Dimensional discrete entropy power inequalities for log-concave random vectors

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***Abstract*—We prove the following discrete generalised Entropy Power Inequality (EPI) for isotropic log-concave sums of independent identically distributed random vectors $X_1, \dots, X_{n+1}$ on $\mathbb{Z}^d$:**

$$H\Big(\sum_{i=1}^{n+1} X_i\Big) \geq H\Big(\sum_{i=1}^{n} X_i\Big) + \frac{d}{2}\log\Big(\frac{n+1}{n}\Big) + o(1),$$

where $o(1)$ vanishes as $H(X_1) \to \infty$. Moreover, for the $o(1)$-term we obtain a rate of convergence $O\Big(H(X_1)e^{-\frac{1}{d}H(X_1)}\Big)$, where the implied constants depend on d and n. This generalises to $\mathbb{Z}^d$ the one-dimensional result of the second named author (2023). As in dimension one, our strategy is to establish that the discrete entropy is close to the differential entropy of the sum after adding n independent and identically distributed uniform random vectors on $[0,1]^d$ and to apply the continuous EPI. However, in dimension $d \geq 2$, more involved tools from convex geometry are needed. One of our technical tools is a dimensional analogue to a result of Bobkov, Marsiglietti and Melbourne (2022), which bounds the maximum probability of a log-concave p.m.f. in terms of the inverse of the determinant of the covariance matrix and may be of independent interest.

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I. Introduction

A. The Entropy Power Inequality

The *differential entropy* of an $\mathbb{R}^d$-valued random vector X with density f is defined as $h(X) = -\int_{\mathbb{R}^d} f(x)\log f(x)dx$, if the integral exists. When X is supported on a strictly lower-dimensional set than $\mathbb{R}^d$ (and hence does not have a density with respect to the d-dimensional Lebesgue measure), we set $h(X) = -\infty$. The *entropy power* of X is defined by $N(X) = e^{2h(X)/d}$.

Let $X_1, \dots, X_n$ be i.i.d. continuous random variables. A generalisation of the classical EPI of Shannon [1] and Stam [2] is due to Artstein, Ball, Barthe and Naor [3] and states that

$$h\Big(\sum_{i=1}^{n+1} X_i\Big) \geq h\Big(\sum_{i=1}^{n} X_i\Big) + \frac{d}{2}\log\Big(\frac{n+1}{n}\Big). \tag{1}$$

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Later, other proofs and generalisations were given by Shlyakhtenko [4], Tulino and Verdú [5] and Madiman and Barron [6] among others. The generalised EPI can also be interpreted as the monotonic increase of entropy along the Central Limit Theorem [7].

Let A be a discrete (finite or countable) set and let X be a random variable supported on A with probability mass function (p.m.f.) p on A. The discrete (Shannon) entropy of X is $H(X) = -\sum_{x\in A} p(x)\log p(x)$. An exact analogue of (1) cannot be true for discrete random variables as can be seen by taking $n = 1, d = 1$ and considering deterministic or even close to deterministic random variables. Nevertheless, Tao [8], using ideas from additive combinatorics, proved that for any independent and identically distributed random variables X_1, X_2 taking values in a finite subset of a torsion-free (abelian) group

$$H(X_1 + X_2) \geq H(X_1) + \frac{1}{2}\log 2 - o(1), \tag{2}$$

where $o(1) \to 0$ as $H(X_1) \to \infty$. For $n = 1$, this can be seen as a one-dimensional discrete analogue of (1).

Besides (2), discrete versions of the EPI have been studied by several authors. In [9], [10] a discrete EPI similar to (2) was proved, with a worse constant than $\frac{1}{2}\log 2$, but with sharp quantitative bounds for the $o(1)$-term (see also [11] for a detailed discussion on connections with that work). An EPI for the binomial family was proved in [12]; in [13, Theorem 4.6] it was shown that $e^{2H(X)} - 1$ is superadditive with respect to convolution on the class of uniform distributions on finite subsets of the integers (see also [14] for extensions to Rényi entropy and [15] for dimensional extensions). Finally the constant $\frac{1}{2}\log 2$ was shown to be improvable for the class of uniform distributions in [16]. Discrete Gaussians, which can informally be seen as "extremisers" of our main result, have been studied in the context of differential privacy in [17].

In addition to (2), Tao conjectured that for any independent and identically distributed random variables $X_1, \dots, X_{n+1}$ taking values in a finite subset of a torsion-free group

$$H\Big(\sum_{i=1}^{n+1} X_i\Big) \geq H\Big(\sum_{i=1}^{n} X_i\Big) + \frac{1}{2}\log\Big(\frac{n+1}{n}\Big) - o(1), \tag{3}$$

as $H(X_1) \to \infty$ depending on n. A special case of this conjecture was recently proven in [18], where it is shown that (3) is satisfied by log-concave random variables on the integers with explicit rate for the $o(1)$ that is exponential in $H(X_1)$:

Theorem 1 ([18]): Let $X_1, \dots, X_n$ be i.i.d. log-concave random variables on $\mathbb{Z}$. Then

$$H\left(\sum_{i=1}^{n+1} X_i\right) \geq H\left(\sum_{i=1}^{n} X_i\right) + \frac{1}{2}\log\left(\frac{n+1}{n}\right) - O_n\Big(\frac{H(X_1)}{e^{H(X_1)}}\Big),$$

as $H(X_1) \to \infty$.

