DANA SCOTT

THE OXFORD YEARS

A CELEBRATION
OF SIMPLICITY

BACKGROUND

Set Theory

Arithmetic

Recursion theory

Model theory

Modal logic

Constructive logic

Models for λ-calculus

Domains & programming semantics

These dominated the Oxford years.

Toposes

Domains

What is a model of the λ-calculus?

Traditional answer:

Take $(D, \cdot)$: an applicative structure.

Consider the closed terms of the λ-calculus with constants from D.

Take an interpretation $t \longmapsto [\![t]\!] \in D$ of these terms such that the laws of the λ-calculus hold.

That's it!

Note:

1. This is what is called a

<u>λ-algebra</u>

2. Since terms are given by an inductive definition, we can make the definition <u>look better</u> (or worse) as in Tarski's definition of truth.

What is a model of the λ-calculus?

Scott's answer:

An object D in a cartesian closed category equipped with a retraction

$$D^D \lhd D$$

Precise connexion:
(Rough statement)

- $(D, \cdot)$ is a λ-algebra (in the traditional sense) iff D is the global sections of a model in Scott's sense.
- $(D, \cdot)$ is a λ-model iff D is the global sections of a Scott model with enough points.
- $(D, \cdot)$ is a $\lambda\eta$-model iff D is the global sections of a Scott model with $D = D^D$.

Outline of proof
(perverse?)

D the traditional model

Consider $M = \{ d \in D \mid d = \lambda x. dx \}$ and give it the structure of a monoid under composition

$$x, y \longmapsto x \circ y = \lambda z.\, x(yz)$$

Consider $Sets_M$ = category of right M-sets
= presheaves on M,
and write $D_\bullet$ for the generic object = right regular representation.

Check that $\Gamma(D_\bullet) = D$.

Consider the idempotent $e = \lambda xy.\, xy$ in M

$$e : x \longmapsto \lambda y.\, xy$$

and split it (i.e. its image under Yoneda) in $Sets_M$ to get

$$E_\bullet = \{ d \in D \mid d = \lambda xy.\, dxy \} \subseteq D_\bullet$$

under the standard action.

Show that

$$E_\bullet \cong D_\bullet^{D_\bullet} \quad \text{in } Sets_M.$$

Done we have $D_\bullet^{D_\bullet} \lhd D_\bullet$ in $Sets_M$

with $\Gamma(D_\bullet) = D$

(and interpretations obviously correspond)

Genericity : $E_\circ \cong D_\circ^{D_\circ}$

$$D_\circ^{D_\circ} = \left\{ f : \overset{M}{D_\circ} \times \overset{M}{D_\circ} \longrightarrow \overset{M}{D_\circ} \;\middle|\; f(m,n)_\circ d = f(m_\circ d, n_\circ d) \right\}$$

$$\begin{array}{c} E_\circ \longrightarrow D_\circ^{D_\circ} \\ \hline a \longmapsto f_a \end{array} \qquad f_a(m,n) = \lambda x.\ a\,(mx)(nx)$$

$$\begin{array}{c} D_\circ^{D_\circ} \longrightarrow E_\circ \\ \hline f \longmapsto \hat{f} \end{array} \qquad \hat{f} = \lambda xy.\ \big(f(\mathrm{fst}, \mathrm{snd})\big)\langle x, y\rangle$$

$$\widehat{f_a} = a \qquad f_{\hat{f}} = f$$

Note: A little light coding needed for products in λ-calculus.

$(\mathrm{fst}, \mathrm{snd})$ is a "generic pair" in $D_\circ \times D_\circ$.

The proof reflects the fact that also $D_\circ \times D_\circ \lhd D_\circ$

Basic Perspective

Let $\mathbb{D}$ = category of retracts of D
= " " idempotents of M
= Karoubi envelope of M
= Cauchy completion of M

$\mathbb{D} \hookrightarrow \mathrm{Sets}_M$ is a cartesian closed subcategory containing $D_\circ$ with $D_\circ^{D_\circ} \lhd D_\circ$

(Immediate from $D_\circ \times D_\circ \lhd D_\circ$

$D_\circ^{D_\circ} \lhd D_\circ$)

Why is Scott's answer good?

