Glueing and Orthogonality for Models of Linear Logic

Martin Hyland*
DPMMS
University of Cambridge

Andrea Schalk†
Department of Computer Science
University of Manchester

February 8, 2001

Abstract

We present the general theory of the method of glueing and associated technique of orthogonality for constructing categorical models of all the structure of linear logic: in particular we treat the exponentials in detail. We indicate simple applications of the methods and show that they cover familiar examples.

1 Introduction

This paper is a contribution to the model theory of linear logic. We give a concrete account of some central techniques for constructing categorical models. To some extent these are implicit in the literature. However we give here a proper abstract formulation of glueing and orthogonality and so make clear the wide range of possible applications. We have focussed in particular on the exponentials of linear logic and demonstrate how even this structure may be handled quite generally.

The techniques we exhibit have a variety of applications, for example to proof theoretic questions. However we shall not discuss those in detail at this time. Rather we concentrate on showing how more or less familiar models arise by applications of these techniques. We are particularly interested in applications to the theory of Abstract Games. That will be the subject of a companion paper. For a general outline see [37].

We are aware of more abstract formulations and extensions of some of our results, but we do not strive for maximal generality. Rather we hope to convey the scope and flavour of the techniques.

The paper is organised as follows. In Section 2, we explain the notions of static categorical models for linear logic, and give a range of examples. Classical linear logic is self-dual, and we give a brief account of how corresponding categories arise in Section 3. With this background in place we turn to the main topics of the paper. Our first central technique is that of glueing. We give a number of versions in Section 4, and briefly discuss applications. As a technique for constructing models, glueing becomes truly effective when extended by our second central technique. In Section 5 we identify a notion of orthogonality for maps in a glued category,

*DPMMS, Centre for Math. Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB. Voice: +44 (0)1223 337986. Fax: +44 (0)1223 337920. Email: M.Hyland@dpmms.cam.ac.uk

†Department of Computer Science, University of Manchester, Oxford Road, Manchester, M13 9PL. Voice: +44 (0)161 275 6194. Fax: +44 (0)161 275 6204. Email: A.Schalk@cs.man.ac.uk

and show how it can be used to construct various subcategories. When the glued category is a model for linear logic so usually are our orthogonality categories. We close with a discussion of examples.

2 Models of linear logic

In this section we make precise the notion of model for linear logic with which we shall be concerned. We discuss maps between models, survey some examples and close by giving a simple construction on models.

2.1 Categorical models

In this paper we are concerned with static models for intuitionistic and more particularly classical linear logic.

Definition 1 *A* categorical model of intuitionistic linear logic *consists of a category which*

- *is symmetric monoidal closed;*
- *has finite products;*
- *is equipped with a linear exponential comonad.*

To model the classical calculus we additionally require a strong duality. So a model for classical linear logic *consists of a category which*

- *is $*$-autonomous;*
- *has finite products and (so) finite coproducts;*
- *is equipped with a linear exponential comonad and (so) a linear exponential monad.*

Note that in each case there are three components to the categorical structure corresponding to the *multiplicative*, *additive* and *exponential* structure of linear logic. We use this pattern when presenting models and constructions of models throughout the paper. Sometimes a model of intuitionistic linear logic has coproducts. We allow for this possibility in our treatment but we do not consider other possibilities such as full intuitionistic linear logic.

A model of intuitionistic linear logic is *affine* when $\mathbf{I} \longrightarrow \mathbf{1}$ is an isomorphism. We say that a model of classical linear logic is *validates MIX* when it supports the mix rule in the sense of Cockett and Seely [19]: then we have a sensible map $\mathsf{mix}\colon \bot \longrightarrow \mathbf{I}$ and so coherent maps $A \otimes B \longrightarrow A \parr B$.

A model of classical linear logic is a model of intuitionistic logic with a strong duality; the duality implies the existence of dual structures: coproducts where before we had only products, a second tensor to model par, and a monad such that every free coalgebra comes with a (natural) commutative monoid structure (with respect to par) modelling ?, or 'why not'; but we do not need to dwell on these aspects.

