# Gaussian mixtures: convexity properties and CLT rates for the entropy and Fisher information

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# Outline

# Entropy

$X_1, X_2$ independent RVs

- Entropy Power Inequality (EPI)

$$h(\sqrt{1-t}X_1 + \sqrt{t}X_2) \geq (1-t)h(X_1) + th(X_2)$$

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Q: Where is it maximized?

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- EPI special case
- No special case known (of $X_1, X_2$)!

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- $g_X \propto e^{-|x|^p}$ (p- stable, $p \in (0,2]$)
- Cauchy $\propto \frac{1}{\pi(1+x^2)} \Rightarrow \frac{Z_1}{|Z_2|}$

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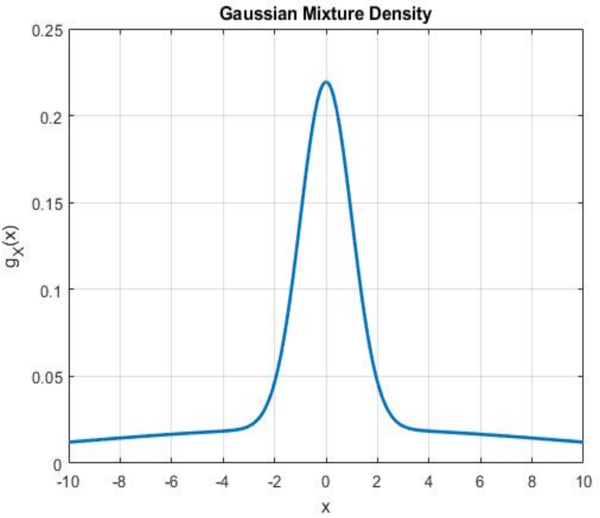


Figure: $\mathbb{P}(Y = 1) = \mathbb{P}(Y = 10) = 0.5$

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- In [Costa '85]: strengthened EPI for stable laws

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Cramér-Rao (C-R):

$$I(X) \geq \frac{1}{\mathrm{Var}(X)} \quad (d = 1)$$

# Fisher Information

## Proposition

*Let* $X = Y \cdot Z$ *a GM*, $\mathrm{Var}(X) = \mathbb{E}[Y^2]$

$$\frac{1}{\mathbb{E}[Y^2]} \overset{C-R}{\leq} I(X) \leq \mathbb{E}\left[\frac{1}{Y^2}\right]$$

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Proof.

$$f(x) = \mathbb{E}\left[\frac{1}{\sqrt{2\pi Y^2}} e^{-\frac{x^2}{2Y^2}}\right], f'(x) = -x\mathbb{E}\left[\frac{1}{Y^2\sqrt{2\pi Y^2}} e^{-\frac{x^2}{2Y^2}}\right]$$

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$$I(X) = \int \frac{(f'(x))^2}{f(x)} dx \leq \int x^2 \mathbb{E}\left[\frac{1}{Y^4\sqrt{2\pi Y^2}} e^{-\frac{x^2}{2Y^2}}\right] dx = \mathbb{E}\left[\frac{1}{Y^2}\right] \qquad \square$$

# III. Fisher Information

(GM) in $\mathbb{R}^d$: $\quad X = \mathbf{Y} \cdot Z$,
$Z \sim \mathcal{N}(0, \mathrm{I}_{d\times d})$, $\mathbf{Y} \in \mathbb{R}^{d\times d}$, symmetric, $\mathbb{P}(\mathbf{Y} \succeq 0) = 1$

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$R(x, y) = \frac{x^2}{y}$ is jointly convex on $\mathbb{R}^2_+ \Rightarrow I(X) = \int_{\mathbb{R}^d} \frac{|\nabla f(X)|^2}{f(X)} dx$ is a convex functional of $f$

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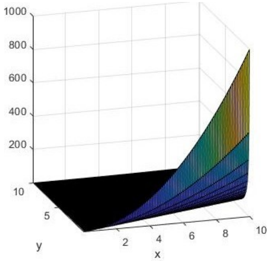