This statement can be interpreted as a type of "approximate" monotonicity in the *discrete* entropic CLT [19], [20]. We recall here that an integer valued random variable X with p.m.f. p is said to be *log-concave* if, for every $k \in \mathbb{Z}$,

$$p(k)^2 \geq p(k-1)p(k+1). \tag{4}$$

Our goal is to provide a d-dimensional extension of Theorem 1 (see Theorem 3). In dimension $d > 1$, it is not obvious what the suitable definition of log-concavity should be. We discuss this in Section I-B.

Moreover, we prove a d-dimensional analogue (Theorem 5) of the following result, due to Bobkov, Marsiglietti and Melbourne [21], who also studied discrete versions of the EPI (up to multiplicative constants) for Rényi entropies of log-concave distributions and which is an important tool in the one-dimensional case:

Theorem 2: [21, Theorem 1.1] If a random variable X follows a discrete log-concave p.m.f. f on $\mathbb{Z}$, then

$$\max_{k\in\mathbb{Z}} f(k) \leq \frac{1}{\sqrt{1+4\sigma^2}}, \tag{5}$$

where $\sigma^2 = \mathrm{Var}(X)$.

B. Notations and definitions

Big- and small-O notation. Let f be a real-valued function and g another strictly positive function. We write $f = O(g)$ if there exist positive absolute constants N, C such that $|f(x)| \leq Cg(x)$ for every $x \geq N$. If N, C are absolute up to a parameter d, we write $f = O_d(g)$. Analogously, we write $f = \Omega(g)$, if $|f(x)| \geq Cg(x)$ for every $x \geq N$. If $f = \Omega(g)$ and $f = O(g)$, we write $f = \Theta(g)$ (with the analogous definitions for Ω_d and Θ_d). When it is more convenient, we will write $f \lesssim_d g$ for $f = O_d(g)$ and $f \simeq_d g$ for $f = \Theta(g)$. We write $f(x) = o(g(x))$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

Log-concavity in $\mathbb{Z}^d$. For $d > 1$ there are more than one definitions of log-concavity that have been used in different contexts. We will use the following:

Definition 1 (Discrete log-concavity): A function $p : \mathbb{Z}^d \to \mathbb{R} \cup \{+\infty\}$ is said to be log-concave if it is log-concave extensible [22], that is there exists a continuous log-concave function $f : \mathbb{R}^d \to \mathbb{R}$ given by $f(x) = e^{-V(x)}$ where $V : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is convex, such that $f(k) = p(k) \quad \forall k \in \mathbb{Z}^d$. We refer to f as the *extension* of p.

We say that a random vector X with values in $\mathbb{Z}^d$ and p.m.f. $p : \mathbb{Z}^d \to [0, 1]$ is log-concave if p is log-concave.

Definition 1 is quite general as it implies several other notions of log-concavity. For a detailed study of convex extensible functions the reader is referred to Murota [22]. We also note that for $d = 1$, this definition is equivalent to the usual definition (4) [21].

Isotropic functions. The *covariance matrix*, $\mathrm{Cov}(f)$, of an integrable function $f : \mathbb{R}^d \to \mathbb{R}_+$ is defined by

$$[\mathrm{Cov}(f)]_{ij} := \frac{\int_{\mathbb{R}^d} x_i x_j f(x)dx}{\int_{\mathbb{R}^d} f(x)dx} - \frac{\int_{\mathbb{R}^d} x_i f(x)dx \int_{\mathbb{R}^d} x_j f(x)dx}{\left(\int_{\mathbb{R}^d} f(x)dx\right)^2}$$

for $1 \leq i, j \leq d$. We say that f is *isotropic* if $\mathrm{Cov}(f) = \sigma^2 \mathrm{I}_d$, for some $\sigma > 0$, where I_d is the $d \times d$ identity matrix. Let K be a convex body in $\mathbb{R}^d$. Then its covariance matrix is $\mathrm{Cov}(K) := \mathrm{Cov}(\mathbb{1}_K)$. The convex body K is called *isotropic* if $\mathbb{1}_K$ is isotropic. In the discrete case $p : \mathbb{Z}^d \to \mathbb{R}_+$, the covariance matrix $\mathrm{Cov}(p)$ is defined analogously with the integrals being replaced by sums.

Definition 2: A family $\{f_\sigma\}_{\sigma\in\mathbb{R}_+}$ of non-negative functions on $\mathbb{R}^d$ is *almost isotropic* if, as $\sigma \to \infty$,

$$\begin{aligned} \mathrm{Cov}(f_\sigma)_{i,j} &= \sigma^2 + O(\sigma) \quad \text{for } i = j, \\ &= O(\sigma) \quad \text{for } i \neq j. \end{aligned}$$

We are interested in log-concave densities f, for which $\det \mathrm{Cov}(f) \to \infty$. Thus, when we write that f is almost isotropic, it is meant that f represents a family of functions with underlying parameter $\sigma := \det \mathrm{Cov}(f)^{\frac{1}{2d}}$.

