Linear Algebra: Example Sheet 1

The first 12 questions cover the course and should ensure good understanding of the course: the remainder are provided for amusement, or as a challenge, according to taste.

1. Suppose that T and U are subspaces of the vector space V. Show that $T \cup U$ also a subspace of V if and only if either $T \leq U$ or $U \leq T$.

2. Let T, U, W be subspaces of V.
(i) Give explicit counter-examples to the following statements.
(a) $T + (U \cap W) = (T + U) \cap (T + W)$. (b) $(T + U) \cap W = (T \cap W) + (U \cap W)$.
(ii) Show in both (a) and (b) that the equality can be replaced by a valid inclusion of one side in the other.

3. Show that if $T \leq W$, then $(T + U) \cap W = (T \cap W) + (U \cap W)$.
Deduce that in general one has $T \cap (U + (T \cap W)) = (T \cap U) + (T \cap W)$.

4. If α and β are linear maps from U to V, show that $\alpha + \beta$ is linear and that
$$\mathrm{Im}(\alpha + \beta) \leq \mathrm{Im}\alpha + \mathrm{Im}\beta \qquad \text{and} \qquad \ker(\alpha + \beta) \geq \ker\alpha \cap \ker\beta.$$
Show by example that each inclusion may be strict.

5. Suppose that $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a base for V. Which of the following are also bases?
(a) $\{\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \ldots, \mathbf{e}_{n-1} + \mathbf{e}_n, \mathbf{e}_n\}$.
(b) $\{\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \ldots, \mathbf{e}_{n-1} + \mathbf{e}_n, \mathbf{e}_n + \mathbf{e}_1\}$.
(c) $\{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, \ldots, \mathbf{e}_{n-1} - \mathbf{e}_n, \mathbf{e}_n\}$.
(d) $\{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, \ldots, \mathbf{e}_{n-1} - \mathbf{e}_n, \mathbf{e}_n - \mathbf{e}_1\}$.
(e) $\{\mathbf{e}_1 - \mathbf{e}_n, \mathbf{e}_2 + \mathbf{e}_{n-1}, \ldots, \mathbf{e}_n + (-1)^n\mathbf{e}_1\}$.

6. For each of the following pairs of vector spaces (V, W) over $\mathbb{R}$, either give an isomorphism $V \to W$ or show that no such isomorphism can exist. (Here P denotes the space of polynomial functions $\mathbb{R} \to \mathbb{R}$, and $C[a, b]$ denotes the space of continuous functions defined on the closed interval $[a, b]$.)
(a) $V = \mathbb{R}^4$, $W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_2 + x_3 + x_4 + x_5 = 0\}$.
(b) $V = \mathbb{R}^5$, $W = \{p \in P : \deg\ p \leq 5\}$.
(c) $V = C[0, 1]$, $W = C[-1, 1]$.
(d) $V = C[0, 1]$, $W = \{f \in C[0, 1] : f(0) = 0, f \text{ continuously differentiable }\}$.
(e) $V = \mathbb{R}^2$, $W = \{\text{solutions of } \ddot{x}(t) + x(t) = 0\}$.
(f) $V = \mathbb{R}^4$, $W = C[0, 1]$.
(g) $V = P$, $W = \mathbb{R}^{\mathbb{N}}$.

7. Let
$$\begin{aligned} U &= \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_3 + x_4 = 0,\ 2x_1 + 2x_2 + x_5 = 0\}, \\ W &= \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_5 = 0,\ x_2 = x_3 = x_4\}. \end{aligned}$$
Find bases for U and W containing a basis for $U \cap W$ as a subset. Give a basis for $U + W$ and show that
$$U + W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + 2x_2 + x_5 = x_3 + x_4\}.$$

8. Find the ranks of the following matrices A, and give bases for the kernel and image of the linear maps $\mathbf{x} \mapsto A\mathbf{x}$.
$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad ; \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad ; \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

9. Let $\alpha : U \to V$ be a linear map between two finite dimensional vector spaces and let W be a vector subspace of U. Show that the restriction of α to W is a linear map $\alpha|_W : W \to V$ which satisfies
$$\mathrm{r}(\alpha) \geq \mathrm{r}(\alpha|_W) \geq \mathrm{r}(\alpha) - \dim(U) + \dim(W)\ .$$
Give examples to show that either of the two inequalities can be an equality.