The only part of the structure which perhaps needs explanation is that of a *linear exponential comonad.* This is a monoidal comonad $(!, \varepsilon, \delta)$ such that each free coalgebra is equipped with the structure of (co)commutative comonoid (e, d), naturally with respect to maps of coalgebras. Another way of expressing the condition is to say that the image of the free functor ! in the category of coalgebras consists of commutative comonoids and comonoid maps. In fact, it follows that the category of all coalgebras is a category of comonoids and comonoid morphisms. For the intuitionistic case, the structures involved are given in [11, 12], or see the corresponding extended exercise in [35]. For a detailed step-by-step development and all diagrams required to commute, see [15, 16]. The most concise definition (but one not usually found in the literature) is the following.

Definition 2 *We say that a symmetric monoidal category has a* linear exponential comonad *if it has a monoidal comonad equipped so that the category of coalgebras is a category of commutative comonoids (with respect to $\otimes$).*

It should be noted that if a symmetric monoidal closed (or $*$-autonomous) category admits free commutative comonoids then it certainly has a linear exponential comonad.

In our treatment we shall not bother to distinguish between what is the 'same' structure in different categories; for example we shall use $\mathbf{I}$ for the unit and $\otimes$ for the tensor product both in $\mathbf{C}$ and in models constructed from $\mathbf{C}$.

2.2 Degenerate models

A crucial role in the theory of categorical models is played by the class of degenerate models. In these the tensor ($\otimes$) and par ($\parr$) of linear logic are identified; further the additives 'with' (&) and 'or' ($\oplus$) are identified, and so are finally the two exponentials 'of course' (!) and 'why not' (?). We make this precise as follows.

Definition 3 *A* degenerate model for classical linear logic *consists of a category which*

- *is compact closed;*
- *has finite biproducts;*
- *has a bi-exponential.*

The notion of compact closed category (see Kelley and Laplaza [39]) and of biproducts (see MacLane [45]) are well-known, so we just indicate what is our notion of bi-exponential. By this we mean a self-dual functor on a compact closed category equipped with the structure of a linear exponential comonad and so dually with that of a linear exponential monad.

2.3 Maps of models

We need to consider functors between models of linear logic in order to make precise connections between the various models which we consider. Sometimes we encounter functors which preserve structure. (While this is not unproblematic in principle it will be for us as structure is preserved on the nose in the few cases where it is preserved at all. So we do not need the general analysis of (Blackwell, Kelly and Power [17, 40]). We shall however find very many more instances of a weaker notion. We shall have functors which are monoidal in the usual sense and also *linearly distributive* with respect to the linear exponential comonads.

Definition 4 *Let* $\mathbf{C}$*,* $\mathbf{D}$ *be models for linear logic. The functor* $F\colon \mathbf{C} \longrightarrow \mathbf{D}$ *is* linearly distributive[1] *if and only if* F *is monoidal (with structure* $m_{\mathbf{I}}, m_{C,C'}$*) and is equipped with a distributive law in the sense of Beck* [9] *(see also* [48]*)* $\kappa\colon !F \longrightarrow F!$ *respecting the comonoid structure, in the sense that*

$$\begin{tikzcd}
\mathbf{I} & !F(C) \arrow[l, "e_{F(C)}"'] \arrow[r, "d_{F(C)}"] \arrow[d, "\kappa_C"] & !F(C)\otimes !F(C) \arrow[d, "m_{!C,!C}\cdot(\kappa_C\otimes\kappa_C)"] \\
F(I) \arrow[from=u, "m_{\mathbf{I}}"'] & F(!C) \arrow[l, "F(e_C)"] \arrow[r, "F(d_C)"'] & F(!C\otimes !C)
\end{tikzcd}$$

commutes.

2.4 Examples

Examples derived from basic set theory

Sup lattices. Let $\bigvee$-**Lat** denote the category of sup lattices. It may be most familiar as a basic tool in topos theory (see Joyal and Tierney [38]). Concretely it is the category of complete lattices and $\bigvee$-preserving maps; abstractly it can be identified with the Eilenberg-Moore category of algebras for the power set monad. $\bigvee$-**Lat** is a model of classical linear logic.

MULTIPLICATIVE STRUCTURE. The category $\bigvee$-**Lat** was identified as $*$-autonomous by Barr [6]. The tensor product $A \otimes B$ classifies maps $A \times B \longrightarrow C$ preserving suprema in each component; and the linear function space $B \multimap C$ is the lattice of all $\bigvee$-preserving maps from B to C with the pointwise order. Barr notes explicitly that $\bigvee$-**Lat** is not compact closed. Note, however, that $\mathbf{I} = \bot$, so the model is affine.