# Fisher Information

## Proposition (Bobkov, '22)

*The Fisher information satisfies "Jensen's" inequality: let $\pi$ be a probability measure on the space of probability densities in $\mathbb{R}^n$. Then, if $f$ is any mixture of densities $f = \int g\pi(dg)$,*

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## Proposition

*The Fisher information matrix is convex as a matrix-valued functional and satisfies "Jensen's" inequality: if $f$ is any mixture of densities $f = \int g\pi(dg)$,*

$$\mathcal{I}(f) \preceq \int \mathcal{I}(g)\pi(dg)$$

*provided that $\int \|\mathcal{I}(g)\|_{\mathrm{OP}} < \infty$*

# Fisher Information

*Proof.* $R(x, \lambda) = \frac{xx^T}{\lambda}$ is jointly operator convex + Bobkov's proof with $\langle \mathcal{I}(X)x, x \rangle$ $\square$

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$$\Rightarrow$$

$$\mathcal{I}(\mathbf{Y}Z) \preceq \mathbb{E}_{\mathbf{Y}}\mathcal{I}(Z_{\mathbf{Y}}) = \mathbb{E}(\mathbf{Y}\mathbf{Y}^T)^{-1},$$

where $Z_\Sigma \sim \mathcal{N}(0, \Sigma\Sigma^T)$

# CLT rates

$(a_1, \ldots, a_n) : \sum_{i=1}^n a_i^2 = 1$, $X_i = Y_i Z_i$ GMs

$$S_n = \sum_{i=1}^n a_i X_i = \sum_{i=1}^n a_i Y_i Z_i \stackrel{d}{=} \left(\sum a_i^2 Y_i^2\right)^{1/2} Z$$

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- $$I(S_n) \leq \mathbb{E}\left[\frac{1}{\sum a_i^2 Y_i^2}\right]$$

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Standardised Fisher information:

$$J_{st}(S_n) := \mathrm{Var}(S_n) I(S_n) - 1 \leq \left(\mathbb{E} Y_1^2\right) \mathbb{E}\left[\frac{1}{\sum a_i^2 Y_i^2}\right] - 1$$

# CLT rates

## Corollary

*Suppose* $\mathbb{E}Y_1^{2+2\delta}, \mathbb{E}Y_1^{-2-2\delta} < \infty$. *Then*

$$J_{st}(S_n) \leq C_{Y_1} \|a\|_{2+2\delta}^{\frac{2\delta}{(1+\delta)}}$$

# CLT rates

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- E.g. $a_i = \frac{1}{\sqrt{n}}$, then

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- $O(\|a\|_4^4)$ in [Artstein, Ball, Barthe, Naor, '04] for weighted sums under much stronger moment assumptions

# CLT rates

**Corollary (General dimension)**

*If* $\mathbb{E}\|\mathbf{Y}\mathbf{Y}^T\|_{\mathrm{OP}}^{1+\delta} < \infty$ *and* $\mathbb{E}\|\left(\mathbf{Y}\mathbf{Y}^T\right)^{-1}\|_{\mathrm{OP}}^{1+\delta} < \infty$,

$$\|\mathrm{Cov}(S_n)^{\frac{1}{2}}\mathcal{I}(S_n)\mathrm{Cov}(S_n)^{\frac{1}{2}} - \mathrm{I}_d\|_{\mathrm{OP}} \leq C(\mathbf{Y})\log^{\delta}(d+1)\|a\|_{2+2\delta}^{\frac{2\delta}{1+\delta}}$$

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- Convergence of the Fisher information matrix
- First such result!

# An Open Question

- Is

$$I\left(\sqrt{t}X_1 + \sqrt{1-t}X_2\right)$$

minimized at $t = \frac{1}{2}$ ? More generally, is

$$I\left(\sum_{i=1}^{n} a_i X_i\right)$$

minimized at $a_i = \frac{1}{\sqrt{n}}$?

Thank you!