# COMPLEX ANALYSIS EXAMPLES 1

**Lent 2010** **J. M. E. Hyland**

These questions are a mix of questions from recent versions of the course together with some specific to my own take on the material. The questions are not all equally difficult.
I welcome both comments and corrections which can be sent to `m.hyland@dpmms.cam.ac.uk`.

**1**. Let $T : \mathbb{C} = \mathbb{R}^2 \to \mathbb{R}^2 = \mathbb{C}$ be a real linear map. Show that $T$ can be written $Tz = Az + B\overline{z}$ for unique $A, B \in \mathbb{C}$. Deduce that $T$ is complex linear if and only if $B = 0$ if and only if $T$ is complex differentiable.

**2** . Show that $f(z) = |z|^2$ is complex differentiable at $z = 0$ but nowhere else in $\mathbb{C}$. What about $f(z) = |z|$? What about $\overline{z}$?

**3.** Find all analytic functions of the form $f(z) = f(x + iy) = u(x) + iv(y)$.

**4**. (i) Verify by direct calculation that the Cauchy-Riemann equations hold both for the exponential function $\exp(z)$ and for the principle branch $\log(z)$ of the logarithm.
(ii) Show that the function $u(x, y) = \cos x \cosh y$ is harmonic. Find a conjugate harmonic function $v(x, y)$, that is, a function $v$ such that $u$ and $v$ together satisfy the Cauchy-Riemann equations. Find (explicitly in terms of $z$) an analytic function $f(z)$ with real part equal to $u$.

**5**. Let $f : D \to \mathbb{C}$ be analytic. Show that if any of the real part $\Re f$, imaginary part $\Im f$, modulus $|f|$, or argument $\arg f$ of $f$ are constant then so is $f$.

**6**. (i) Find the set of complex numbers $z$ for which $|e^z| > 1$, the set of those for which $|e^{iz}| > 1$ and the set of those for which $|e^z| \leq e^{|z|}$.
(ii) Find the zeros of $1 + e^z$, $\cosh z$, $\sinh z$ and $\sin z + \cos z$.

**7**. Write down the power series expansion for the principal branch of the logarithm about 1. What is its radius of convergence? Show that the series does indeed give the principle branch where defined.

**8**. (i) Suppose that $f : D \to \mathbb{C}$ is differentiable at $a \in D$ with $f'(a) = b$. Let $\overline{D} = \{z|\overline{z} \in D\}$ and define $g : \overline{D} \to \mathbb{C}$ by $g(z) = \overline{f(\overline{z})}$. Show that $g : \overline{D} \to \mathbb{C}$ is complex differentiable at $\overline{a} \in \overline{D}$ with $g'(\overline{a}) = \overline{b}$.
(ii) Suppose that $f : \mathbb{C} \to \mathbb{C}$ is analytic. Can you find $\int_{|z|=r} f(\overline{z}) d\overline{z}$? What about $\int_{|z|=r} \overline{f(z)} d\overline{z}$?

**9**. Let $\mathbb{C}^-$ be the complex plane with $(-\infty, 0]$ removed. Define $l : \mathbb{C}^- \to \mathbb{C}$ by $l(z) = \displaystyle\int_{[1,z]} \frac{dw}{w}$.
(i) Show that $l'(z) = \dfrac{1}{z}$ and deduce that $l(z)$ is the principal branch of the logarithm.
(ii) Compute $l(z)$ directly by integrating first along the straight line from 1 to $|z|$ and then round the circle centre 0 radius $|z|$ to $z$.

**10**. Suppose that $h : \mathbb{C} \to \mathbb{C}^\times$ is an analytic function with no zeros. Show that there is an analytic function $H : \mathbb{C} \to \mathbb{C}$ such that $h(z) = \exp H(z)$ for all $z$.

**11**. By considering $\int_\gamma z^{-1}dz$ for $\gamma$ the evident natural parametrization of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, show that for $a$, $b$ real and positive,

$$\int_0^{2\pi}\frac{dt}{a^2\cos^2 t+b^2\sin^2 t}=\frac{2\pi}{ab}\,.$$

**12**. (i) Use the Cauchy integral formula to calculate $\displaystyle\int_{|z|=1}\frac{e^{\alpha z}}{2z^2-5z+2}dz$ with $\alpha\in\mathbb{C}$.

(ii) By considering the real part of a suitable complex integral, show that for all $r\in(0,1)$,

$$\int_0^{\pi}\frac{\cos n\theta}{1-2r\cos\theta+r^2}d\theta=\frac{\pi r^n}{1-r^2}\,.$$

**13**. Using partial fractions calculate $\displaystyle\int_{|z|=2}\frac{dz}{z^2+1}$ and $\displaystyle\int_{|z|=2}\frac{dz}{z^2-1}$ from the integral formula. Are the answers an accident? Formulate and prove a general result
Suppose that $p(z)=(z-a_1)\cdots(z-a_n)$ is a polynomial with $n$ distinct roots. How many distinct values can $\displaystyle\int_\gamma\frac{dz}{p(z)}$ take for simple closed cuves $\gamma$ not passing through any $a_i$?

**14**. Does the function $\dfrac{z}{1+z^2}$ have an anti-derivative in $|z|>1$? What about in $|z|<1$? Justify your answers. Now what about the function $\dfrac{1}{1+z^2}$?

**15**. Suppose that $f(z):\mathbb{C}\to\mathbb{C}$ is analytic and bounded. Take $a\neq b\in\mathbb{C}$, and consider $\displaystyle\int_{|z|=r}\frac{f(z)}{(z-a)(z-b)}dz$ for $r>\max(|a|,|b|)$. Give an upper bound for the absolute value of this integral. Using partial fractions evaluate the integral. Hence deduce that $f(a)=f(b)$.

**16**. Let $f$ be an entire function such that for some $a\in\mathbb{C}$ and $r>0$, $f$ takes no values in the disc $D(a,r)$. Prove that $f$ must be constant.
Let $u$ and $v$ be the real and imaginary parts of an entire function $f$. Deduce that any of the conditions $|u|>|v|$, $u+v>0$ or $uv>0$ throughout $\mathbb{C}$ implies that $f$ is constant.

**17**. A period of an analytic function $f:\mathbb{C}\to\mathbb{C}$ is a complex number $a$ with $f(z+a)=f(z)$ for all $z\in\mathbb{C}$. Suppose that $a$ and $b$ are two non-zero periods of an entire $f$, with $a/b$ not real. Show that $f$ is constant.

**18**. Let $f(z)=\sum_0^\infty a_n z^n$ have radius of convergence $R$. Show that for $0\le r<R$,

$$\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^2d\theta=\sum_0^\infty|a_n|^2r^{2n}\,.$$

(i) Show that if $|f(z)|$ attains its maximum at 0 then it must be constant.
(ii) By considering $f(z)=1+z+\cdots+z^{n-1}$ show that

$$\int_0^{2\pi}\Big(\frac{\sin(n\theta/2)}{\sin\theta/2}\Big)^2d\theta=2\pi n\,.$$

.