mathfrak{p}, \mathbb{Z}/n\mathbb{Z}) \times \prod_{\mathfrak{p}} H^1(K_\mathfrak{p}, \mu_n) \to \mathbb{Z}/n\mathbb{Z}.$$

The products are understood to be restricted products over all primes $\mathfrak{p}$ of $K$, restricted with respect to the subgroups $H^1_{nr}(K_\mathfrak{p}, \mathbb{Z}/n\mathbb{Z})$, resp. $H^1_{nr}(K_\mathfrak{p}, \mu_n)$, see (3.3.7). This means that an element of the product lies in the unramified subgroups for all but finitely many $\mathfrak{p}$. For $\chi = (\chi_\mathfrak{p})$ in the first and $\alpha = (\alpha_\mathfrak{p})$ in the second product, the pairing is given by

$$(\chi, \alpha) = \sum_{\mathfrak{p}} \chi_\mathfrak{p}(\mathrm{rec}_K(\alpha_\mathfrak{p})).$$

*Proof.* By (3.3.7), $H^1_{nr}(K_\mathfrak{p}, \mu_n)$ is the orthogonal complement of $H^1_{nr}(K_\mathfrak{p}, \mathbb{Z}/n\mathbb{Z})$ whenever the residue characteristic of $K_\mathfrak{p}$ does not divide $n$. Since there are only finitely many $\mathfrak{p}$ which do divide $n$, this ensures that the sum is finite. Therefore the pairing is well-defined. Perfectness follows right away from local Tate duality, see (3.3.4). □

**Theorem 5.1.2.** The images of

$$\mathrm{Res} : H^1(K, \mathbb{Z}/n\mathbb{Z}) \to \prod_{\mathfrak{p}} H^1(K_\mathfrak{p}, \mathbb{Z}/n\mathbb{Z})$$

and

$$\mathrm{Res} : H^1(K, \mu_n) \to \prod_{\mathfrak{p}} H^1(K_\mathfrak{p}, \mu_n)$$

are mutual orthogonal complements with respect to the pairing above.
This is equivalent to the exactness of the sequences

$$H^1(K, \mu_n) \xrightarrow{\mathrm{Res}} \prod_{\mathfrak{p}} H^1(K_\mathfrak{p}, \mathbb{Z}/n\mathbb{Z})^\vee \to H^1(K, \mathbb{Z}/n\mathbb{Z})^\vee$$

and

$$H^1(K,\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\text{Res}} \prod_{\mathfrak{p}} H^1(K_{\mathfrak{p}},\mu_n)^\vee \to H^1(K,\mu_n)^\vee.$$

*Proof.* We will prove the theorem by showing that the first sequence is exact. The exactness of the second sequence will follow by Pontryagin duality and local Tate duality.

By the long exact sequence of cohomology of $0 \to \mu_n \to \overline{K} \xrightarrow{\cdot n} \overline{K} \to 0$ and by Hilbert's Theorem 90 we see that $H^1(K,\mu_n) = K^\times/(K^\times)^n$, analogously to the local case.

By local Tate duality, $H^1(K_{\mathfrak{p}},\mathbb{Z}/n\mathbb{Z})^\vee \cong H^1(K_{\mathfrak{p}},\mu_n)$, and so the restriction map becomes $\text{Res} : K^\times/(K^\times)^n \to \prod_{\mathfrak{p}} K_{\mathfrak{p}}^\times/(K_{\mathfrak{p}}^\times)^n$. Therefore Coker(Res) equals $C_K/C_K^n$, where $C_K^n = I_K^n K^\times/K^\times$ is the norm group $N_{L/K}C_L$ of the maximal abelian extensions $L/K$ of exponent $n$. Let us now show that $C_K/C_K^n$ is isomorphic to $H^1(K,\mathbb{Z}/n\mathbb{Z})^\vee$. By (4.2.7), $\text{Gal}(L/K) \cong C_K/C_K^n$. But we also have

$$H^1(K,\mathbb{Z}/n\mathbb{Z}) = \text{Hom}(G_K,\mathbb{Z}/n\mathbb{Z}) \cong \text{Hom}(G_K/G_K^n,\mathbb{Q}/\mathbb{Z}) \cong \text{Hom}(C_K/C_K^n,\mathbb{Q}/\mathbb{Z}).$$