10. Let $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by $\alpha : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Find the matrix representing α relative to the base $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ for both the domain and the range.
Write down bases for the domain and range with respect to which the matrix of α is the identity.

11. Find the reduced column echelon form of the matrices:

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix} ; \qquad \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix} ;$$

and describe the spaces spanned by their columns. In case the matrix is invertible give its inverse.

12. Let Y and Z be subspaces of the finite dimensional vector spaces V and W respectively. Show that $R = \{\theta \in \mathcal{L}(V, W) : \theta(\mathbf{x}) \in Z \text{ for all } \mathbf{x} \in Y\}$ is a subspace of $\mathcal{L}(V, W)$. What is the dimension of R?

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13. Let S be the vector space of real sequences $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ and define a map $\Delta : S \to S$ by

$$\Delta : \mathbf{x} \mapsto \mathbf{y} \qquad \text{where} \qquad y_n = x_{n+1} - x_n \ .$$

Show that Δ is linear and describe its kernel and image. Similarly describe the kernel and image of Δ^2 (the composite of Δ with itself). What about Δ^3?

14. X and Y are linearly independent subsets of a vector space V; no member of X is expressible as a linear combination of members of Y, and no member of Y is expressible as a linear combination of members of X. Is the set $X \cup Y$ necessarily linearly independent? Give a proof or counterexample.

15. Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_s\}$ be linearly independent subsets of a vector space V, and suppose $r \leq s$. Show that it is possible to choose distinct indices $i_1, i_2, \ldots, i_r$ from $\{1, 2, \ldots, s\}$ such that, if we delete each $\mathbf{y}_{i_j}$ from Y and replace it by $\mathbf{x}_j$, the resulting set is still linearly independent.

16. Let U be a vector subspace of $\mathbb{R}^N$ (where N is finite). Show that there is a finite subset I of $\{1, 2, \ldots, N\}$ for which the subspace $W = \langle\{\mathbf{e}_i : i \in I\}\rangle$ is a complementary subspace to U in $\mathbb{R}^N$.

17. Let $\alpha : U \to V$ and $\beta : V \to W$ be maps between finite dimensional vector spaces, and suppose that $\ker(\beta) = \mathrm{Im}(\alpha)$. Show that bases may be chosen for U, V and W with respect to which α and β have matrices

$$\begin{pmatrix} I_r & O \\ O & O \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} O & O \\ O & I_{n-r} \end{pmatrix}$$

respectively, where $\dim(V) = n$, $r = \mathrm{r}(\alpha)$ and I_k is the identity $k \times k$ matrix .

18. (i) Let $\alpha : V \to V$ be an endomorphism of a finite dimensional vector space V. Set $r_i = \mathrm{r}(\alpha^i)$. Show that $r_i \geq r_{i+1}$ and that $(r_i - r_{i+1}) \geq (r_{i+1} - r_{i+2})$.
(ii) Suppose that $\dim(V) = 5$, $\alpha^3 = 0$, but $\alpha^2 \neq 0$. What possibilities are there for $\mathrm{r}(\alpha)$ and $\mathrm{r}(\alpha^2)$?

19. Let T, U, V, W be vector spaces over the same field and let $\alpha : T \to U, \beta : V \to W$ be fixed linear maps. Show that the mapping $\Phi : \mathcal{L}(U, V) \to \mathcal{L}(T, W)$ which sends θ to $\beta \circ \theta \circ \alpha$ is linear. If the spaces are finite-dimensional and α and β have rank r and s respectively, find the rank of Φ.

20. An $n \times n$ magic square is a square matrix whose rows, columns and two diagonals all sum to the same quantity. Find the dimension of the space of $n \times n$ magic squares.

Comments, corrections and queries can be sent to me at `m.hyland@dpmms.cam.ac.uk`.