When its not clear from the context, we write $\mathrm{Cov}_{\mathbb{R}}(f)$ and $\mathrm{Cov}_{\mathbb{Z}}(f)$ to distinguish between the continuous and discrete covariance matrix of f.

C. Main results and paper outline

Our first main result is the following:

Theorem 3: Let $n \geq 1$. For any i.i.d. random vectors $X_1, \dots, X_n$ on $\mathbb{Z}^d$ such that the sums $X_1 + \cdots + X_n$ and $X_1 + \cdots + X_{n+1}$ are log-concave with almost isotropic extension, we have as $H(X_1) \to \infty$,

$$H\left(\sum_{i=1}^{n+1} X_i\right) \geq H\left(\sum_{i=1}^{n} X_i\right) + \frac{d}{2}\log\left(\frac{n+1}{n}\right) - O_{d,n}\left(\frac{H(X_1)}{e^{\frac{H(X_1)}{d}}}\right).$$

Theorem 3 readily follows from the next theorem together with the generalised EPI in $\mathbb{R}^d$.

Theorem 4: Let $n \geq 1$. For any i.i.d. random vectors $X_1, \dots, X_n$ on $\mathbb{Z}^d$ such that the sum $X_1 + \cdots + X_n$ is log-concave with almost isotropic extension, we have as $H(X_1) \to \infty$,

$$h\Big(\sum_{i=1}^{n} X_i + \sum_{i=1}^{n} U_i\Big) = H\Big(\sum_{i=1}^{n} X_i\Big) + O_{d,n}\left(\frac{H(X_1)}{e^{\frac{1}{d}H(X_1)}}\right), \tag{6}$$

where $U_1, \dots, U_n$ are i.i.d. continuous uniforms on $[0, 1]^d$.

Remark 1: Let $X_1, \dots, X_n$ be i.i.d. random vectors on $\mathbb{Z}^d$ such that their sum is log-concave with almost isotropic extension. Then, since $\mathrm{Cov}\Big(\sum_{i=1}^n X_i\Big) = n\mathrm{Cov}(X_1)$, X_1 is also almost isotropic. Denote the covariance matrix of X_1 by

K_{X_1}, and let U_1 a uniform on the open box $[0,1]^d$. Then, using that the Gaussian distribution maximises the entropy under fixed covariance matrix,

$$H(X_1) = h(X_1 + U_1) \leq \frac{d}{2}\log\left(\det\left(K_{X_1} + \frac{1}{12}I_{\rm d}\right)^{\frac{1}{d}} 2\pi e\right),$$

which is $\simeq \frac{d}{2}\log\left(\det\left(K_{X_1}\right)^{\frac{1}{d}} 2\pi e\right)$. Thus, $H(X_1) \to \infty$ implies $\det(K_{X_1}) \to \infty$ and therefore it suffices to prove (6) with $o(1) \to 0$ as $\det(K_{X_1}) \to \infty$.

In the one-dimensional version (Theorem 1), Theorem 2 of [21] is used. Our last main result is Theorem 5 below, which gives a generalisation to dimensions $d > 1$ of that result.

Theorem 5 may also be seen as a discrete analogue of a dimensional upper bound on the *isotropic constant* $L_f := \left(\frac{\max_{\mathbb{R}^d} f}{\int_{\mathbb{R}^d} f}\right)^{\frac{1}{d}} \det(\mathrm{Cov}(f))^{\frac{1}{2d}}$. The study of L_f is a subject of central importance in convex geometry. A lower bound on L_f can be deduced from maximum entropy property of the Gaussian distribution and the fact that $e^{-h(X)} \leq \max(f)$: $L_f \geq e^{-\frac{h(X)}{d}} \det(\mathrm{Cov}(f))^{\frac{1}{2d}} \geq \frac{1}{\sqrt{2\pi e}}$. It is well known that a dimensional upper bound also exists and we denote by L_d the maximum of the isotropic constants L_f among log-concave functions f in $\mathbb{R}^d$. The latter is related to the famous *hyperplane conjecture* (or slicing problem), one of the central questions in this field, as the hyperplane conjecture is equivalent to the *isotropic constant conjecture* [23, Theorem 3.1.2], which states that there exists a universal constant C such that for any dimension d, one has $L_d \leq C$. A recent result of Klartag [24] gives the best known constant depending on the dimension d for the slicing problem and thus also the best upper bound known for L_d:

$$L_d \leq C\sqrt{\log d}\,, \tag{7}$$

for some absolute constant C.