- There is a huge gain in mathematical insight.

Traditional answer is profoundly uninformative.

cf Tarski's definition of truth

Models for higher order logic, type theory

- It clarifies odd points in traditional treatments.

λ-algebra – general case

λ-model – D has enough points.

highlights the strangeness of modern preference for λ-models.

- It is more general.

D in a ccc is not determined by its points.

- One can work effectively with it.

  - Checking one has a model
  - One can find new models from old (Freyd's 'curious derived structures')
  - One can uncover additional structure (Theorem of Paul Taylor.)

- It is part of a big family of ideas

E.g. Pitts on 'Polymorphism is set-theoretic' but most of all

- IT IS SIMPLE!

So why has it not caught on?

Example

<u>Theorem</u> (Paul Taylor) For any model D of the λ-calculus, the category $\mathbb{D}$ of retracts is relatively cartesian closed – in an interesting way! (cf. Nancy McCracken.)

Abstract explanation

- We have $D^D \lhd D$ in a topos $\mathcal{D} = \mathrm{Sets}_M$
- For $I \in \mathcal{D}$ define $\mathbb{D}(I)$ = category of retracts of $\Delta_I D$.
- Then $\mathbb{D}(I) \hookrightarrow \mathcal{D}/I$ gives a cartesian closed subfibration of the standard $\mathcal{D}^{\to} \to \mathcal{D}$
- Restrict over $\mathbb{D}$ to get the fibration claimed to be relatively cartesian closed (i.e. has Π, Σ along display maps)
- The display maps are $J \xrightarrow{\alpha} I$ in $\mathbb{D}$ such that

$$\begin{tikzcd}
J \arrow[rr, shift left, "i"] \arrow[rr, phantom, "\lhd"] \arrow[dr, "\alpha"'] & & D \times I \arrow[ll, shift left, "p"] \arrow[dl, "\mathrm{snd}"] \\
& I &
\end{tikzcd}$$

NB. $I \in \mathbb{D}$ implies $J \in \mathbb{D}$ here.

Proof:-

Take $\begin{pmatrix} P \\ \downarrow \\ J \end{pmatrix} \in \mathbb{D}(J) \hookrightarrow \mathcal{D}/J$ and show that the product $\Pi_\alpha \begin{pmatrix} P \\ \downarrow \\ J \end{pmatrix} \in \mathcal{D}/I$ is in fact in $\mathbb{D}(I)$.

① Since $\begin{pmatrix} P \\ \downarrow \\ J \end{pmatrix} \triangleleft \Delta_J D$ we have $\Pi_\alpha \begin{pmatrix} P \\ \downarrow \\ J \end{pmatrix} \triangleleft \Pi_\alpha \Delta_J D$.

② Abstract calculation $\Rightarrow$

$$\Pi_\alpha \Delta_J D \triangleleft \Pi_{snd} \Delta_{D \times I} D$$

③

$$\Pi_{snd} \Delta_{D \times I} D \cong \Delta_I (D^D)$$

④ As $D^D \triangleleft D$, $\quad \Delta_I (D^D) \triangleleft \Delta_I D$

This shows $\Pi_\alpha \begin{pmatrix} P \\ \downarrow \\ J \end{pmatrix} \triangleleft \Delta_I D$ as required.

OXFORD THEMES

Toposes and constructive logic

(toposes as models for HOIL)

Domains and programming semantics

(solution of recursive domain equations)

Connected by the idea of <u>classifying toposes</u>

(Duality: $\begin{pmatrix}\text{Points}\\ \text{Models}\end{pmatrix}$ vs (Prop$^{\underline{n}}$s))

NB. Two incursions of logic.

AND ONWARDS

Simple explanations:

Models of type theory

Theory of choice sequences (Fourman)

Realizability

Synthetic domain theory

His influence lingers on
in the UK
+ especially at Edinburgh!