ANALYSIS II EXAMPLES 2

Michaelmas 2005 **J. M. E. Hyland**

Again the Basic Questions focus on the examinable component of the course, while with Additional Questions are for those wishing to take things further. The questions are not equally difficult; most of the hardest are among the Additional Questions.
The sheet is almost identical to last year's sheet, itself a modification of a sheet prepared by Gabriel Paternain.
Please send comments and corrections to `m.hyland@dpmms.cam.ac.uk`.

1. Let $\alpha\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Show that

$$\sup\{\|\alpha(\mathbf{x})\| : \mathbf{x} \in \mathbb{R}^n,\ \|\mathbf{x}\| \le 1\} = \inf\{k \in \mathbb{R} : k \text{ is a Lipschitz constant for } \alpha\}\ .$$

Show that the function which sends α to the common value $\|\alpha\|$ of these two expressions is a norm on the vector space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ of all linear maps $\mathbb{R}^n \to \mathbb{R}^m$. [This is the *operator norm* on $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$.] Show also that

$$\|\alpha\| = \sup\{\|\alpha(\mathbf{x})\| : \mathbf{x} \in \mathbb{R}^n,\ \|\mathbf{x}\| = 1\} = \sup\{\|\alpha(\mathbf{x})\|/\|\mathbf{x}\| : \mathbf{x} \in \mathbb{R}^n,\ \mathbf{x} \ne \mathbf{0}\}\,.$$

2. Let ℓ_c be the space of all real sequences $(x_n)_{n=1}^{\infty}$ such that all but finitely many of the x_n are zero. With the natural (pointwise) addition and scalar multiplication ℓ_c is a real vector space. Find two norms in ℓ_c which are not Lipschitz equivalent, showing explicitly that they are norms. Can you find uncountably many norms which are not Lipschitz equivalent?

3. Prove again the following facts about convergence of sequences in an arbitrary normed space:
(i) If $(x_n) \to x$ and $(y_n) \to y$, then $(x_n + y_n) \to x + y$.
(ii) If $(x_n) \to x$ and $\lambda \in \mathbb{R}$, then $(\lambda x_n) \to \lambda x$.
(iii) If $x_n = x$ for all $n \ge n_0$, then $(x_n) \to x$.
(iv) If $(x_n) \to x$, then any subsequence (x_{n_i}) of (x_n) also converges to x.

4. Which of the following subsets of $\mathbb{R}^2$ are (a) open, (b) closed? (And why?)

(i) $\{(x, 0) : 0 \le x \le 1\}$. (ii) $\{(x, 0) : 0 < x < 1\}$. (iii) $\{(x, y) : y \ne 0\}$.
(iv) $\{(x, y) : x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$. (v) $\{(x, y) : xy = 1\}$.

5. Let E be a subset of $\mathbb{R}^n$ which is both open and closed. Show that E is either the whole of $\mathbb{R}^n$ or the empty set. [Method: suppose for a contradiction that $x \in E$ but $y \in \mathbb{R}^n \setminus E$. Define a function $f : [0, 1] \to \mathbb{R}$ by setting $f(t) = 1$ if the point $tx + (1 - t)y$ belongs to E, and $f(t) = 0$ otherwise; now recall a suitable theorem from Analysis I.]

6. (i) Show that the mapping $\mathbb{R}^{2n} \to \mathbb{R}^n$ which sends a $2n$-dimensional vector

$$(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$$

to

$$(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

is continuous. Deduce that if f and g are continuous functions from $E \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$, then so is their (pointwise) sum $f + g$.

(ii) By considering a suitable function $\mathbb{R}^{n+1} \to \mathbb{R}^n$, give a similar proof that if f is a continuous $\mathbb{R}^m$-valued function on $E \subseteq \mathbb{R}^n$, and λ is a continuous real-valued function on E, then the pointwise scalar product λf (i.e. the function whose value at x is $\lambda(x).f(x)$) is continuous on E.

7. If A and B are subsets of $\mathbb{R}^n$, we write $A+B$ for the set $\{a+b : a \in A, b \in B\}$. Show that if A and B are both closed and one of them is bounded, then $A+B$ is closed. Give an example in $\mathbb{R}^1$ to show that the boundedness condition cannot be omitted. If A and B are both open, is $A+B$ necessarily open? Justify your answer.