The upper bound (7) together with the general lower bound give $\max_{x\in\mathbb{R}^d} f(x) \simeq_d \det\Big(\mathrm{Cov}(f)\Big)^{-\frac{1}{2}}$. Our discrete analogue of (7) (and dimensional analogue of (5)) reads:

Theorem 5: Suppose p is a log-concave p.m.f. on $\mathbb{Z}^d$ with almost isotropic extension and covariance matrix $\mathrm{Cov}(p)$. Then there exists a constant C'_d depending on the dimension only, such that

$$\max_{k\in\mathbb{Z}^d} p(k) \leq \frac{C'_d}{\det\Big(\mathrm{Cov}_{\mathbb{Z}^d}(p)\Big)^{\frac{1}{2}}},$$

provided that $\det(\mathrm{Cov}_{\mathbb{Z}^d}(p))$ is large enough depending on d.

Our method for proving Theorem 5 is to use the corresponding continuous result. To this end and in order to exploit (7), we obtain the approximations
$\left|\int_{\mathbb{R}^d} f - \sum_{\mathbb{Z}^d} f\right| = o_d(1)$, $\left|\int_{\mathbb{R}^d} xf - \sum_{k\in\mathbb{Z}^d} kf(k)\right| = O_d(1)$, and $\left|\det\Big(\mathrm{Cov}_{\mathbb{Z}^d}(f)\Big) - \det\Big(\mathrm{Cov}_{\mathbb{R}^d}(f)\Big)\right| = O_d(\sigma^{2d-1})$, as $\sigma \to \infty$. This is done in Section II. Although our results hold under the more general almost isotropicity assumption, for better illustration of the ideas, we assume first that the continuous extension f is isotropic. In Remark 3, we describe how to remove this extra assumption. The proofs of Theorems 3 and 4 are analogous to their one-dimensional counterparts from [18]. We describe the key differences in Section III. For the missing details, the reader is referred to the full version of our paper. Finally, in Section IV we conclude with a brief discussion of our assumptions and of some open questions.

II. Theorem 5 for isotropic f

A. Ball's bodies

A family of convex bodies was introduced by Ball [25]. We refer to the book [23] for the definition and properties of these bodies. Using some of their fundamental properties ([23, Proposition 2.5.3] and the inclusion relations [23, Proposition 2.5.7]), as well as [26, Theorem 4.1], we obtain the following technical lemma, whose proof we omit.

Lemma 1: Let $d \geq 1$ be an integer. There exist two constants $0 < C'_d < C_d$ such that for any $f : \mathbb{R}^d \to \mathbb{R}_+$ centered, isotropic, log-concave density and for every $\theta \in \mathbb{S}^{d-1}$,

$$C'_d \leq \left(\int_0^\infty dr^{d-1} f(r\theta)dr\right)^{\frac{1}{d}} \leq C_d, \tag{8}$$

where C_d and C'_d are constants depending only on d.

B. Sum of maxima of isotropic log-concave functions

The following lemma, whose proof is also omitted, is a consequence of Lemma 8 and shows that log-concave isotropic densities have exponential tails.

Lemma 2 (Concentration Lemma): Let $c_d := 3^{\frac{1}{d}} C_d$, where C_d is the constant from (8). Then, for every log-concave, isotropic, centered density function f and for every $x \in \mathbb{R}^d$ such that $\|x\|_2 \geq c_d / f(0)^{\frac{1}{d}}$,

$$f(x) \leq f(0) 2^{-\|x\|_2 \frac{f(0)^{\frac{1}{d}}}{c_d}}. \tag{9}$$

The following lemma bounds the sum of the maxima of a log-concave density using our previous concentration result.

Lemma 3: Let f be a centered, isotropic, log-concave density on $\mathbb{R}^d$ with covariance $\sigma^2 \mathrm{I_d}$. Then, as $\sigma \to \infty$,

$$\sum_{l\in\mathbb{Z}^{d-1}} \max_{k\in\mathbb{Z}} f(k,l) = O_d\Big(\frac{1}{\sigma}\Big). \tag{10}$$

Proof: Set $\lambda = c_d f(0)^{-1/d}$, with c_d as given by Lemma 2. Then, from the definition of L_f, its bounds and the inequality $f(0) \geq e^{-d}\max(f)$ from [27], we have $\lambda \simeq \sigma$. Thus, we have

$$\begin{aligned}&\sum_{l\in\mathbb{Z}^{d-1}} \max_{k\in\mathbb{Z}} f(k,l)\\ &= \sum_{l\in\mathbb{Z}^{d-1}, \|l\|_\infty \leq \lambda} \max_{k\in\mathbb{Z}} f(k,l) + \sum_{l\in\mathbb{Z}^{d-1}, \|l\|_\infty > \lambda} \max_{k\in\mathbb{Z}} f(k,l),\end{aligned}$$

where the first sum can be trivially bounded above by $O_d(\frac{1}{\sigma})$. Using the tails estimates of Lemma 2 and the fact that for $l \in \mathbb{Z}^{d-1}$, one has $\|l\|_2 \geq \|l\|_\infty$ and $\|l\|_2 \geq \frac{\|l\|_1}{\sqrt{d-1}} \geq \frac{\|l\|_1}{\sqrt{d}}$ the second sum can be bounded above as

$$\sum_{\|l\|_\infty > \lambda} \max_{k\in\mathbb{Z}} f(k,l) \leq f(0) \sum_{\|l\|_\infty > \lambda} 2^{-\frac{\|l\|_1}{\lambda\sqrt{d}}},$$

which can be calculated to be $O_d(\frac{1}{\sigma})$. □

C. Discrete approximation of the integral, mean and covariance

In this section, we will approximate the isotropic constant of an isotropic, log-concave density $f \in \mathbb{R}^d$ by its discrete analogue. The following proposition allows us to approximate the integral of an isotropic log-concave function by its sum.