ADDITIVE STRUCTURE. $\bigvee$-**Lat** has biproducts so the additive structure is degenerate. In fact $\bigvee$-**Lat** has infinite biproducts. (Note moreover that if a category is enriched in $\bigvee$-**Lat** and has products (or coproducts) then these are biproducts.)

EXPONENTIAL STRUCTURE. We draw attention to the warning contained in Section 11 of Barr [8]. This should certainly make one doubt whether free comonoids exist in a category like $\bigvee$-**Lat**. However the infinite biproducts come to the rescue. Most simply we can define a free $\parr$-monoid by the natural formula

$$\bot \oplus A \oplus (A \parr A) \oplus \cdots$$

and as the infinite $\oplus$ is a product we get an easy multiplication. Equalizing the actions of the symmetric groups on the factors gives commutativity. The dual to this is the free commutative comonoid constructed exactly via the illegitimate formula of Barr [8].

Sets and relations. The category **Rel** of sets and relations is the Kleisli category of free algebras for the power set monad. Hence it can be identified with a full subcategory of $\bigvee$-**Lat**. All the linear logic structure restricts from $\bigvee$-**Lat** to **Rel**, which is a degenerate model for linear logic. It is presented as such in Barr [8].

[1] Note that in the presence of products a linearly distributive functor lifts to a functor between the cartesian closed Kleisli categories of (co)free coalgebras.

Posets and relations. The category **RPos** of posets and relations, that is 'relational profunctors', has posets as objects; and maps from a poset A to a poset B are relations $R\colon A \longrightarrow\!\!\!\!\!\!+\ B$ such that

$$a' \geq a \; R \; b \geq b' \qquad \text{implies} \qquad a' \; R \; b'.$$

RPos is a degenerate model for linear logic. Again we can identify **RPos** with a full subcategory of $\bigvee$-**Lat** and it inherits the structure of a (degenerate) model for linear logic.

Linear logical predicates and totality spaces. The categories **LLP** of linear logical predicates and **Tot** of totality spaces were first considered by Loader (see [43, 44]). Loader gave full completeness results for his categories and these were reconsidered by Tan [51].

The objects of **LLP** are sets A equipped with a pair U, X of collections of subsets of A; that is, $U, X \subseteq \mathsf{P}(A)$. A map from (A, U, X) to (B, V, Y) is a relation $R\colon A \longrightarrow\!\!\!\!\!\!+\ B$ such that

$$\begin{aligned} R \cdot u \in V \text{ for all } u \in U \qquad & \text{and} \\ R^{\mathsf{op}} \cdot y \in X \text{ for all } y \in Y. & \end{aligned}$$

Tot is the full subcategory of **LLP** on objects of $(A; U, X)$ with $A = \bigcup U = \bigcup X$ and satisfying

$$\begin{array}{lll} u \in U & \text{if and only if} & u \cap x \text{ is a singleton for all } x \in X; \\ x \in X & \text{if and only if} & u \cap x \text{ is a singleton for all } u \in U. \end{array}$$

We shall show that **LLP** and **Tot** arise as examples of our general constructions. Hence they are automatically models of classical linear logic and we omit the concrete details.

Examples derived from linear algebra

Vector spaces. Let **Vec** be the category of vector spaces and linear maps (over an arbitrary field). **Vec** is a model for intuitionistic linear logic.

MULTIPLICATIVE STRUCTURE. **Vec** is symmetric monoidal closed with the standard tensor product and linear function space.

ADDITIVE STRUCTURE. The direct sum of vector spaces is a biproduct. (This slight degeneracy is typical of linear algebra.)

EXPONENTIAL STRUCTURE. The existence of the free commutative coalgebra (that is comonoid) $!V$ on a vector space V is not quite obvious. It follows readily enough by an adjoint functor argument, but the standard construction [49] is more elementary. First for any coalgebra H, the dual H^* is automatically an algebra; and one has either directly ([49]) or by an easy adjoint functor argument an adjoint $(-)^\circ\colon \mathbf{Alg} \longrightarrow \mathbf{CoAlg}^{\mathsf{op}}$ so that

$$\mathbf{Alg}(A, H^*) \cong \mathbf{CoAlg}(H, A^\circ)$$

naturally in A and H. Now the symmetric algebra $S(V)$ gives the free commutative algebra (that is monoid) on a vector space V, and it follows at once that $S(V^*)^\circ$ is the free commutative coalgebra on the double dual V^{**}. Then the free commutative coalgebra $!V$ on V is universal in the following diagram

$$\begin{tikzcd} !V \arrow[r] \arrow[d] & S(V^*)^\circ \arrow[d] \\ V \arrow[r] & V^{**}, \end{tikzcd}$$

where $!V \longrightarrow S(V^*)^\circ$ is a map of coalgebras. (So in this instance it is the union of the subcoalgebras of $S(V^*)^\circ$ which factor through V.)