8. Let $f : \mathbb{R}^n \to \mathbb{R}^m$, and let E, F be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Determine which of the following statements are always true and which may be false, giving a proof or a counterexample as appropriate. [N.B. For the counterexamples, it suffices to take $n = m = 1$.]

(i) If $f^{-1}(F)$ is closed whenever F is closed, then f is continuous.
(ii) If f is continuous, then $f^{-1}(F)$ is closed whenever F is closed.
(iii) If f is continuous, then $f(E)$ is open whenever E is open.
(iv) If f is continuous, then $f(E)$ is bounded whenever E is bounded.
(v) If $f(E)$ is bounded whenever E is bounded, then f is continuous.

9. In lectures we proved that if E is a closed and bounded set in $\mathbb{R}^n$, then any continuous function defined on E has bounded image. Prove the converse: if every continuous real-valued function on $E \subseteq \mathbb{R}^n$ is bounded, then E is closed and bounded.
Now suppose that every bounded continuous real-valued function on $E \subseteq \mathbb{R}^n$ attains its bounds. Does it again follow that E is closed and bounded?

10. Consider the vector space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ of all linear maps $\mathbb{R}^n \to \mathbb{R}^m$, equipped with the operator norm defined in Question 1.

(i) Show that if $\|\alpha\| < \varepsilon$ then all the entries in the matrix A representing α (with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$) have absolute value less than ε. Conversely, show that if all entries of the matrix A have absolute value less than ϵ, then the norm of the linear map α represented by A is less than $nm\varepsilon$.

(ii) Deduce that convergence for sequences of linear maps is equivalent to 'entry-wise' convergence of the representing matrices, and so that $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is complete. How else might you prove this?

(iii) If $\alpha : \mathbb{R}^n \to \mathbb{R}^m$ and $\beta : \mathbb{R}^m \to \mathbb{R}^p$ are linear maps, show that the norm of the composite $\beta \circ \alpha$ is less than or equal to the product $\|\beta\|.\|\alpha\|$.

(iv) Now specialize to the case $n = m$. Show that if α is an endomorphism of $\mathbb{R}^n$ satisfying $\|\alpha\| < 1$, then the sequence whose kth term is $\iota + \alpha + \alpha^2 + \cdots + \alpha^{k-1}$ converges (here ι denotes the identity mapping), and deduce that $\iota - \alpha$ is invertible.

(v) Deduce that if α is invertible then so is $\alpha - \beta$ whenever $\|\beta\| < \|\alpha^{-1}\|^{-1}$. Hence conclude that the set of invertible linear maps is open in $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$.

11. Suppose that $g : [0,1] \times [0,1] \to \mathbb{R}$ is a continuous function. Show that one can define a map $G : C[0,1] \to C[0,1]$ by setting

$$(Gf)(x) = \int_0^1 g(x,t) f(t) dt$$

for $f \in C[0,1]$.

Show further that $G : C[0,1] \to C[0,1]$ is a linear map, and that G is a continuous map with respect to the uniform (sup) norm. Is it continuous with respect to the L^1 norm? (Can you justify your answer? If not at least have a guess!)

12. Recall from lectures the normed space ℓ^2. The Hilbert cube is the subset of ℓ^2 consisting of all $(x_n)_{n=1}^{\infty} = (x_1, x_2, x_3, \cdots)$ such that for each n, $|x_n| \le 1/n$. Show that the Hilbert cube is closed in ℓ^2, and that it has the Bolzano-Weierstrass property, that is, any sequence in the Hilbert cube has a convergent subsequence in it. (So the Hilbert cube is *compact*).

Additional Questions

13. Let $R[0,1]$ be the space of Riemann integrable functions on $[0,1]$.

(i) Why is $R[0,1]$ equipped with $\|f\| = \int_0^1 |f(t)|dt$ not a normed space? What could you imagine doing to remedy the situation?

(ii) Is $R[0,1]$ equipped with $\|f\| = \sup\{|f(t)| : t \in [0,1]\}$ a normed space? Is it complete?

14. (i) Let $C_b(\mathbb{R})$ be the space of continuous bounded functions $f : \mathbb{R} \to \mathbb{R}$ equipped with the uniform (sup) norm. Show that $C_b(\mathbb{R})$ is complete.

