Dimensional discrete entropy power inequalities for log-concave random vectors

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Outline

1. The Entropy Power Inequality (EPI)
2. Discrete random variables
3. Discrete log-concavity in higher dimensions
4. An open question

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- Wide literature

EPI and the CLT

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$$h\Big(\frac{\sum_{i=1}^n X_i}{\sqrt{n}}\Big) \to h(Z) = \frac{1}{2} \log\left(2\pi e \mathrm{Var}(X_1)\right) \Rightarrow \mathrm{CLT}$$

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[Artstein, Ball, Barthe, Naor '04]:

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However,

$$H(X_1 + X_2) \geq H(X_1) + \frac{1}{2} \log 2 - o(1) \quad \text{[Tao, '10]}$$

where $o(1) \to 0$ as $H(X_1) \to \infty$

Discrete random variables

Conjecture [Tao, '10]

For any $n \geq 1$

$$H(X_1 + \ldots + X_{n+1}) \geq H(X_1 + \ldots + X_n) + \frac{1}{2}\log\left(\frac{n+1}{n}\right) - o(1)$$

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- Motivation ($n = 1$): Polar coding

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A random variable $X \sim p$ on $\mathbb{Z}$ is said to be log-concave if

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Theorem (G. '23)

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$$H\left(\sum_{i=1}^{n+1} X_i\right) \geq H\left(\sum_{i=1}^{n} X_i\right) + \frac{1}{2}\log\left(\frac{n+1}{n}\right) - O\Big(H(X_1)e^{-H(X_1)}\Big)$$

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- Discrete convex analysis $\to$ [Murota]

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Theorem

*For any i.i.d. random vectors $X_1, \dots, X_{n+1}$ on $\mathbb{Z}^d$ with *almost isotropic extension, such that both $\sum_{i=1}^{n} X_i$ and $\sum_{i=1}^{n+1} X_i$ are log-concave*

$$H(X_1 + \cdots + X_{n+1}) \geq H(X_1 + \cdots + X_n) + \frac{d}{2} \log\left(\frac{n+1}{n}\right) - o(1)$$

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- More general EPIs analogous to [Madiman, Barron, '07]

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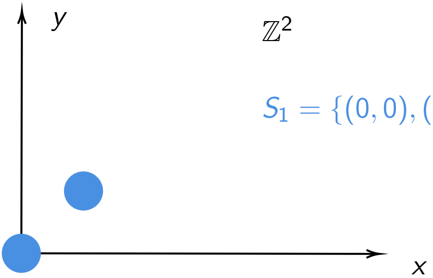

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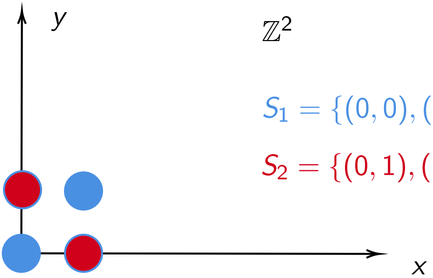

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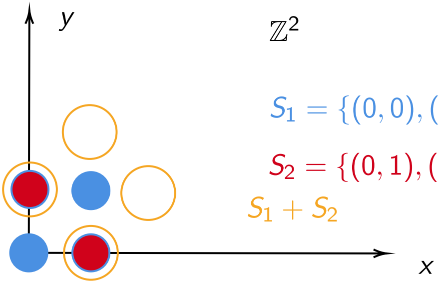

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Is log-concavity preserved under self-convolution?

Thank you!