By definition, $\text{Hom}(C_K/C_K^n,\mathbb{Q}/\mathbb{Z}) = (C_K/C_K^n)^\vee$, so finally

$$H^1(K,\mathbb{Z}/n\mathbb{Z})^\vee \cong ((C_K/C_K^n)^\vee)^\vee \cong C_K/C_K^n,$$

as required. By going through the definitions one can check that this isomorphism, due to Pontryagin duality, is canonical, so we indeed have $\text{Coker(Res)} = H^1(K,\mathbb{Z}/n\mathbb{Z})^\vee$. □

**Definition 5.1.3.** Let $K$ be a global field and let $S$ be a finite set of primes of $K$. Define $Ш(K,\mu_n)$ and $Ш(K,\mathfrak{p}\notin S,\mu_n)$ by the exactness of

$$0 \to Ш(K,\mu_n) \to H^1(K,\mu_n) \xrightarrow{\text{Res}} \prod_{\mathfrak{p}} H^1(K_{\mathfrak{p}},\mu_n)$$

and

$$0 \to Ш(K,\mathfrak{p}\notin S,\mu_n) \to H^1(K,\mu_n) \xrightarrow{\text{Res}} \prod_{\mathfrak{p}\notin S} H^1(K_{\mathfrak{p}},\mu_n),$$

respectively.

We say that the Hasse principle holds for all $\mathfrak{p}$, resp. for $\mathfrak{p}\notin S$ when $Ш(K,\mu_n)$, resp. $Ш(K,\mathfrak{p}\notin S,\mu_n)$ is zero.

**Proposition 5.1.4.** The following diagram, where $\text{Coker}^1(K,S,\mathbb{Z}/n\mathbb{Z})$ is defined to be the cokernel of $\text{Res} : H^1(K,\mathbb{Z}/n\mathbb{Z}) \to \prod_{\mathfrak{p}\in S} H^1(K_{\mathfrak{p}},\mathbb{Z}/n\mathbb{Z})$, is commutative and exact.

$$\begin{tikzcd}
 & & 0 & \\
Ш^1(K, \mathfrak{p} \notin S, \mu_n) \arrow[r, hook] & H^1(K, \mu_n) \arrow[r] & \prod_{\mathfrak{p} \notin S} H^1(K_\mathfrak{p}, \mu_n) \arrow[u] & \\
Ш^1(K, \mu_n) \arrow[r, hook] \arrow[u, hook] & H^1(K, \mu_n) \arrow[r] \arrow[u, equal] & \prod_{\mathfrak{p}} H^1(K_\mathfrak{p}, \mu_n) \arrow[r] \arrow[u] & H^1(K, \mathbb{Z}/n\mathbb{Z})^\vee \\
 & & \prod_{\mathfrak{p} \in S} H^1(K_\mathfrak{p}, \mu_n) \arrow[r, "\sim"] \arrow[u] & \prod_{\mathfrak{p} \in S} H^1(K_\mathfrak{p}, \mathbb{Z}/n\mathbb{Z})^\vee \arrow[u] \\
 & & 0 \arrow[u] & \mathrm{Coker}^1(K, S, \mathbb{Z}/n\mathbb{Z})^\vee \arrow[u] \\
 & & & 0 \arrow[u]
\end{tikzcd}$$

*Proof.* The commutativity of the left square is straightforward. In the middle square, the top and bottom horizontal maps are $\prod_{\mathfrak{p} \notin S} \mathrm{Res}$ and $\prod_{\mathfrak{p}} \mathrm{Res}$ respectively. Commutativity follows. To show that the bottom right hand square commutes, we can use the fact that $\prod_{\mathfrak{p}} H^1(K_\mathfrak{p}, \mu)$ is canonically isomorphic to $\prod_{\mathfrak{p}} H^1(K_\mathfrak{p}, \mathbb{Z}/n\mathbb{Z})^\vee$ by local Tate duality. The top horizontal map is equal to the composition of this isomorphism with $\mathrm{Res}^\vee$. Since the right hand vertical map is also equal to $\mathrm{Res}^\vee$, the square commutes. The exactness of the 4-term sequence follows from (5.1.2) and the isomorphism given by the bottom right hand arrow follows from local Tate duality. Exactness at the remaining places follows by definition. □

**Proposition 5.1.5.** There exists an exact sequence

$$0 \to Ш^1(K, \mu_n) \to Ш^1(K, \mathfrak{p} \notin S, \mu_n) \to \mathrm{Coker}^1(K, S, \mathbb{Z}/n\mathbb{Z})^\vee \to 0.$$

*Proof.* Apply the snake lemma to the above diagram. □

## 5.2 The Grunwald-Wang Theorem

**Proposition 5.2.1.** If the map

$$H^1(K, \mu_n) \to \prod_{\mathfrak{p} \notin S} H^1(K_\mathfrak{p}, \mu_n)$$

is injective, then the map

$$H^1(K, \mathbb{Z}/n\mathbb{Z}) \to \prod_{\mathfrak{p} \in S} H^1(K_\mathfrak{p}, \mathbb{Z}/n\mathbb{Z})$$

must be surjective.

