COMPLEX ANALYSIS EXAMPLES 2

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These questions are a mix of questions from recent versions of the course together with some specific to my own take on the material. The questions are not all equally difficult.
I welcome both comments and corrections which can be sent to `m.hyland@dpmms.cam.ac.uk`.

1. By considering $\int_\gamma z^{-1}dz$ for γ the evident natural parametrization of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, show that for a, b real and positive,

$$\int_0^{2\pi}\frac{dt}{a^2\cos^2 t+b^2\sin^2 t}=\frac{2\pi}{ab}\,.$$

2. Using partial fractions calculate $\displaystyle\int_{|z|=2}\frac{dz}{z^2+1}$ and $\displaystyle\int_{|z|=2}\frac{dz}{z^2-1}$ from the integral formula. Are the answers an accident? Formulate and prove a result for polynomilas of arbitrary degree.
Now for an optional further twist, suppose that $p(z)=(z-a_1)\cdots(z-a_n)$ is a polynomial with n distinct roots. How many distinct values can $\displaystyle\int_\gamma\frac{dz}{p(z)}$ take for simple closed cuves γ not passing through any a_i?

3. Does the function $\dfrac{z}{1+z^2}$ have an anti-derivative in $|z|>1$? What about in $|z|<1$? Justify your answers. Now what about the function $\dfrac{1}{1+z^2}$?

4. Suppose that $f(z):\mathbb{C}\to\mathbb{C}$ is analytic and bounded. Take $a\neq b\in\mathbb{C}$, and consider $\displaystyle\int_{|z|=r}\frac{f(z)}{(z-a)(z-b)}dz$ for $r>\max(|a|,|b|)$. Give an upper bound for the absolute value of this integral. Using partial fractions evaluate the integral. Hence deduce that $f(a)=f(b)$.

5. Let f be an entire function such that for some $a\in\mathbb{C}$ and $r>0$, f takes no values in the disc $D(a,r)$. Prove that f must be constant.
Let u and v be the real and imaginary parts of an entire function f. Deduce that any of the conditions $|u|>|v|$, $u+v>0$ or $uv>0$ throughout $\mathbb{C}$ implies that f is constant.

6. A period of an analytic function $f:\mathbb{C}\to\mathbb{C}$ is a complex number a with $f(z+a)=f(z)$ for all $z\in\mathbb{C}$. Suppose that a and b are two non-zero periods of an entire f, with a/b not real. Show that f is constant.

7. Let $f(z)=\sum_0^\infty a_nz^n$ have radius of convergence R. Show that for $0\le r<R$,

$$\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^2d\theta=\sum_0^\infty|a_n|^2r^{2n}\,.$$

(i) Show that if $|f(z)|$ attains its maximum at 0 then it must be constant.
(ii) By considering $f(z)=1+z+\cdots+z^{n-1}$ show that

$$\int_0^{2\pi}\Big(\frac{\sin(n\theta/2)}{\sin\theta/2}\Big)^2d\theta=2\pi n\,.$$

8. Find the radii of convergence about 0 of the functions

$$\tan z\,, \qquad z\cot z\,, \qquad \frac{z}{e^z-1}\,.$$

By definition $\dfrac{z}{e^z-1} = \sum_0^\infty \dfrac{B_n}{n!} z^n$ where B_n are the Bernoulli numbers.

(i) Show that the Bernoulli numbers are rational, and that $\dfrac{z}{e^z-1} + \dfrac{z}{2}$ is an even function so that $B_{2k+1} = 0$ for $k \geq 1$. (In an alternative convention the odd Bernoulli numbers are not mentioned.)

(ii) Explicit calculation will show that the first few Bernoulli numbers are small. Why is it evident that the Bernoulli numbers are unbounded?

9. Suppose that f is a non-constant analytic function with power series expansion $f(z) = \sum_0^\infty c_n z^n$ in $|z| < R$. For $r < R$ let γ_r be the image of the circle $|z| = r$ under f, and consider the winding number $n(\gamma_r, c_0)$ of this curve about $f(0) = c_0$. Take n least with $n > 0$ and $c_n \neq 0$. Show that for $r > 0$ sufficiently small, $n(\gamma_r, c_0) = n$.

10. What are the zeros of the function $\sin \dfrac{1}{1-z}$ in the unit disc $D(0,1)$? Is what you find consistent with the isolation of zeros of analytic functions?

11. Show that the power series $\sum_1^\infty z^{n!}$ defines an analytic function in $D(0,1)$. Show that f cannot be analytically continued to any domain properly condaining $D(0,1)$. [Hint: consider $z = \exp 2\pi i p/q$ with $p/q \in \mathbb{Q}$.]

12. Let $f : D \to \mathbb{C}$ be analytic. Suppose that the real part $\Re f$ of f has a local maximum at some $a \in D$. Show that f is constant.

13. (i) Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function with $f(\frac{1}{n}) = \frac{1}{n}$ for all positive integers n. Show that $f(z) = z$.

(ii) Show that there is no analytic function $f : D(0,2) \to \mathbb{C}$ with $f(\frac{1}{n}) = \frac{1}{n+1}$ for all positive integers n.

14. (i) Let $f : \mathbb{C} \to \mathbb{C}$ be analytic. Suppose that $f(n) = n^2$ for all $n \in \mathbb{Z}$. Does it follow that $f(z) = z^2$?

(ii) Let $g : \mathbb{C} \to \mathbb{C}$ be analytic except possibly at some singularities. Suppose that $g(n) = \frac{1}{n^2}$ for all $n \in \mathbb{Z}$. What more do you need to know to identify g?

15. Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial function of degree $n \geq 1$. Show that $|f(z)| \to \infty$ as $|z| \to \infty$, and deduce that $|f(z)|$ takes a minimum at some $a \in \mathbb{C}$. Hence deduce that f has a zero in $\mathbb{C}$.
[This may appear to use a theorem of the course, but it should be clear that once you have the minimum you can proceed with bare hands.]

16. Find the Laurent expansion in powers of z of the function $1/(z^2-3z+2)$ in each of the domains

$$\{z \in C : |z| < 1\} \qquad \{z \in C : 1 < |z| < 2\} \qquad \{z \in C : |z| > 2\}\,.$$

What is its Laurent expansion in powers of $(z-1)$ in the domain $\{z : 0 < |z-1| < 1\}$?

17. Classify the singularities of each of the following functions.

$$\frac{z}{\sin z}\,, \quad \frac{1}{z^4+z^2}\,, \quad \cos\frac{\pi}{z^2}\,, \quad \frac{1}{z^2}\cos\frac{\pi z}{z+1}\,.$$

18. Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function.

(i) Show that f is a polynomial of degree $\leq n$ if and only if for some constant M we have $|f(z)| < M(1+|z|^n)$ for all $z \in \mathbb{C}$.

(ii) Show that f is a polynomial of positive degree if and only if $|f(z)| \to \infty$ as $|z| \to \infty$.