Proposition 1: Let $f : \mathbb{R}^d \to \mathbb{R}$ be a log-concave isotropic density function with covariance matrix of the form $\sigma^2 \mathrm{I_d}$. Then

$$\left| \int_{\mathbb{R}^d} f dx - \sum_{k \in \mathbb{Z}^d} f(k) \right| = O_d\Big(\frac{1}{\sigma}\Big) \ .$$

Proof: Consider the one dimensional case first. Let $f : \mathbb{R} \to \mathbb{R}$ be an isotropic log-concave function and let suppose that the maximum of f is attained at $x_0 \in \mathbb{R}$. Let $k_0 \in \mathbb{Z}$ such that $k_0 \leq x_0 < k_0 + 1$. Then, we have

$$\begin{aligned}
&\int_{\mathbb{R}} f = \sum_{k \in \mathbb{Z}} \int_k^{k+1} f(x) dx \\
&\geq \sum_{k < k_0} f(k) + \sum_{k \geq k_0+1} f(k+1) + \min\{f(k_0), f(k_0+1)\} \\
&= \sum_{k \in \mathbb{Z}} f(k) - \max_{\mathbb{Z}} f \ .
\end{aligned}$$

The reverse inequality is analogous and we get $\left| \int_{\mathbb{R}} f - \sum_{k\in\mathbb{Z}} f \right| \leq \max_{\mathbb{R}} f$. We have $\max_{\mathbb{R}}(f) = \frac{L_f}{\sigma} \leq \frac{1}{\sigma}$ [28] and the one-dimensional result follows.

By considering $F(y) := \int_{\mathbb{R}} f(x, y) dx$, an inductive argument can be used to prove the result in any dimension.

□

Remark 2: The hypothesis of isotropicity is necessary. Indeed, taking for instance $d = 2$, one can easily construct (essentially one-dimensional) convex sets that do not contain integer points but whose volumes are increasingly large.

The proofs of Propositions 2 and 3 are similar to that of Proposition 1 and omitted.

Proposition 2: Let $f : \mathbb{R}^d \to \mathbb{R}$ be a centered, isotropic, log-concave density. Then

$$\sum_{k \in \mathbb{Z}^d} k_i f(k) = \int_{\mathbb{R}^d} x_i f + O_d(1) = O_d(1).$$

Proposition 3 below shows that a (continuous) isotropic, log-concave density in $\mathbb{R}^d$ is almost isotropic in the discrete sense, meaning that its discrete covariance matrix has $O(\sigma)$ off-diagonal elements and $\sigma^2 + O(\sigma)$ diagonal elements.

Proposition 3: For every centered, isotropic, log-concave density $f : \mathbb{R}^d \to \mathbb{R}$ with $\mathrm{Cov}(f) = \sigma^2 \mathrm{I_d}$,

$$\sum_{k \in \mathbb{Z}^d} f(k) k_i^2 = \sigma^2 + O_d(\sigma), \text{ for every } \ 1 \leq i \leq d.$$

and for all $i \neq j$

$$\sum_{k \in \mathbb{Z}^d} f(k) k_i k_j = O_d(\sigma).$$

Corollary 1: Let f be a centered, isotropic, log-concave density in $\mathbb{R}^d$, with $\mathrm{Cov}(f) = \sigma^2 \mathrm{I}_d$. Then

$$\det\Big(\mathrm{Cov}_{\mathbb{Z}^d}(f)\Big) = \sigma^{2d} + O_d(\sigma^{2d-1}).$$

Proof: By Proposition 3

$$\begin{aligned}
\det\Big(\mathrm{Cov}_{\mathbb{Z}^d}(f)\Big) &= \sum_{\tau} \mathrm{sgn}(\tau) \prod_{i=1}^d \mathrm{Cov}_{\mathbb{Z}^d}(f)_{i\tau(i)} \\
&= \sigma^{2d} + O(\sigma^{2d-1}),
\end{aligned}$$

since for τ being the identity permutation $\mathrm{sgn}(\tau) \prod_{i=1}^d \mathrm{Cov}_{\mathbb{Z}^d}(f)_{i\tau(i)} = \Big(\sigma^2 + O(\sigma)\Big)^d$ and for any other permutation τ, $\prod_{i=1}^d \mathrm{Cov}_{\mathbb{Z}^d}(f)_{i\tau(i)} = O_d(\sigma^{2d-2})$.

□

Finally, Corollary 2 below is a version of Theorem 5 with the assumption that the continuous function f is isotropic. It can be observed from the proof that the dependence of of C'_d on d can be taken to be the same as in the continuous case (cf. (7)). It can also be seen that due to use of Corollary 1, σ needs to be taken at least $\Omega(d!)$. We do not know whether this is the best (lowest) rate.