*Proof.* This follows immediately from (5.1.5). □

Naturally we will ask when it is the case that the upper restriction map in (5.2.1) is injective. This question is interesting in its own right: for a local or global field $K$ we have $H^1(K, \mu_n) \cong K^\times/(K^\times)^n$, so the restriction map becomes

$$K^\times/(K^\times)^n \to \prod_{\mathfrak{p}\notin S} K_\mathfrak{p}^\times/(K_\mathfrak{p}^\times)^n.$$

Now we can reformulate the question as follows:

Let $K$ be a global field and assume that $\alpha \in K$ is an $n$-th power locally for all primes except possibly for the finitely many primes in $S$. Does it follow that $\alpha$ is an $n$-th power in $K$?

We will now see that the answer is yes for most global fields $K$, except for those which constitute the so-called *special case*. In the original 1933 proof of what is now the Grunwald-Wang theorem, Wilhelm Grunwald failed to notice this special case altogether. This was remedied by Shianghao Wang who published a corrected proof in 1948.

**Theorem 5.2.2.** Let $K$ be a global field. The map

$$H^1(K, \mu_n) \to \prod_{\mathfrak{p}\notin S} H^1(K_\mathfrak{p}, \mu_n)$$

is injective, i.e. $Ш^1(K, \mathfrak{p} \notin S, \mu_n) = 0$, except in the special case, which is characterised as follows:

- $K$ is a number field,
- $-1$, $2+\eta_r$ and $-(2+\eta_r)$ are not squares in $K$, where $\eta_r = \zeta_{2^r} + \zeta_{2^r}^{-1}$ for $\zeta_{2^r}$ a primitive $2^r$-th root of unity, and we furthermore have $\eta_r \in K$ but $\eta_{r+1} \notin K$,
- $n = 2^t m$, where $m$ is odd and $t > r$, and
- $S_0 \subseteq S$, where $S_0$ is the set of those primes $\mathfrak{p}$ lying above 2 for which $-1$, $2+\eta_r$, and $-(2+\eta_r)$ are not squares in $K_\mathfrak{p}$.

We denote such a special case by $(K, n, S)$.
In the special case, $Ш^1(K, \mathfrak{p} \notin S, \mu_n)$ is of order 2. Therefore it holds for any global field $K$ that an element that is a $2k$-th power locally everywhere must at least be a $k$-th power globally.

*Proof.* For the cases of $K$ a function field or $K$ a number field with $K(\zeta_{2^t})/K$ is cyclic, [1], IX.1 gives a clear account. The proof is long but uses only elementary algebraic number theory, except for the following fact: if a normal extension of global fields $L/K$ satisfies $L_\mathfrak{p} = K_\mathfrak{p}$ for all but finitely many primes $\mathfrak{p} \in K$, then $L = K$. This follows from global class field theory, see [7], VI.3.8 or [1], V.2.
When $K(\zeta_{2^t})/K$ is not cyclic, the result follows by carefully considering different cases and through tedious but elementary calculations. See [1], X.1. □

In fact, an element that is an $n$-th power locally almost everywhere can fail to be an $n$-th power globally in two distinct ways:

Either (Case 1) an element might be an $n$-th power locally almost everywhere but not an $n$-th power locally everywhere, or (Case 2) it might be an $n$-th power locally everywhere but not an $n$-th power globally. We give an example for each case.