(ii) Let $C_0(\mathbb{R})$ be the subspace of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \to 0$ as $|x| \to \infty$. Is $C_0(\mathbb{R})$, equipped with the uniform norm, complete?

(iii) Let $C_c(\mathbb{R})$ be the subspace of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ for $|x|$ sufficiently large. Is $C_c(\mathbb{R})$, equipped with the uniform norm, complete?

15. Recall the normed space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ equipped with the operator norm. We now write the elements of $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ as matrices A, B, ...

(i) Show that in the operator norm the sum

$$\sum_{r=0}^{\infty} A^r/r!$$

converges to a matrix (and so linear map) $\exp A$.

(ii) Show that if A and B commute, then $\exp(A + B) = \exp A \exp B$.

(iii) What happens when A and B do not commute? (If you want to know more, look up the Campbell-Hausdorff formula.)

16. Let $T : E \to F$ be a linear map between normed spaces. Prove that the following are equivalent.

(i) T is continuous.

(ii) T is continuous at $\mathbf{0}$.

(iii) There is $0 < K < \infty$ such that $\|T(x)\| \le K\|x\|$ for all $x \in E$.

In such circumstances, T is a *bounded linear operator*. (What is the moral of this equivalence?)

Let $\mathcal{B}(E, F)$ be the space of bounded linear operators equipped with the usual operator norm $\|T\| = \sup\{\|T(x)\| : x \in E \text{ and } \|x\| \le 1\}$. Show that if F is complete then so is $\mathcal{B}(E, F)$.

17. A special case of the space $\mathcal{B}(E, F)$ of bounded linear operators gives the *dual* E^* of a normed space. It is defined to be $E^* = \mathcal{B}(E, \mathbb{R})$, the space of *bounded linear functionals* with the operator norm. Now recall from lectures the normed spaces ℓ^p, $1 \le p \le \infty$.

(i) Show that the dual of ℓ^1 is isomorphic to ℓ^∞.

(ii) Show that the dual of ℓ^2 is isomorphic to ℓ^2.

Now let $c_0 = \{(x_n)_{n=1}^\infty : x_n \to 0 \text{ as } n \to \infty\}$, equipped with the sup norm. Is c_0 complete? Identify the dual of c_0. Is the dual of ℓ^∞ isomorphic to ℓ^1? (Maybe you do not have enough background for the last point, but have a guess!)

18. Here is a different take on the relationship between ℓ^1 and ℓ^∞.

(i) Let $(x_i)_{i=1}^\infty$ be such that, for all $(y_i)_{i=1}^\infty \in \ell^\infty$, $\sum_i x_i y_i$ converges. Show that $(x_i)_{i=1}^\infty \in \ell^1$.

(ii) Let $(y_i)_{i=1}^\infty$ be such that, for all $(x_i)_{i=1}^\infty \in \ell^1$, $\sum_i x_i y_i$ converges. Show that $(y_i)_{i=1}^\infty \in \ell^\infty$.

(iii) Suppose that $\mathbf{x}^{(n)} = (x_i^{(n)})_{i=1}^\infty$ is a sequence in ℓ^1 such that, for all $(y_i)_{i=1}^\infty \in \ell^\infty$, $\sum_i x_i^{(n)} y_i \to 0$ as $n \to \infty$. Does $\mathbf{x}^{(n)} \to \mathbf{0}$ in ℓ^1?

(iv) Suppose that $\mathbf{y}^{(n)} = (y_i^{(n)})_{i=1}^\infty$ is a sequence in ℓ^∞ such that, for all $(x_i)_{i=1}^\infty \in \ell^1$, $\sum_i x_i y_i^{(n)} \to 0$ as $n \to \infty$. Does $\mathbf{y}^{(n)} \to \mathbf{0}$ in ℓ^∞?

19. Suppose that E is a normed space in which the unit ball $\{x : \|x\| \le 1\}$ is compact (in the sense that the Bolzano-Weierstrass Theorem holds for it). Show that E is finite dimensional.

20. Let E be a normed space. Can there exist bounded linear operators $S, T : E \to E$ such that $S \circ T - T \circ S = I$ where $I : E \to E$ is the identity?