We can extend these ideas in a couple of different directions. First we can consider categories of modules over general commutative rings. Secondly we can consider more general linear categories: for example categories of superspaces and of graded vector spaces.

Finite dimensional vector spaces. Let **FdVec** be the category of finite dimensional vector space and linear maps. If K is of characteristic 2 then $\mathbf{FdVec}_K$ is a degenerate model of linear logic. The multiplicative and additive structure is inherited from **FdVec**. The commutative comonoid comonad on **Vec** takes one outside **FdVec**. There is a way to retrieve the situation in characteristic 2. Observe that the exterior algebra $\bigwedge(V)$ is generally skew-commutative (so that the product $a \wedge b$ of homogeneous elements a and b satisfies $a \wedge b = (-1)^{\deg a \deg b}(b \wedge a)$); but as a result in characteristic 2 this product is commutative. So $\bigwedge$ is a linear exponential monad. But $\bigwedge$ is self-dual so $\bigwedge$ is also a linear exponential comonad.

Examples from domain theory

Scott domains and linear maps. Let **LinDom** be the category defined as follows. The objects are *Scott domains*, that is, bounded complete algebraic dcpos with bottom. If A is a Scott domain we write A_o for its poset of finite elements; A is obtained from A_o by adding directed limits. The maps are *linear maps*, that is, functions which preserve all existing suprema (that is all suprema bounded above, including $\bigvee \emptyset = \bot$). **LinDom** is an affine model for intuitionistic linear logic.

MULTIPLICATIVE STRUCTURE. The tensor product $A \otimes B$ classifies maps $A \times B \longrightarrow C$ linear in each argument; the linear function space $B \multimap C$ consists of all linear maps from B to C under the pointwise order. This gives a symmetric monoidal closed structure.

ADDITIVE STRUCTURE. The product of domains is the usual product.

EXPONENTIAL STRUCTURE. We can describe the linear exponential comonad in terms of finite elements. Given A we let $(!A)_o$ be the poset obtained by freely adding bounded finite suprema to A_o. We complete $(!A)_o$ with respect to directed limits to give $!A$.

The Kleisli category for the comonad is the usual category of Scott domains and continuous maps. Other cartesian closed categories of domains and continuous maps can be treated in the same way.

Scott domains and strict maps. The familiar example of Scott domains and (continuous but) strict maps, **StrictDom**, is a simpler version of the previous one. Again we have an affine model for intuitionistic linear logic. The tensor product $A \otimes B$ is the smash product $A \wedge B$, the linear function space $B \multimap C$ consists of all strict maps from B to C under the pointwise order, the product of domains is the usual product, the linear exponential comonad is given by lifting. The Kleisli category for the comonad is again the usual category of Scott domains and continuous maps.

Examples from stable domain theory

dI domains and linear maps. Let **dIDom** be the category of dI domains (see [13, 14]) and stable linear maps: stability is the familiar condition

$$f(a \wedge b) = f(a) \wedge f(b)$$

for all compatible a and b. **dIDom** is a model for intuitionistic linear logic.

MULTIPLICATIVE STRUCTURE. The tensor product $A \otimes B$ classifies maps $A \times B \longrightarrow C$ stable and linear in each argument; the linear function space $B \multimap C$ consists of all stable linear maps from B to C under the stable order. This gives a symmetric monoidal closed structure.

ADDITIVE STRUCTURE. The product of dI domains is the usual product of domains.

EXPONENTIAL STRUCTURE. A dI domain A is generated by the subposet A_p of prime algebraic elements. We construct $!A$ so that $(!A)_p = A_o$, but with the discrete order. Then $!A$ consists of all subsets of A_o whose join in A exists, ordered by subset inclusion.

The Kleisli category for the comonad is the usual category of dI domains and stable continuous maps.

Qualitative domains. A qualitative domain (see [28]) (R, U) is a set R equipped with a collection U of subsets of R satisfying

- $\emptyset \in U$ and $\{r\} \in U$ for all $r \in R$;
- $u' \subseteq u \in U$ implies $u' \in U$;
- U is closed under directed unions.