Case 1: The Hasse principle can fail even for $K = \mathbb{Q}$. We have $\eta_2 = 0 \in \mathbb{Q}$ but $\eta_3 = \sqrt{2} \notin \mathbb{Q}$. Clearly $-1, 2 + \eta_2 = 2$, and $-(2 + \eta_2) = -2$ are not squares in $\mathbb{Q}$. Therefore set $n = 2^3 \cdot 1 = 8$. Trivially, the only prime above 2 is 2. By Hensel's lemma, $x$ is a square in $\mathbb{Q}_2$ if and only if $x$ is a square modulo 8. Therefore $-1, 2$, and $-2$ are non-squares in $\mathbb{Q}_2$ and $(\mathbb{Q}, 8, \{2\})$ is in the special case.
16 is not an 8-th power in $\mathbb{Q}_2$, since its 2-adic valuation is $v_2(16) = 4$, which is not divisible by 8. Now let $p \neq 2$. 16 is an 8-th power in $\mathbb{Q}_p$ if and only if $X^8 - 16$ has a root in $\mathbb{Q}_p$. Factorising $X^8 - 16$ as $(X^2 - 2)(X^2 + 2)(X^2 - 2X + 2)(X^2 + 2X + 2)$, we see that this is equivalent to requiring that one of $2, -2$, and $-1$ be a square in $\mathbb{Q}_p$. Modulo $p$ one of them is certainly a square due to the multiplicativity of the Legendre symbol. Now we only need to apply Hensel's lemma to see that 16 is an 8-th power in $\mathbb{Q}_p$ for all $p \neq 2$. Lastly, 16 is clearly an 8-th power in $\mathbb{R}$.
Thus the restriction map $\mathbb{Q}^\times/(\mathbb{Q}^\times)^8 \to \prod_{p \neq 2} \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^8$ is not injective as it contains 16 in its kernel.

Case 2: Elementary methods show that 16 is not an 8-th power in $\mathbb{Q}(\sqrt{7})$. But 16 is an 8-th power locally everywhere: this follows from the example above for $p \neq 2$ and from the equality $\mathbb{Q}_2(\sqrt{7}) = \mathbb{Q}_2(\sqrt{-1})$, which is a consequence of Hensel's lemma, for $p = 2$. Note that 2 is inert in $\mathbb{Q}(\sqrt{7})$ because $2 \mid 28 = \operatorname{disc}(\mathbb{Q}(\sqrt{7}))$. Since 16 is an 8-th power, $-1, 2$, and $-2$ are all squares in $\mathbb{Q}_2(\sqrt{7})$. so the fourth condition of the special case becomes vacuous for the single prime $\mathfrak{q}$ lying above 2. Thus we can even take $S = \emptyset$ and see that $\mathbb{Q}(\sqrt{7})^\times/(\mathbb{Q}(\sqrt{7})^\times)^8 \to \prod_\mathfrak{p} \mathbb{Q}(\sqrt{7})_\mathfrak{p}^\times/(\mathbb{Q}(\sqrt{7})_\mathfrak{p}^\times)^8$. is not injective.

**Theorem 5.2.3.** (Grunwald-Wang)
Let $K$ be a global field that is not in the special case as defined above. Let $S$ be a finite set of primes of $K$ and assume there are cyclic extensions $L_\mathfrak{p}/K_\mathfrak{p}$ for every prime $\mathfrak{p} \in S$. Then there exists a cyclic extension $L/K$ which has completions $L_\mathfrak{p}/K_\mathfrak{p}$ for all $\mathfrak{p} \in S$. Furthermore

$$[L : K] = \operatorname{lcm}_{\mathfrak{p} \in S} [L_\mathfrak{p}/K_\mathfrak{p}].$$

*Proof.* Let $[L_\mathfrak{p}/K_\mathfrak{p}] = n_\mathfrak{p}$ and let $n = \mathrm{lcm}_{\mathfrak{p}\in S}\, n_\mathfrak{p}$. The cyclic extension $L_\mathfrak{p}/K_\mathfrak{p}$ is defined uniquely by a local character $\chi_\mathfrak{p} \in H^1(K_\mathfrak{p}, \mathbb{Z}/n\mathbb{Z})$ with kernel $\mathrm{Gal}(\overline{K}_\mathfrak{p}/L_\mathfrak{p})$ and image $\mathbb{Z}/\frac{n}{n_\mathfrak{p}}\mathbb{Z}$. Since we are not in the special case, the restriction map

$$H^1(K, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\mathrm{Res}} \prod_{\mathfrak{p}\in S} H^1(K_\mathfrak{p}, \mathbb{Z}/n\mathbb{Z})$$

is surjective by (5.1.5). So for a tuple of local characters $(\chi_\mathfrak{p})_{\mathfrak{p}\in S}$ we can find a preimage $\chi : G_K \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$. This global character corresponds to a cyclic extension $L/K$ of degree $n$ where $L := \overline{K}^{\mathrm{Ker}\,\chi}$, the field fixed by the kernel of $\chi$. By the definition of the restriction map we have $\mathrm{Gal}(L_\mathfrak{p}/K_\mathfrak{p}) \hookrightarrow \mathrm{Gal}(L/K)$ for all $\mathfrak{p} \in S$, as required. □

## 5.3 Generalisations

The Grunwald-Wang theorem and the related theory have been generalised in various directions. We give an overview of some of those results here.