Corollary 2 (**Theorem 5 for isotropic** f)**:** Let p be log-concave p.m.f. on $\mathbb{Z}^d$, whose continuous log-concave extension, say f, is isotropic. Then there exists a constant C'_d that depends on the dimension only, such that

$$\max_{k \in \mathbb{Z}^d} p(k) \leq \frac{C'_d}{\det\Big(\mathrm{Cov}(p)\Big)^{\frac{1}{2}}}$$

provided that $\sigma := \det\Big(\mathrm{Cov}_{\mathbb{R}}(f)\Big)$ is large enough depending on d.

Proof: Since p is extensible log-concave, there exists a continuous log-concave function f (not necessarily a density) such that $f(k) = p(k)$ for all $k \in \mathbb{Z}^d$ and by assumption f is isotropic. Thus

$$\max_{k \in \mathbb{Z}^d} p(k) \leq \max_{x \in \mathbb{R}^d} f(x) = \frac{L_f^d \int_{\mathbb{R}^d} f}{\sigma^d} \leq \frac{C_d \int_{\mathbb{R}^d} f}{\sigma^d} \ , \qquad (11)$$

where C_d is an upper bound of L_f^d. But by Proposition 1 and Corollary 1 applied to $\frac{f(\cdot)}{\int f}$, $\int f = 1 + O_d(\frac{1}{\sigma})$ and $\sigma^d = \Big(\det\Big(\mathrm{Cov}(p)\Big) + O_d(\sigma^{2d-1})\Big)^{\frac{1}{2}}$, provided that σ is large enough depending on d. □

Remark 3: The assumptions of Lemma 1 can be relaxed: using convex geometric tools to bound the inradius and circumradius of convex bodies in not necessarily isotropic position, we can show that $\int_0^\infty r^{d-1} f(r\theta) dr$ can always be bounded below by $\gtrsim_d f(0)\big(\sqrt{\lambda_{\min}(\mathrm{Cov}(f))}\big)^d$ and above by $\lesssim_d f(0)\big(\sqrt{\lambda_{\max}(\mathrm{Cov}(f))}\big)^d$. Then $\int_0^\infty r^{d-1} f(r\theta) dr = \Theta_d(1)$, as $\sigma := \det(\mathrm{Cov}(f)) \to \infty$.

The isotropicity assumption was only used in Lemma 1, which in turn allowed us to prove the concentration Lemma 2 which was repeatedly evoked in this section to bound the sums of maxima of log-concave functions and thus also the error terms. By the previous remarks, a version of Lemma 2 holds true for almost isotropic densities, although with possibly different constants and for large enough σ. Therefore, all the results of Section II hold true under this assumption as well and therefore also Theorem 5.

III. Discrete EPI in $\mathbb{Z}^d$

The proof of Theorem 4 makes use of a different generalisation of Theorem 2, Lemma 4 below, which is a direct consequence of Lemma 3.

Lemma 4: Fix $d \geq 1$ and let p be a log-concave p.m.f. on $\mathbb{Z}^d$ with almost isotropic extension. Then, for every $1 \leq i \leq d$,

$$\sum_{k\in\mathbb{Z}^d} |p(k) - p(k-e_i)| = O_d\left(\frac{1}{\det\left(\mathrm{Cov}(p)\right)^{\frac{1}{2d}}}\right),$$

where $e_i \in \mathbb{Z}^d$ is defined as the vector with the i-th coordinate 1 and all the other coordinates 0.

Proof: This follows by the observation for any one-dimensional unimodal non-negative real sequence $(a_n)_{n\in\mathbb{Z}}$, one has $\sum_{n\in\mathbb{Z}} |a_n - a_{n-1}| \leq 2\max_{n\in\mathbb{Z}} a_n$ and (10). □

The following result is a dimensional generalisation of [18, Lemma 2].

Lemma 5: Fix $n, d \geq 1$ and let S be a log-concave random vector on $\mathbb{Z}^d$ with almost isotropic extension. Let $U_1, \ldots, U_n$ be i.i.d. continuous uniforms on the unit cube. Let f_n denote the density of $S + \sum_{i=1}^n U_i$ and p_S the p.m.f. of S. Then, for each $k \in \mathbb{Z}^d, x \in k + [0,1)^d$ if we define $g(k,x) = f_n(x) - p_S(k)$, then

$$\sum_{k\in\mathbb{Z}^d} \sup_{x\in k+[0,1)^d} g(k,x) = O_d\Big(\frac{1}{\det\big(\mathrm{Cov}(p_S)\big)^{\frac{1}{2d}}}\Big).$$

Proof: The proof is a dimensional adaptation of argument from [18]. We omit the details. □

Proof of Theorem 4: The proof is analogous to [18, Theorem 1], the difference being that we apply the dimensional analogues; Lemma 5 now bounds the error term when approximating the density with the p.m.f. and Theorem 5 shows that the maximum probability as well as the maximum of the density are bounded by $O_{d,n}(\frac{1}{\sigma^d})$. □

Proof of Theorem 3: Follows directly by the continuous EPI (1), and by Theorem 4 applied to the differential entropies on both sides. □