Thus U is a dI domain with prime elements $\{r\}$ for $r \in R$. Let **QDom** be the full subcategory of **dIDom** consisting of qualitative domains. **QDom** inherits the structure of a model of intuitionistic linear logic from **dIDom**. (Note that the exponential of any dI domain is in fact a qualitative domain.)

Coherence spaces. A coherence space (see [29, 30]) is a graph (that is a reflexive relation $\frown\!\!\!\!\!\smile$) on a set R. Given $(R, \frown\!\!\!\!\!\smile)$ let U be the collection of cliques in R. Then (R, U) is a qualitative domain. Let **Coh** be the full subcategory of **QDom** on the coherence spaces. This category inherits the structure of a model of intuitionistic linear logic, but now more is true: The graph $(R, \asymp)$ dual to $(R, \frown\!\!\!\!\!\smile)$ gives a qualitative domain (R, X) where X is the collection of co-cliques in $(R, \frown\!\!\!\!\!\smile)$. This induces a duality on **Coh** so that **Coh** is in fact a model of classical linear logic. Its structure can, of course, be given entirely based in terms of sets (with coherence structure) and relations, which is the format usually found in the literature. We will identify the structure in yet another way in Section 5.

Categories of games

Simple games. Let **Gam** be the category whose objects are games in which Opponent starts and whose maps are (partial deterministic) strategies for Player in the linear function space. (This simple category is described in detail in [35], see also [2].) **Gam** is an affine model for intuitionistic linear logic.

MULTIPLICATIVE STRUCTURE. The tensor product $A \otimes B$ is the game obtained by playing A and B in parallel; the linear function space $B \multimap C$ is the game obtained by playing the cogame $B^{\perp}$ (interchanging Opponent and Player) in parallel with C.

ADDITIVE STRUCTURE. In the product $A \times B$, Opponent chooses to play in one or other of A and B and then the game continues in that component.

EXPONENTIAL STRUCTURE. Play in $!A$ is in effect play in an infinite sequence of copies of A. At any stage Opponent can play in any game already begun or can choose to start the next version of A in the sequence.

Sequential algorithms. Filiform concrete data structures [21] can be regarded as games. Let **FCDS** be the category of games **Gam** but equipped with a more sophisticated structure making it a model of intuitionistic linear logic. We take the same multiplicative and additive structure as before but change the exponential to the following.

CURIEN EXPONENTIAL. This exponential is obtained by allowing Opponent to play many strands of the game A. At any point Opponent may return to an earlier Player move and play a fresh response to it; Player must always respond to the last Opponent move (and therefore cannot change strands). The Kleisli category for this comonad is the category of filiform concrete data structures and sequential algorithms.

Games with protocols. The extreme case is a category **InnGam** of games and *innocent* strategies. This can be formulated to give an affine model for intuitionistic logic, though what is published does not quite do this. The corresponding Kleisli category is the basis for a semantics for PCF [33, 46] . Innocence involved two distinct protocols, *visibility* and *bracketing* (a stack discipline). Each on its own leads to a model for intuitionistic linear logic.

History-free games. A particularly interesting example is a category **HFGam** of games and history-free strategies [3, 4, 5]. This gives a model for intuitionistic linear logic, but without products. The corresponding co-Kleisli category again is a basis for a semantics for PCF. (The lack of products leads to an interesting point. The simple dualization (see Section 3) $\mathbf{HFGam}^{\mathsf{d}}$ is not a model for classical linear logic (even disregarding the problem of the additives). But there is a category apparently very like it [3] which is.)

Graph games. Recently we have discovered a category **GGam** of graph games, that is games played on directed graphs rather than simply trees of positions. This produces yet another affine model for intuitionistic linear logic. There is a close relation between this and categories of abstract games which model classical linear logic [37].

2.5 Comonoid indexing

For completeness we mention one simple technique for constructing models which we have found useful in the theory of abstract games [37].

