COMPLEX ANALYSIS EXAMPLES 1

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These questions are a mix of questions from recent versions of the course together with some specific to my own take on the material. The questions are not all equally difficult.
I welcome both comments and corrections which can be sent to `m.hyland@dpmms.cam.ac.uk`.

1. Let $T : \mathbb{C} = \mathbb{R}^2 \to \mathbb{R}^2 = \mathbb{C}$ be a real linear map. Show that T can be written $Tz = Az + B\overline{z}$ for unique $A, B \in \mathbb{C}$. Deduce that T is complex linear if and only if $B = 0$ if and only if T is complex differentiable.

2. Show that $f(z) = |z|^2$ is complex differentiable at $z = 0$ but nowhere else in $\mathbb{C}$. What about $f(z) = |z|$? What about $\overline{z}$?

3. Suppose that $f : D \to \mathbb{C}$ is differentiable at $a \in D$ with $f'(a) = b$. Let $\overline{D} = \{z | \overline{z} \in D\}$ and define $g : \overline{D} \to \mathbb{C}$ by $g(z) = \overline{f(\overline{z})}$. Show that $g : \overline{D} \to \mathbb{C}$ is complex differentiable at $\overline{a} \in \overline{D}$ with $g'(\overline{a}) = \overline{b}$.

4. Find all analytic functions of the form $f(z) = f(x + iy) = u(x) + iv(y)$.

5. Under what condition is the function $u(x, y) = ax^2 + 2bxy = cy^2$ the real part of an analytic function $f(z)$? Assuming the condition is satisfied give such an f

6. (i) Verify by direct calculation that the Cauchy-Riemann equations hold both for the exponential function $\exp(z)$ and for the principle branch $\log(z)$ of the logarithm.
(ii) Show that the function $u(x, y) = \cos x \cosh y$ is harmonic. Find a conjugate harmonic function $v(x, y)$, that is, a function v such that u and v together satisfy the Cauchy-Riemann equations. Find (explicitly in terms of z) an analytic function $f(z)$ with real part equal to u.

7. Let $f : D \to \mathbb{C}$ be analytic. Show that if any of the real part $\Re f$, imaginary part $\Im f$, modulus $|f|$, or argument $\arg f$ of f are constant then so is f.

8. (i) Find the set of complex numbers z for which $|\exp(z)| > 1$, the set of those for which $|\exp(iz)| > 1$ and the set of those for which $|\exp(z)| \leq e^{|z|}$.

(ii) Find the zeros of $1 + \exp z$, $\cosh z$, $\sinh z$ and $\sin z + \cos z$.

9. Write down the power series expansion for the principal branch of the logarithm about 1. What is its radius of convergence? Show that the series does indeed give the principle branch where defined.

10. What is the power i^i if in the definition of z^α we use the principal branch of the logarithm? What do we get with other branches? What about 1^i and i^1 What about e^i? Is it true that $\exp z = e^z$?

11. (i) Show that the transformation $w = \sin z$ maps the z-plane onto the w-plane. Where is this transformation locally conformal?

(ii) Show that $w = \sin z$ maps lines $y = b$ where $b \neq 0$ to ellipses in the w-plane. What happens to the line $y = 0$? What are the images of curves $x = a$?

(iii) Exhibit a domain mapped conformally onto its image by $w = \sin z$.

12. What are the images of ∞, i and 0 under the transformation $w = \dfrac{z-i}{z+i}$? That is enough information to determine the image of $\{z : \Re z > 0 \text{ and } \Im z > 0\}$. What is it? And what are the images of the other quarter planes? Finally what is the image of the quadrant $\{z \in D(0,1) : \Re z > 0 \text{ and } \Im z > 0\}$? And why?

13. Can you find conformal isomorphisms between the following domains and the unit disc?
(i) $\{z \in D(0,1) : \Re z > 0 \text{ and } \Im z > 0\}$.
(ii) $\{z : \Re z < 0 \text{ and } 0 < \Im z < \pi\}$.
(iii) $\{z : |z-1| < \sqrt{2} \text{ and } |z+1| < \sqrt{2}\}$.
(iv) $\{z : |z+1| > 1 \text{ and } |z+2| < 2\}$.
(v) $\{z : |z+1| > 0 \text{ and } |z+2| < 2\}$.

14. (i) Show that a Möbius transformation $w = \dfrac{az+b}{cz+d}$ maps the upper half plane to itself if and only if up to scalar multiple a, b, c and d are real with $ad - bc > 0$.

(ii) Show that a Möbius transformation maps the unit disc to itself if and only if it is of the form $w = \lambda \dfrac{z-b}{\bar{b}z-1}$ where $|b| < 1$ and $|\lambda| = 1$.

(iii) Characterise the Mobius transformations mapping the upper half plane to the unit disc.

15. (i) Find a Möbius transformation which sends the region between the two circles $|z| = 1$ and $|z-1| = 5/2$ into an annulus $\{w : 1 < |w| < R\}$. What is R? Did you have a choice of R?

(ii) Find a Möbius transformation which sends the unbounded region outside both of the circles $|z-5| = 4$ and $|z+5| = 4$ into an annulus $\{w : 1 < |w| < R\}$. Again what is R? And again did you have a choice?

16. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is analytic. Can you find $\int_{|z|=r} f(\bar{z})d\bar{z}$? What about $\int_{|z|=r} \overline{f(z)}d\bar{z}$? (Here $d\bar{z}$ has the natural interpretation $re^{-i\theta}d\theta$.

17. Let $\mathbb{C}^-$ be the complex plane with $(-\infty, 0]$ removed. Define $l : \mathbb{C}^- \to \mathbb{C}$ by $l(z) = \displaystyle\int_{[1,z]} \frac{dw}{w}$.

(i) Show that $l'(z) = \dfrac{1}{z}$ and deduce that $l(z)$ is the principal branch of the logarithm.

(ii) Compute $l(z)$ directly by integrating first along the straight line from 1 to $|z|$ and then round the circle centre 0 radius $|z|$ to z.

18. Suppose that $h : \mathbb{C} \to \mathbb{C}^\times$ is an analytic function with no zeros. Show that there is an analytic function $H : \mathbb{C} \to \mathbb{C}$ such that $h(z) = \exp H(z)$ for all z.