Remark 4: Applying the generalised EPI of [6] we can also obtain the following generalisation:

For $n \geq 1$, let $X_1, \ldots, X_n$ be i.i.d. random vectors. Let $\mathcal{C}$ be an arbitrary collection of subsets of $\{1, \ldots, n\}$ and r be the maximum number of sets in $\mathcal{C}$ in which any one index appears. Suppose that all the sums $\sum_{j\in S} X_j : S \in \mathcal{C} \cup \{1, \ldots, n\}$ are log-concave with almost isotropic extension. Then,

$$e^{\frac{2}{d}H(\sum_{i=1}^n X_i)} \geq \frac{1}{r}\sum_{S\in\mathcal{C}} e^{\frac{2}{d}H(\sum_{j\in S} X_j) - O_{d,n}\left(\frac{H(X_1)}{e^{\frac{1}{d}H(X_1)}}\right)}. \quad (12)$$

IV. Concluding Remarks and Open Questions

1) For $d = 1$ discrete log-concavity is preserved under convolution (e.g. [29]). However, for $d > 1$ log-concavity may not be preserved in general. Indeed, note that the support of a discrete log-concave function is $\mathbb{Z}^d$-convex, i.e. $\mathrm{conv}(A) \cap \mathbb{Z}^d = A$, the support of a convolution is the Minkowski sum of the supports and consider two log-concave distributions supported on $S_1 = \{(0,0),(1,1)\}$ and $S_2 = \{(0,1),(1,0)\}$ respectively [22, Example 3.15]. In this case, the Minkowski sum $S_1 + S_2$ is not $\mathbb{Z}^2$-convex. However, it is elementary to show that $\sum_{i=1}^n A$ is $\mathbb{Z}^d$-convex for any $\mathbb{Z}^d$-convex set A. We do not know if our definition of log-concavity is preserved under self-convolution. It is therefore natural to ask the following question:

   Question 1: Let X_1, X_2 be i.i.d. log-concave random vectors on $\mathbb{Z}^d$. Is $X_1 + X_2$ log-concave? Furthermore, is $\sum_{i=1}^n X_i$ log-concave for every n?

   If the answer to Question 1 is positive, a sufficient condition for the assumptions of Theorems 3 and 4 would be that X_1 is log-concave. Answering Question 1 even for quantised multivariate Gaussians seems to be non-trivial. Nevertheless, the following simple example shows that the answer is positive for quantised isotropic multivariate Gaussians, yielding an example that satisfies the assumptions of Theorems 3 and 4.

   Example 1: Let X_1 be a random variable with p.m.f. on $\mathbb{Z}^d$ proportional to $e^{-\frac{|k|^2}{2\sigma^2}}$; that is X_1 is a multivariate centered isotropic Gaussian quantised on $\mathbb{Z}^d$. Then X_1 has the distribution of $(Z_1, \ldots, Z_d)$, where Z_i are i.i.d. one-dimensional quantised centered Gaussians with variance σ^2. Note that if p is a p.m.f. on $\mathbb{Z}^d$ which is a product of log-concave p.m.f.s $\{p_i\}_{i=1}^d$ on $\mathbb{Z}$, $p(k) = \prod_{i=1}^d p^{(i)}(k_i), \quad k = (k_1, \ldots, k_n) \in \mathbb{Z}^d$, then p is clearly log-concave on $\mathbb{Z}^d$.

   Therefore, since log-concavity is preserved under convolution in dimension one [21] and the coordinates of $\sum_{i=1}^n X_i$ are independent, it follows that $\sum_{i=1}^n X_i$ is log-concave for every n.

2) For Theorem 5 and therefore for Theorems 3 and 4 as well, we have assumed that there exists a continuous extension f, which is almost isotropic. We have then shown that f is also almost isotropic in the discrete sense. It would be more natural to start with the assumption that the discrete p.m.f. is isotropic. However, in this case we would not be able to use the continuous toolkit to prove our concentration lemma. Nevertheless, we suspect that if the discrete p.m.f. is isotropic, then there exists an almost isotropic continuous extension and therefore our results would still hold under this assumption:

   Question 2: Let X be a log-concave random vector with p.m.f. p on $\mathbb{Z}^d$. Assume that p is isotropic (respectively almost isotropic). Is there a continuous log-concave extension of p on $\mathbb{R}^d$, which is isotropic (respectively almost isotropic)?

References

[1] C. E. Shannon, "A mathematical theory of communication," *The Bell system technical journal*, vol. 27, no. 3, pp. 379–423, 1948.

[2] A. J. Stam, "Some inequalities satisfied by the quantities of information of Fisher and Shannon," *Information and Control*, vol. 2, no. 2, pp. 101–112, 1959.

[3] S. Artstein, K. Ball, F. Barthe, and A. Naor, "Solution of Shannon's problem on the monotonicity of entropy," *Journal of the American Mathematical Society*, vol. 17, no. 4, pp. 975–982, 2004.

