Convergence of percolation on random quadrangulations

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Outline

Part I: Introduction — percolation and random planar maps

Part II: SLE$_6$ on Brownian surfaces

Part III: Proof ideas
Part I: Introduction
Percolation review

- Graph $G = (V, E)$, $p \in (0, 1)$.
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![Graph Diagram]
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Variants: site percolation, face percolation, etc...
Critical bond percolation on a box in $\mathbb{Z}^2$ with side-length 1000, conformally mapped to $D$. Shown are the clusters which touch the boundary.
Results on planar lattices

- **Kesten**: $p_c = \frac{1}{2}$ for bond percolation on the $\square$-lattice

- **Smirnov**: The exploration path between open and closed sites in critical site percolation on the $\triangle$-lattice converges to SLE$_6$ as the mesh size tends to 0.

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This talk is about proving the convergence of percolation on random planar maps.
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- Interested in **uniformly random quadrangulations** with $n$ faces — **random planar map** (RPM).

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What is the structure of a typical quadrangulation when the number of faces is large? How many are there?

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\frac{2 \times 3^n}{(n+1)(n+2)} \binom{2n}{n}.
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Random quadrangulation with 25,000 faces

(Simulation due to J.F. Marckert)
Topologies for quadrangulations

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- □ of the disk with $\partial$-length $2\ell$
- Infinite $\partial$-length local limit: uniform infinite half-planar quadrangulation (UIHPQ)
Gromov-Hausdorff topology

The **Hausdorff distance** between closed sets $A_1, A_2$ in a metric space is

$$d_H(A_1, A_2) = \inf\left\{ \epsilon > 0 : A_2 \subseteq A_1(\epsilon) \quad \text{and} \quad A_1 \subseteq A_2(\epsilon) \right\}.$$
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$$d_{GH}(X_1, X_2) = \inf \{ d_H(\iota_1(X_1), \iota_2(X_2)) \}$$

where the infimum is taken over all metric spaces $W$ and isometric embeddings $\iota_j : X_j \to W$ for $j = 1, 2.$
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* Gromov-Hausdorff-Prokhorov-uniform: metric space + measure + path
  $$d_{GHPU}(X_1, X_2) = \inf\{d_H(\iota_1(X_1), \iota_2(X_2)) + d_P(\iota_1^*\mu_1, \iota_2^*\mu_2) + d_\infty(\iota_1(\gamma_1), \iota_2(\gamma_2))\}$$
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(Simulation due to J.F. Marckert)
Large scale structure of random quadrangulations

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There exists a unique limit in distribution: **the Brownian map** (Le Gall, Miermont)
Convergence results toward Brownian surfaces

**General principle:** Uniformly random planar $\Box$’s with $n$ faces with distances rescaled by $n^{-1/4}$ converge to Brownian surfaces in the Gromov-Hausdorff-Prokhorov topology (metric space + measure).
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- \( \square \) of the disk (simple boundary, random area) \( \rightarrow \) Brownian disk (Gwynne-M.)
Percolation on random planar maps

- **Angel:** $p_c = \frac{1}{2}$ for site percolation on a random $\triangle$

Open faces are adjacent if they share an edge. Closed faces are adjacent if they share a vertex.

Percolation thresholds for many other types of maps have been computed (c.f. Angel-Curien, Menard-Nolin, Richlier...)

We will consider critical $p = p_c = \frac{3}{4}$ face percolation on a random $\square$. 

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14 / 28
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Percolation exploration path

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**Perspective:** It is a *random path* on a *random metric space*
Main result

Theorem (Gwynne-M.)

The exploration path for critical face percolation on a random \( \square \) of the disk with boundary length \( 2\ell \) converges as \( \ell \to \infty \) to a random path on a random metric space with respect to the Gromov-Hausdorff-Prokhorov-uniform topology.

The limit is SLE\(_6\) on a Brownian disk.

Comments:
▶ Universal strategy: works for any random planar map model provided one has certain technical inputs.
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Part II: $\text{SLE}_6$ on a Brownian surface
Schramm-Loewner evolution (SLE)

- Random fractal curve in a planar domain

Critical percolation, hexagonal lattice
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- Characterized by conformal invariance and domain Markov property

- Indexed by a parameter $\kappa > 0$
- Simple for $\kappa \in (0, 4]$, self-intersecting for $\kappa \in (4, 8)$, space-filling for $\kappa \geq 8$

- Dimension: $1 + \kappa/8$ for $\kappa \leq 8$

- Some special $\kappa$ values:
  - $\kappa = 2$ LERW
  - $\kappa = 8/3$ SAW
  - $\kappa = 3$ Ising
  - $\kappa = 16/3$ FK-Ising
  - $\kappa = 4$ GFF level lines
  - $\kappa = 6$ Percolation
  - $\kappa = 12$ Bipolar orientations
  - …

