Michaelmas Term 2003 J. M. E. Hyland

Linear Algebra: Jordan Normal Form

One can regard the concrete proof of the existence of Jordan Normal Form (JNF) as consisting of three parts. First there is the decomposition into generalised eigenspaces. Then there is an analysis of (bases for) nilpotent endomorphisms. Finally we put things together to get the JNF

1 Generalised Eigenspaces

The following decomposition is relatively straightforward to establish. (Essentially it depends on the Chinese Remainder Theorem.) Suppose that $\alpha : V \to V$ is an endomorphism of a finite dimensional complex vector space V; and suppose that its minimal polynomial is

$$m(t) = (t - \lambda_1)^{d_1}(t - \lambda_2)^{d_2} \dots (t - \lambda_k)^{d_k}$$

so we have k distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$. Then V is the direct sum

$$V = V(\lambda_1) \oplus V(\lambda_2) \oplus \dots \oplus V(\lambda_k)$$

of the *generalised eigenspaces*

$$V(\lambda_i) = \ker((\alpha - \lambda_i I)^{d_i}) \qquad \text{for } i = 1, 2, \dots, k.$$

Furthermore each generalised eigenspace $V(\lambda_i)$ is α-invariant, that is α maps $V(\lambda_i)$ to $V(\lambda_i)$, and the endomorphism

$$\alpha_i : V(\lambda_i) \to V(\lambda_i) \ ; \ \mathbf{v} \to \alpha(\mathbf{v})$$

has minimal polynomial $(t - \lambda_i)^{d_i}$. In particular we see that each $\alpha_i - \lambda_i$ is nilpotent, so we are reduced to constructing a good base for a nilpotent endomorphism.

2 Nilpotent Endomorphisms

Now suppose that we have a nilpotent endomorphism α of a finite dimensional vector space V, so that there is a natural number d with $\alpha^{d-1} \neq 0$ but $\alpha^d = 0$. Then we have a manifestly increasing sequence

$$\{\mathbf{0}\} = K_0 \leq K_1 \leq K_2 \leq \dots \leq K_{d-1} \leq K_d = V$$

of subspaces $K_i = \ker(\alpha^i)$ with $\alpha(K_{i+1}) \leq K_i$. Note that this must be a strictly increasing sequence of subspaces as generally $\ker(\alpha^i) = \ker(\alpha^{i+1})$ implies $\ker(\alpha^{i+1}) = \ker(\alpha^{i+2})$ and so on. In particular it follows easily that $e \leq \dim(V) = n$. (It is easy to see just by looking at exponents that the minimal polynomial must divide characteristic polynomial; this gives yet another argument for the Cayley-Hamilton Theorem.) What we are about to do confirms that the JNF is determined by the various nullities in question.

Conceptually our first task is to find the cyclic blocks of size d. There are two essentially equivalent methods. 1) Look at $\mathrm{Im}\alpha^{d-1}$. (It is a space of eigenvectors.) Take a basis $\mathbf{f}_1, \dots, \mathbf{f}_r$, and let $\alpha^{d-1}(\mathbf{e}_i) = \mathbf{f}_i$. We see readily that

$$V = K_{d-1} \oplus W_{d-1} \qquad \text{where} \qquad W_{d-1} = \langle \mathbf{e}_1, \dots, \mathbf{e}_r \rangle \, .$$

For we clearly have $K_{d-1} \cap W_{d-1} = \{\mathbf{0}\}$; and if $\mathbf{x} \in V$ we can write $\alpha^{d-1}(\mathbf{x}) = \sum x_i \mathbf{f}_i$ and then

$$\mathbf{x} = (\mathbf{x} - \sum x_i \mathbf{e}_i) + \sum x_i \mathbf{x}_i \qquad \text{with} \qquad (\mathbf{x} - \sum x_i \mathbf{e}_i) \in K_{d-1} \qquad \text{and} \qquad \sum x_i \mathbf{e}_i \in W_{d-1} \, .$$

2) Take a complement $V = K_{d-1} \oplus W_{d-1}$ for K_{d-1} in V. Take a basis $\mathbf{e}_1, \ldots, \mathbf{e}_r$ for W_{d-1}. Since $K_{d-1} \cap W_{d-1} = \{\mathbf{0}\}$ it follows that setting $\alpha^{d-1}(\mathbf{e}_i) = \mathbf{f}_i$ we get a basis $\mathbf{f}_1, \ldots, \mathbf{f}_r$ for $\mathrm{Im}\alpha^{d-1}$.

Either way we get cyclic subspaces

$$\langle \mathbf{e}_i \rangle_\alpha = \langle \mathbf{e}_i, \alpha(\mathbf{e}_i), \ldots, \alpha^{d-1}(\mathbf{e}_i) \rangle ;$$

and a little argument will show independence so that we have a direct sum

$$\langle \mathbf{e}_1 \rangle_\alpha \oplus \cdots \oplus \langle \mathbf{e}_r \rangle_\alpha .$$

This gives us r cyclic subblocks of size d.

Now we seek the cyclic subblocks of size $d-1$. Again there are two ways to look at things.

1) Look at $\mathrm{Im}\alpha^{d-2}$. We already have independent elements $\alpha^{d-2}(\mathbf{e}_i)$, $\alpha^{d-1}(\mathbf{e}_i)$, the latter being eigenvectors. We can extend this to a basis of $\mathrm{Im}\alpha^{d-2}$ by elements which are also eigenvectors; and we find elements which go to these new elements under α^{d-2}. These together with the $\alpha(\mathbf{e_i})$ will be the basis for a space W_{d-2} with $K_{d-1} = K_{d-2} \oplus W_{d-2}$; and on their own they will give the cyclic subspaces of size $d-2$.