[4] D. Shlyakhtenko, "A free analogue of Shannon's problem on monotonicity of entropy," *Adv. Math.*, vol. 208, no. 2, pp. 824–833, 2007. [Online]. Available: https://doi.org/10.1016/j.aim.2006.03.014

[5] A. M. Tulino and S. Verdú, "Monotonic decrease of the non-Gaussianness of the sum of independent random variables: a simple proof," *IEEE Trans. Inform. Theory*, vol. 52, no. 9, pp. 4295–4297, 2006. [Online]. Available: https://doi.org/10.1109/TIT.2006.880066

[6] M. Madiman and A. Barron, "Generalized entropy power inequalities and monotonicity properties of information," *IEEE Transactions on Information Theory*, vol. 53, no. 7, pp. 2317–2329, 2007.

[7] A. R. Barron, "Entropy and the central limit theorem," *The Annals of Probability*, pp. 336–342, 1986.

[8] T. Tao, "Sumset and inverse sumset theory for Shannon entropy," *Combinatorics, Probability and Computing*, vol. 19, pp. 603–639, 07 2010.

[9] S. Haghighatshoar, E. Abbe, and E. Telatar, "Adaptive sensing using deterministic partial Hadamard matrices," in *2012 IEEE International Symposium on Information Theory Proceedings*. Ieee, 2012, pp. 1842–1846.

[10] ——, "A new entropy power inequality for integer-valued random variables," *IEEE Transactions on Information Theory*, vol. 60, no. 7, pp. 3787–3796, 2014.

[11] L. Gavalakis, "Discrete generalised entropy power inequalities for log-concave random variables," in *2023 IEEE International Symposium on Information Theory (ISIT)*, 2023, pp. 42–47.

[12] P. Harremoés, C. Vignat *et al.*, "An entropy power inequality for the binomial family," *JIPAM. J. Inequal. Pure Appl. Math*, vol. 4, no. 5, p. 93, 2003.

[13] M. Madiman, L. Wang, and J. O. Woo, "Entropy inequalities for sums in prime cyclic groups," *SIAM Journal on Discrete Mathematics*, vol. 35, no. 3, pp. 1628–1649, 2021.

[14] M. Madiman, J. Melbourne, and C. Roberto, "Bernoulli sums and Rényi entropy inequalities," *Bernoulli*, vol. 29, no. 2, pp. 1578–1599, 2023.

[15] M. Madiman, L. Wang, and J. O. Woo, "Majorization and Rényi entropy inequalities via Sperner theory," *Discrete Mathematics*, vol. 342, no. 10, pp. 2911–2923, 2019.

[16] J. O. Woo and M. Madiman, "A discrete entropy power inequality for uniform distributions," in *2015 IEEE International Symposium on Information Theory (ISIT)*. IEEE, 2015, pp. 1625–1629.

[17] C. L. Canonne, G. Kamath, and T. Steinke, "The discrete Gaussian for differential privacy," *Advances in Neural Information Processing Systems*, vol. 33, pp. 15 676–15 688, 2020.

[18] L. Gavalakis, "Approximate discrete entropy monotonicity for log-concave sums," *Combinatorics, Probability and Computing*, p. 1–14, 2023.

[19] L. Gavalakis and I. Kontoyiannis, "Entropy and the discrete central limit theorem," *Stochastic Processes and their Applications*, p. 104294, 2024.

[20] ——, "The entropic central limit theorem for discrete random variables," in *2022 IEEE International Symposium on Information Theory (ISIT)*, 2022, pp. 708–713.

[21] S. G. Bobkov, A. Marsiglietti, and J. Melbourne, "Concentration functions and entropy bounds for discrete log-concave distributions," *Combinatorics, Probability and Computing*, vol. 31, no. 1, pp. 54–72, 2022.

[22] K. Murota, "Discrete convex analysis: monographs on discrete mathematics and applications." *Computing Reviews*, vol. 45, no. 6, p. 339, 2004.

[23] S. Brazitikos, A. Giannopoulos, P. Valettas, and B.-H. Vritsiou, *Geometry of isotropic convex bodies*. American Mathematical Soc., 2014, vol. 196.

[24] B. Klartag, "Logarithmic bounds for isoperimetry and slices of convex sets," *arXiv preprint arXiv:2303.14938*, 2023.

[25] K. Ball, "Logarithmically concave functions and sections of convex sets in $\mathbb{R}^n$," *Studia Math*, vol. 88, no. 1, pp. 69–84, 1988.

[26] R. Kannan, L. Lovász, and M. Simonovits, "Isoperimetric problems for convex bodies and a localization lemma," *Discrete & Computational Geometry*, vol. 13, pp. 541–559, 1995.

[27] M. Fradelizi, "Sections of convex bodies through their centroid," *Archiv der Mathematik*, vol. 69, no. 6, pp. 515–522, 1997.

[28] ——, "Hyperplane sections of convex bodies in isotropic position," *Beiträge Algebra Geom.*, vol. 40, no. 1, pp. 163–183, 1999.

[29] S. Hoggar, "Chromatic polynomials and logarithmic concavity," *Journal of Combinatorial Theory, Series B*, vol. 16, no. 3, pp. 248–254, 1974.