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Critical percolation, hexagonal lattice
Loewner's equation: if $\eta$ is a non self-crossing path in $H$ with $\eta(0) \in \mathbb{R}$ and $g_t$ is the Riemann map from the unbounded component of $H \setminus \eta([0, t])$ to $H$ normalized by $g_t(z) = z + o(1)$ as $z \to \infty$, then

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t} \quad \text{where} \quad g_0(z) = z \quad \text{and} \quad W_t = g_t(\eta(t)).$$
**Loewner's equation**: if \( \eta \) is a non self-crossing path in \( \mathbb{H} \) with \( \eta(0) \in \mathbb{R} \) and \( g_t \) is the Riemann map from the unbounded component of \( \mathbb{H} \setminus \eta([0, t]) \) to \( \mathbb{H} \) normalized by \( g_t(z) = z + o(1) \) as \( z \to \infty \), then

\[
\frac{\partial_t g_t(z)}{g_t(z) - W_t} = 2,
\]

where \( g_0(z) = z \) and \( W_t = g_t(\eta(t)) \).

\((\star)\)

**SLE\(_\kappa\) in \( \mathbb{H} \)**: The random curve associated with \((\star)\) with \( W_t = \sqrt{\kappa}B_t \), \( B \) a standard Brownian motion.
SLE$_\kappa$

\[ \eta(t) \quad \eta(s) \]

\[ g_t \]

\[ g_t(\eta(s)) \]

\[ W_t = g_t(\eta(t)) \]

**Loewner's equation:** if $\eta$ is a non self-crossing path in $\mathbb{H}$ with $\eta(0) \in \mathbb{R}$ and $g_t$ is the Riemann map from the unbounded component of $\mathbb{H} \setminus \eta([0, t])$ to $\mathbb{H}$ normalized by $g_t(z) = z + o(1)$ as $z \to \infty$, then

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**SLE$_\kappa$ in $\mathbb{H}$:** The random curve associated with (\star) with $W_t = \sqrt{\kappa}B_t$, $B$ a standard Brownian motion. Other domains: apply conformal mapping.
Simulations due to Tom Kennedy.
What about $\text{SLE}_6$ on a Brownian surface?

- SLE is a random curve defined \textit{on a simply connected domain in} $\mathbb{C}$.
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- This is necessary to define $\text{SLE}_6$ on a Brownian surface.
Embedding Brownian surfaces into $\mathbb{C}$

- It is conjectured that if one takes a uniformly random planar map and then embeds it "conformally" into $\mathbb{C}$ (using, e.g., circle packing) then the maps will converge to an embedding of the limiting Brownian surface into $\mathbb{C}$.

\[ \psi \]
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- Is this the right definition? It is if it is the scaling limit of percolation ...
Part III: Proof ideas
Proof overview

Proof has two steps:

- Construct subsequential limits of the percolation exploration
- Characterization theorem which singles out $\text{SLE}_6$ on a Brownian surface
Peeling exploration

A “peeling” of a map is a Markovian “exploration” in which faces are revealed one at a time.

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Holes cut out from \( \infty \) are independent □ of the disk given their \( \partial \)-length.

Have a hole □ of \( \partial \)-length \( k \) with probability \( \approx k^{-5/2} \).

The left/right \( \partial \)-length processes converge to independent stable-3/2 Lévy processes.

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Jason Miller (Cambridge)
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A subsequential limit of the percolation exploration is a random path on a Brownian surface with the following properties:

- Its left/right boundary lengths evolve as independent $3/2$-stable Lévy processes.
- The holes it cuts out are conditionally independent Brownian disks.
- The unexplored region is a Brownian surface.

It turns out that these three properties characterize SLE$_6$ on a Brownian surface.

Proved using the connection between Brownian surfaces and Liouville quantum gravity / GFF.
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Where are we now?

Convergence results for planar maps (RPM) decorated with a statistical physics model to SLE on a random surface.
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- Self-avoiding walks on RPM (Gwynne, M.)
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**Peanosphere sense** (Duplantier, M., Sheffield)
- FK-weighted RPM with $q \in (0, 4)$
  - Infinite volume (Sheffield)
  - finite volume (Gwynne, Mao, Sun and Gwynne, Sun)
- Bipolar orientation decorated RPM (Kenyon, M., Sheffield, Wilson)
- Active spanning tree decorated RPM (Gwynne, Kassel, M., Wilson)
- Schnyder woods (Li, Sun, Watson)
Thanks!