Michaelmas Term 2011 J. M. E. Hyland

Linear Algebra: Example Sheet 3

The first 10 questions cover the course as I see it and should ensure good understanding. The remainder deal with a number of mostly minor point which may be instructive.

1. Show that none of the following matrices are conjugate:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   Is the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

   conjugate to any of them? If so, which?

2. Find a basis with respect to which the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$ has Jordan normal form. Hence compute the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}^n$.

3. Let V be a vector space of dimension n and α an endomorphism of V with $\alpha^n = 0$ but $\alpha^{n-1} \neq 0$. Show that there is a vector $\mathbf{x}$ such that $\mathbf{x}$, $\alpha(\mathbf{x})$, $\alpha^2(\mathbf{x})$, ... , $\alpha^{n-1}(\mathbf{x})$ is a basis for V.
   (i) Let $p(t) = a_0 + a_1 t + a_2 t^2 .... + a_k t^k$ be a polynomial. What is the matrix for $p(\alpha)$ with respect to the basis given above?
   (ii) Suppose that β is an endomorphism of V which commutes with α. Show that $\beta = p(\alpha)$ for some polynomial $p(t)$.
   (iii) What can you deduce using (i) and (ii)?

4. Let A be a non-singular square matrix in Jordan normal form. What is the inverse of A? What is the Jordan normal form of the inverse of A?

5. (i) Show that the Jordan normal form of a 3×3 complex matrix is determined by its characteristic and minimal polynomials. Give an example to show that this fails for 4×4 matrices.
   (ii) Let A be a complex 5×5 matrix with $A^4 = A^2 \neq A$. What are the possible minimum and characteristic polynomials? What are the possible Jordan normal forms?

6. Let P_2 be the space of polynomials in x, y of degree ≤ 2 in each variable. (So $\dim P_2 = 9$.)
   Consider the map $D : P_2 \to P_2$ given by

$$D(f) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.$$

   (i) What are the eigenvalues of the endomorphism D? Find the eigenspaces.
   (ii) Determine the Jordan normal form of the endomorphism D.
   (iii) Make a guess about what happens for the n^2-dimensional space P_n space of polynomials in x, y of degree $\leq n$ in each variable.

7. *(This is just a warm-up exercise!)*

   Show that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ form a basis for $\mathbb{R}^3$. Find the dual basis for the dual space $\mathbb{R}^{3*}$.

8. Let V be a 4-dimensional vector space over $\mathbb{R}$, and let $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ be the basis of V^* dual to the basis $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ for V. Determine, in terms of the ξ_i, the bases dual to each of the following:
(a) $\{\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_3\}$;
(b) $\{\mathbf{x}_1, 2\mathbf{x}_2, \frac{1}{2}\mathbf{x}_3, \mathbf{x}_4\}$;
(c) $\{\mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_2 + \mathbf{x}_3, \mathbf{x}_3 + \mathbf{x}_4, \mathbf{x}_4\}$;
(d) $\{\mathbf{x}_1, \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_2 + \mathbf{x}_1, \mathbf{x}_4 - \mathbf{x}_3 + \mathbf{x}_2 - \mathbf{x}_1\}$;
(e) $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4\}$.

9. (i) Show that if $\mathbf{x} \neq \mathbf{y}$ are vectors in the finite dimensional vector space V, then there is a linear functional $\theta \in V^*$ such that $\theta(\mathbf{x}) \neq \theta(\mathbf{y})$.
(ii) Suppose that V is finite dimensional. Let $A, B \leq V$. Prove that $A \leq B$ if and only if $A^o \geq B^o$. Show that $A = V$ if and only if $A^o = \{\mathbf{0}\}$. Deduce that a subset $F \subset V^*$ of the dual space spans V^* just when $f(\mathbf{v}) = 0$ for all $f \in F$ implies $\mathbf{v} = \mathbf{0}$.

10. Let P_n be the space of real polynomials of degree at most n. For $x \in \mathbb{R}$ define $\varepsilon_x \in P_n^*$ by $\varepsilon_x(p) = p(x)$. Show that $\varepsilon_0, \ldots, \varepsilon_n$ form a basis for P_n^*, and identify the basis of P_n to which it is dual.

---

11. Let θ and ϕ be linear functionals on V with the property that $\theta(\mathbf{x}) = 0$ if, and only if, $\phi(\mathbf{x}) = 0$. Show that θ and ϕ are scalar multiples of each other.

12. Let $\alpha : V \to V$ be an endomorphism of a finite dimensional complex vector space and let $\alpha^* : V^* \to V^*$ be its dual. Show that a complex number λ is an eigenvalue for α if, and only if, it is an eigenvalue for α^*. How are the algebraic and geometric multiplicities of λ for α and α^* related? How are the minimal and characteristic polynomials for α and α^* related?

13. Show that the dual of the space P of real polynomials is isomorphic to the space $\mathbb{R}^{\mathbb{N}}$ of all sequences of real numbers, via the mapping which sends a linear form $\xi : P \to \mathbb{R}$ to the sequence $(\xi(1), \xi(t), \xi(t^2), \ldots)$.

    In terms of this identification, describe the effect on a sequence $(a_0, a_1, a_2, \ldots)$ of the linear maps dual to each of the following linear maps $P \to P$:
    (a) The map D defined by $D(p)(t) = p'(t)$.
    (b) The map S defined by $S(p)(t) = p(t^2)$.
    (c) The map E defined by $E(p)(t) = p(t - 1)$.
    (d) The composite DS.
    (e) The composite SD.

    Verify that $(DS)^* = S^* D^*$ and $(SD)^* = D^* S^*$.

14. For A an $n \times m$ and B an $m \times n$ matrix over the field F, let $\tau_A(B)$ denote $\mathrm{tr}AB$.
Show that, for each fixed A, τ_A is a linear map $\mathrm{Mat}_{m,n} \to F$ from the space $\mathrm{Mat}_{m,n}$ of $m \times n$ matrices to F.
Now consider the mapping $A \mapsto \tau_A$. Show that it is a linear isomorphism $\mathrm{Mat}_{n,m} \to \mathrm{Mat}^*_{m,n}$.

Comments, corrections and queries can be sent to me at `m.hyland@dpmms.cam.ac.uk`.