Stability and Equality in the Entropy Power Inequality and in one of its Discrete Counterparts

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Outline

Entropy Power Inequality (EPI)

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- Geometry: Brunn-Minkowski inequality

$$\mathrm{Vol}(A+B)^{\frac{1}{d}} \geq \mathrm{Vol}(A)^{\frac{1}{d}} + \mathrm{Vol}(B)^{\frac{1}{d}}, \quad A, B \subset \mathbb{R}^d$$

Entropy Power Inequality (EPI)

- Stam's proof via de Bruijn's identity [Stam '59]
- Young's inequality [Dembo, Cover, Thomas '91], [Bobkov, Chistyakov 2014]
- Optimal transport proof [Rioul 2017]

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- de Bruijn
- $\lim_{t\to 0} h(X + \sqrt{t}Z) = h(X)$

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BUT

Proposition (Bobkov, Chistyakov 2014)

There exists X *with* $h(X) = 0$ *such that* $h(X + Y) = \infty$ *for every independent* Y

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Proposition

If there exists independent Y such that $h(X+Y) < \infty$ then $h(X+Z) < \infty$ for every $Z \notin \mathcal{C}_{BC}$ independent of X

Entropy Power Inequality (EPI)

Theorem

If there exists an independent Y such that $h(X+Y)<\infty$, then

1. *de Bruijn's identity holds*
2. *$h(X+\sqrt{t}Z)$ is continuous at $t=0$.*

Therefore, if $h(X), h(Y) < \infty$, we have equality in the EPI

$$h(\sqrt{\lambda}X+\sqrt{1-\lambda}Y)=\lambda h(X)+(1-\lambda)h(Y)$$

if and only if X, Y are Gaussian with the same variance

Stability in the EPI?

$$\delta_{\mathrm{EPI}} := h(X_1 + X_2) - h(X) - \frac{1}{2}\log 2 \geq 0 \quad \text{EPI} \quad \text{for IID}$$

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- If $\delta_{\mathrm{EPI}} \to 0$, does $X_1 \Rightarrow Z \sim \mathcal{N}(\mu, \sigma^2)$? Qualitative stability
- $$d(X, Z) \leq C\delta_{\mathrm{EPI}}? \quad \text{Quantitative stability}$$

(In)stability in the EPI

- Stability fails for Wasserstein–2 and relative entropy [Courtade, Fathi, Pananjady 2018] $f(x) = \epsilon\sqrt{\frac{\epsilon}{\pi}}e^{-\frac{\epsilon x^2}{2}} + (1-\epsilon)\sqrt{\frac{1-\epsilon}{\pi}}e^{-\frac{(1-\epsilon)x^2}{2}}$

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$$D(X||Z) \leq C\delta_{\mathrm{EPI}}$$

$+$ transportation distance

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Assume X_1, X_2 IID and

$$\mathbb{E}|X_1|^k < \infty \text{ for some } k > 1.$$

Then

$$\text{for each } \epsilon > 0 \text{ there is a } \delta > 0 \text{ s.t. } \delta_{\mathrm{EPI}} < \delta$$

$$\text{implies } d_{\mathrm{L}}(X_1, G) < \epsilon,$$

for some Gaussian random variable G.

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for some Gaussian random variable G. *Equivalently if* $\delta_{\mathrm{EPI},n} \to 0$ *as* $n \to \infty$, *then*

$$X_{1,n_k} \Rightarrow \text{Gaussian} \quad \text{as } k \to \infty \quad \text{(in distribution)}$$

for any convergent subsequence X_{n_k}

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d_{L} is the Lévy metric that metrizes convergence in distribution

Discrete EPI and Stability

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BUT $H(X_1 + X_2) \geq H(X_1) + \frac{1}{2}\log 2$ fails in general

Discrete EPI and Stability

However,

Theorem (Tao, 2010)

Let X be a RV on a finite subset of a torsion-free group ($\mathbb{Z}, \mathbb{Z}^d, \mathbb{R}$, etc.) Then

$$H(X_1 + X_2) \geq H(X_1) + \frac{1}{2}\log 2 - o(1)$$

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- Additive combinatorics: $A \subset \mathbb{Z}$, $A + A := \{a_1 + a_2 : a_1, a_2 \in A\}$
  - Sumset bounds $|A| \leq |A + B| \leq |A||B|$, etc.
  - Inverse sumset theory If $|A + B| \ll |A|$, then A looks like ...

Definition

A random variable X with PMF p on $\mathbb{Z}$ is called *log-concave* if

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But includes many important distributions, e.g. Bernoulli, Binomial, Poisson, Geometric, Uniform,...

"Stability" in the discrete EPI

Relative entropy: $D(P\|Q) = \sum_x P(x) \log \frac{P(x)}{Q(x)} \geq 0$

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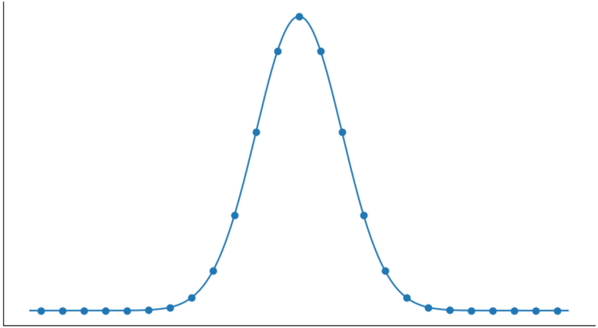
Theorem

Suppose X has a log-concave distribution on the integers. Let $Z^{(\mathbb{Z})}$ a Gaussian with the same mean and variance as X, discretised on $\mathbb{Z}$. Then there are absolute constants C_1, C_2 such that

$$D(X||Z^{(\mathbb{Z})}) \leq C_1\Big(H(X_1+X_2) - H(X_1) - \frac{1}{2}\log 2\Big) + C_2 \frac{\log\big(\mathrm{Var}(X)\big)}{\mathrm{Var}(X)}$$

Discretized Gaussian



Marton's Conjecture

Entropic Marton's Conjecture in $\mathbb{F}_2^n$:

Theorem (Gowers, Green, Manners, Tao, 2023)

If X is a RV on $\mathbb{F}_2^n$, then there is a subgroup H such that

$$H(X+U_H)-\frac{1}{2}H(U_H)-\frac{1}{2}H(X)\leq\frac{11}{2}(H(X_1+X_2)-H(X))$$

Thank you!