Linear Algebra: Example Sheet 2

The first 12 questions cover the course and should ensure good understanding. The remainder vary in difficulty but cover some instructive points.

1. Find the reduced row echelon form of the matrices:

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix}; \qquad \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix};$$

and describe the spaces spanned by the rows.

2. (i) Show that an $n \times n$ matrix is invertible if and only if it is a product of elementary matrices.
(ii) Determine which of the following matrices are invertible, and find the inverses when they exist.

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 3 & 0 \end{pmatrix}.$$

3. Let $\lambda \in F$. Compute the determinant of the $n \times n$ matrix

$$\begin{pmatrix} \lambda & 1 & 1 & \dots & 1 \\ 1 & \lambda & 1 & \dots & 1 \\ 1 & 1 & \lambda & & 1 \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 1 & 1 & \dots & \lambda \end{pmatrix},$$

whose entries are λ down the diagonal and 1 elsewhere.

4. Let A and B be $n \times n$ matrices over a field $\mathbb{F}$. Show that the $(2n \times 2n)$ matrix

$$C = \begin{pmatrix} I & B \\ -A & O \end{pmatrix} \quad \text{can be transformed into} \quad D = \begin{pmatrix} I & B \\ 0 & AB \end{pmatrix}$$

by elementary row operations. By considering the determinants of C and D, obtain another proof that $\det AB = \det A \det B$.

5. For what values of a and b does the system of simultaneous linear equations

$$\begin{aligned} x + y + z &= 1 \\ ax + 2y + z &= b \\ a^2x + 4y + z &= b^2 \end{aligned}$$

have (i) a unique solution, (ii) no solution, (iii) many solutions?

6. (i) Suppose that V is a non-trivial finite dimensional real vector space. Show that there are no endomorphisms α, β of V with $\alpha\beta - \beta\alpha = I$.
(ii) Find endomorphisms of an infinite dimensional real vector space V which do satisfy $\alpha\beta - \beta\alpha = I$.

7. Compute the characteristic polynomials of the matrices

$$\begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Which of the matrices are diagonalizable over $\mathbb{C}$? Which over $\mathbb{R}$?

8. Find the eigenvalues and give bases for the eigenspaces of the following complex matrices:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The second and third matrices commute, so find a basis with respect to which they are both diagonal.

9. Show that if two $n \times n$ real matrices P and Q are conjugate when regarded as matrices over $\mathbb{C}$, then they are conjugate as matrices over $\mathbb{R}$.

10. Suppose that $\alpha \in \mathcal{L}(V, V)$ is invertible. Describe the characteristic and minimal polynomials and the eigenvalues of α^{-1} in terms of those of α.

11. Let α be an endomorphism of a finite dimensional complex vector space. Show that if λ is an eigenvalue for α then λ^2 is an eigenvalue for α^2. Show further that every eigenvalue of α^2 arises in this way. Are the eigenspaces $\ker(\alpha - \lambda I)$ and $\ker(\alpha^2 - \lambda^2 I)$ necessarily the same?

12. Show that an endomorphism $\alpha : V \to V$ of a finite dimensional complex vector space V has 0 as only eigenvalue if and only if it is *nilpotent*, that is, $\alpha^k = 0$ for some natural number k. Show that the minimum such k is at most $\dim(V)$. What can you say if the only eigenvalue of α is 1?

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13. Suppose that $A : \mathbb{C}^n \to \mathbb{C}^n$ has eigenvalues $\lambda_1, \ldots, \lambda_n$. Regard $\mathbb{C}^n \cong \mathbb{R}^{2n}$ as a $2n$-dimensional real vector space, and consider the endomorphism $A : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. What are the complex eigenvalues of this A?

14. Let A be an $n \times n$ matrix all the entries of which are real. Show that the minimum polynomial of A, over the complex numbers, has real coefficients.

15. Let $f(x) = a_0 + a_1 x + \ldots + a_n x^n$, with $a_i \in \mathbb{C}$, and let C be the *circulant* matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_n \\ a_n & a_0 & a_1 & \ldots & a_{n-1} \\ a_{n-1} & a_n & a_0 & \ldots & a_{n-2} \\ \vdots & & & \ddots & \vdots \\ a_1 & a_2 & a_3 & \ldots & a_0 \end{pmatrix}.$$

Show that the determinant of C is $\det C = \prod_{j=0}^{n} f(\zeta^j)$, where $\zeta = \exp(2\pi i/(n+1))$.

16. Let A, B be $n \times n$ matrices, where $n \geq 2$. Show that, if A and B are non-singular, then

$$(i)\ \mathrm{adj}(AB) = \mathrm{adj}(B)\mathrm{adj}(A)\,, \quad (ii)\ \det(\mathrm{adj}A) = (\det A)^{n-1}\,, \quad (iii)\ \mathrm{adj}(\mathrm{adj}A) = (\det A)^{n-2}A\,.$$

What happens if A is singular?

Show that the rank of the matrix $\mathrm{adj}A$ is $\mathrm{r}(\mathrm{adj}(A)) = \begin{cases} n & \text{if } \mathrm{r}(A) = n; \\ 1 & \text{if } \mathrm{r}(A) = n-1; \\ 0 & \text{if } \mathrm{r}(A) \leq n-2. \end{cases}$

17. (i) An endomorphism $\alpha : V \to V$ of a finite dimensional vector space is *periodic* just when $\alpha^k = I$ for some k. Show that a periodic matrix is diagonalisable over $\mathbb{C}$.
(ii) Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be a basis for a vector space V over $\mathbb{C}$. For σ a permutation of $\{1, \ldots, n\}$, define $\widehat{\sigma} : V \to V$ by $\widehat{\sigma}(\mathbf{e}_i) = \mathbf{e}_{\sigma(i)}$. What are the eigenvalues of $\widehat{\sigma}$? [Consider the case when σ is a cycle first?]
(iii) Is every periodic endomorphism of the form $\widehat{\sigma}$ for some choice of permutation σ and basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$?

18. Let $\alpha : V \to V$ be an endomorphism of a finite dimensional vector space V with $\mathrm{tr}(\alpha) = 0$.
(i) Show that, if $\alpha \neq 0$, there is a vector $\mathbf{v}$ with $\mathbf{v}, \alpha(\mathbf{v})$ linearly independent. Deduce that there is a basis for V relative to which α is represented by a matrix A with all of its diagonal entries equal to 0.
(ii) Show that there are endomorphisms β, γ of V with $\alpha = \beta\gamma - \gamma\beta$.

Comments, corrections and queries can be sent to me at `m.hyland@dpmms.cam.ac.uk`.