2) Just take a complement W_{d-2} for K_{d-2} in K_{d-1} which includes $\alpha(W_{d-1})$. Extending a basis for $\alpha(W_{d-1})$ to one for W_{d-2} provides elements which will give the cyclic subspaces of size $d-2$.

So we continue in this way. A succinct explanation is as follows. We show inductively (starting with W_{d-1}, and working downwards) that there are subspaces $W_{d-1}, W_{d-2}, \ldots, W_1, W_0$ which satisfy

$$K_{i+1} = K_i \oplus W_i \qquad \text{and} \qquad \alpha(W_{i+1}) \subset W_i$$

for $i = 0, 1, 2, \ldots, d-1$.

Then we can organise bases for these spaces to give a basis $(\mathbf{u}_m)_{m=1}^n$ for V with $\alpha(\mathbf{u}_m)$ equal to either $\mathbf{u}_{m-1}$ or $\mathbf{0}$ for every m. We have then decomposed V into a direct sum of cyclic subspaces on each of which α acts as in question 4 of example sheet 3. (There are more abstract ways to do all this!) In view of the answer to question 4, the matrix for α looks like

$$\begin{pmatrix} C_1 & 0 & 0 & \ldots & 0 \\ 0 & C_2 & 0 & \ldots & 0 \\ 0 & 0 & C_3 & \ldots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & C_m \end{pmatrix} .$$

where each C_i is a matrix of form K with 1s down the subdiagonal, and m is the dimension of the eigenspace.

3 Jordan Normal Form

Having analysed the nilpotent case, we return to the case of a general α as in Section 1. We apply what we have learnt about nilpotent endomorphisms in Section 2 to the nilpotent endomorphisms $\alpha_i - \lambda_i$. The matrix we get for α is of the form

$$\begin{pmatrix} B_1 & 0 & 0 & \ldots & 0 \\ 0 & B_2 & 0 & \ldots & 0 \\ 0 & 0 & B_3 & \ldots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & B_k \end{pmatrix} .$$

where the blocks B corresponding to the generalised eigenspaces $V(\lambda)$, are themselves of form

$$\begin{pmatrix} C_1 & 0 & 0 & \ldots & 0 \\ 0 & C_2 & 0 & \ldots & 0 \\ 0 & 0 & C_3 & \ldots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & C_m \end{pmatrix} .$$

when we take the bases for each $V(\lambda)$ found in Section 2, so that the C_i are of the form $\lambda I + K$. This gives the *Jordan Normal Form* for α.

4 Worked example

Consider the matrix

$$A = \begin{pmatrix} 3 & 0 & 1 & 0 & 1 & 0 \\ 1 & 3 & 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

One can easily check that the characteristic polynomial is $(t-2)^6$, so there is just one eigenvalue 2. So we consider

$$A - 2I = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 & -1 & 0 \\ 1 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix};$$

then

$$(A - 2I)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

and

$$(A - 2I)^3 = 0.$$

Just by looking at the nullities we can see that there will be cyclic subspaces of dimensions 3, 2 and 1 in the JNF.

First we find a generator for a cyclic subspace of dimension 3. We either see that $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \in \operatorname{Im}(A-2)^2$ and take a preimage $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ say; or we pick perhaps less obviously $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ to generate a complement to $\ker(A-2)^2$. So for a cyclic subspace of dimension 3 we get a basis

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

with the last an eigenvector.

Next we find a generator for a cyclic subspace of dimension 2. Either we look in $\mathrm{Im}(A-2)$ where we already have two linearly independent vectors one an eigenvector; we seek a further vector which is also an eigenvector and get most obviously $\begin{pmatrix}1\\0\\-1\\0\\0\\0\end{pmatrix}$, with preimage $\begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix}$. Alternatively we find the same vector generating, together with the vector $\begin{pmatrix}1\\1\\-1\\1\\0\\1\end{pmatrix}$ which we already have, a complement to $\ker(A-2)$ in $\ker(A-2)^2$. So for a cyclic subspace of dimension 2 we get a basis

$$\begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix} \longmapsto \begin{pmatrix}1\\0\\-1\\0\\0\\0\end{pmatrix}$$

with the last an eigenvector.

Finally we seek a generator for a cyclic subspace of dimension 1. So we either look in F^6 where we already have five independent vectors and find a sixth which is an eigenvector; or else we look for a complement to $\{\mathbf{0}\}$ in $\ker(A-2)$ and a basis for it including the two eigenvectors $\begin{pmatrix}0\\1\\0\\1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}1\\0\\-1\\0\\0\\0\end{pmatrix}$ we already have. Much the same either way, $\begin{pmatrix}0\\0\\0\\1\\0\\1\end{pmatrix}$ seems indicated. It generates a cyclic subspace of dimension 1.

In summary we have a basis (with $A-2$-action indicated)

$$\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix} \longmapsto \begin{pmatrix}1\\1\\-1\\1\\0\\1\end{pmatrix} \longmapsto \begin{pmatrix}0\\1\\0\\1\\0\\0\end{pmatrix} \qquad \begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix} \longmapsto \begin{pmatrix}1\\0\\-1\\0\\0\\0\end{pmatrix} \qquad \begin{pmatrix}0\\0\\0\\1\\0\\1\end{pmatrix}$$

with respect to which A has matrix

$$\begin{pmatrix}2&0&0&0&0&0\\1&2&0&0&0&0\\0&1&2&0&0&0\\0&0&0&2&0&0\\0&0&0&1&2&0\\0&0&0&0&0&2\end{pmatrix}.$$