We know that in the special case, $\text{Ш}^1(K, \mathfrak{p} \notin S, \mu_n)$ is of order 2. We can use this to conclude that for an arbitrary global field $K$, given local characters as in the proof of (5.2.3), one can obtain a global character $\chi : G_K \to \mathbb{Z}/2n\mathbb{Z}$ which restricts to the local characters as required. In fact, the Grunwald-Wang theorem proper can still hold even in the special case if an additional technical condition is satisfied; see [1], X.5.

Furthermore, there exists a converse to (5.2.1).

**Proposition 5.3.1.** If the map

$$H^1(K, \mathbb{Z}/n\mathbb{Z}) \to \prod_{\mathfrak{p}\in S} H^1(K_\mathfrak{p}, \mathbb{Z}/n\mathbb{Z})$$

is surjective, then the map

$$H^1(K, \mu_n) \to \prod_{\mathfrak{p}\notin S} H^1(K_\mathfrak{p}, \mu_n)$$

is injective.

*Proof.* This can be deduced from restricting the pairing in (5.1.1) to $\prod_{\mathfrak{p}\notin S}$ for both arguments and noting that this pairing is non-degenerate. See [4], 4.2. □

The following generalises (5.1.2) from $A = \mathbb{Z}/n\mathbb{Z}$ to arbitrary finite $G_K$-modules $A$.

**Theorem 5.3.2.** (Poitou-Tate)
Let $A$ be a finite $G_K$-module. Then

$$H^1(K, A) \to \prod_{\mathfrak{p}} H^1(K, A) \to H^1(K, A^\vee)^\vee$$

is exact. In fact, exactness can be used to construct a 9-term exact sequence of cohomology groups, called the *Poitou-Tate* exact sequence.

*Proof.* See [8], VIII.6.10. □

The above theorem can be generalised further to the context of *restricted ramification*: One can consider a set $S$ of primes containing all infinite primes and replace every instance of the separable closure $\overline{K}/K$ by the maximal subextension $K_S/K$ which is unramified outside $S$. One can define $S$-idèles and an $S$-idèle class group and get an analogous theory of Galois cohomology for $G_S = \mathrm{Gal}(K_S/K)$. See [8], VIII.3-6.

The Grunwald-Wang theorem itself was generalised from prescribed cyclic extensions to essentially arbitrary abelian extensions by Jürgen Neukirch.

**Theorem 5.3.3.** (Neukirch)
Let $S$ be a finite set of primes of a global field $K$ and let $A$ be a finite abelian group. Let for all $\mathfrak{p} \in S$ finite abelian extensions $L_\mathfrak{p}/K_\mathfrak{p}$ be given such that $\mathrm{Gal}(L_\mathfrak{p}/K_\mathfrak{p})$ can be embedded into $A$. As long as we are not in a certain special case, there exists a global abelian extension $L/K$ with Galois group $A$ such that $L$ has the given completions $L_\mathfrak{p}$ for $\mathfrak{p} \in S$.

*Proof.* See [8], IX.2.8. □

There is a Grunwald-Wang type theorem for abelian varieties.

**Theorem 5.3.4.** (Creutz)
Let $A$ be an abelian variety over a number field $K$. Then there exists a constant $c = c(A, K)$ such that if $n$ is an integer divisible by no prime $p$ less than $c$ and $S$ is any finite set of primes of $K$, then the map $H^1(K, A[n]) \to \prod_{\mathfrak{p} \in S} H^1(K_\mathfrak{p}, A[n])$ is surjective.

*Proof.* See [4], 1.5. □

Lastly we mention an effective version of the Grunwald-Wang theorem due to Song Wang, who does not seem to be related to Shianghao Wang.

**Theorem 5.3.5.** (Wang)
Let $K$ be a global field and $S$ a finite set of primes of $K$. Assume we are not in the special case and assume the global character $\chi : G_K \to \mathbb{Z}/n\mathbb{Z}$ restricts to local characters $\chi_\mathfrak{p}$ for all $\mathfrak{p} \in S$. Then there exists a bound for the conductor $N(\chi)$ of $\chi$ given as a function of $K, n, S$, and $N(\chi_\mathfrak{p})$.

*Proof.* See [12], Theorem A.2. □

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