Assume that K is a comonoid in a monoidal category $\mathbf{C}$, so that we have maps

$$\mathbf{I} \xleftarrow{\;e\;} K \xrightarrow{\;d\;} K \otimes K$$

satisfying the usual identity and associativity equations. Given such a comonoid, tensoring with K induces a comonad on $\mathbf{C}$ in the standard way. We consider the (Kleisli) category $\mathbf{C}_K$ of free coalgebras for this comonad: objects of $\mathbf{C}_K$ are objects of $\mathbf{C}$, maps from C to D in $\mathbf{C}_K$ are given by maps $K \otimes C \longrightarrow D$ in $\mathbf{C}$. We are are interested in finding conditions on $\mathbf{C}$ which make the Kleisli category $\mathbf{C}_K$ a model of linear logic. Since the (co)free functor $\mathbf{C} \longrightarrow \mathbf{C}_K$, the right adjoint of the standard adjunction between $\mathbf{C}$ and $\mathbf{C}_K$, is the identity on objects and takes $f\colon C \longrightarrow D$ in $\mathbf{C}$ to $e \otimes f\colon K \otimes C \longrightarrow \mathbf{I} \otimes D \cong D$ in $\mathbf{C}_K$, most of structure automatically lifts from $\mathbf{C}$ to $\mathbf{C}_K$.

Multiplicative structure. The basic results are straightforward. The proof of the following is routine.

Proposition 2.1 *Let K be a commutative comonoid in a symmetric monoidal category $\mathbf{C}$.*

(i) $\mathbf{C}_K$ is a symmetric monoidal category, and $\mathbf{C}_K \longrightarrow \mathbf{C}$ preserves the structure.

(ii) If $\mathbf{C}$ is also closed, then so is $\mathbf{C}_K$, and $\mathbf{C}_K \longrightarrow \mathbf{C}$ preserves the structure.

(iii) If $\mathbf{C}$ is $*$-autonomous then so is $\mathbf{C}$ and $\mathbf{C}_K \longrightarrow \mathbf{C}$ preserves the structure.

Additive structure. The existence of products on $\mathbf{C}$ is enough to ensure the same for $\mathbf{C}_K$ as the functor $\mathbf{C} \longrightarrow \mathbf{C}_K$ is a right adjoint, and surjective on objects. Coproducts, on the other hand, are slightly more complicated. To obtain a natural isomorphism

$$\mathbf{C}_K(C + D, E) \cong \mathbf{C}_K(C, E) \times \mathbf{C}_K(D, E),$$

we need a natural isomorphism $\mathbf{C}(K \otimes (C+D), E) \cong \mathbf{C}(K \otimes C, E) \times \mathbf{C}(K \otimes D, E)$. So we ask that tensor distributes over sum—which is true for symmetric monoidal closed categories. It is not difficult to show that with this assumption we do indeed get the desired coproducts.

Proposition 2.2 *Let K be a commutative comonoid in a symmetric monoidal category $\mathbf{C}$. If $\mathbf{C}$ has products then so has $\mathbf{C}_K$, and $\mathbf{C} \longrightarrow \mathbf{C}_K$ preserves them.*
If $\mathbf{C}$ has coproducts over which tensor distributes (so in particular if $\mathbf{C}$ has coproducts and is closed) then $\mathbf{C}_K$ has coproducts and $\mathbf{C} \longrightarrow \mathbf{C}_K$ preserves them.

Exponential structure. Assume that $\mathbf{C}$ is a symmetric monoidal category with a linear exponential comonad.

Definition 5 *Let $\mathbf{C}$ be a model for intuitionistic linear logic, and K a commutative comonoid in $\mathbf{C}$. We say that K is an* exponential comonoid *if K is a coalgebra for $!$ and the comonoid structure on K is the one canonically derived from that on $!K$.*

Then $\mathbf{C}_K$ inherits the linear exponential from $\mathbf{C}$ via the functor $\mathbf{C} \longrightarrow \mathbf{C}_K$. By functoriality, the only equations still to be checked are the ones involving generic morphisms in $\mathbf{C}_K$, in other words we just check that all the transformations are natural with respect to maps in $\mathbf{C}_K$.

Proposition 2.3 *Let $\mathbf{C}$ be a model for intuitionistic linear logic and let K be an exponential comonoid in $\mathbf{C}$. Then $\mathbf{C}_K$ has a linear exponential comonad.*

Theorem 2.4 *Let $\mathbf{C}$ be a model for classical (intuitionistic) linear logic and K an exponential comonoid in $\mathbf{C}$. Then $\mathbf{C}_K$ is a model for classical (intuitionistic) linear logic.*

3 Dualization

In models for classical linear logic negation is interpreted by a self-duality. In this section we give a brief survey of some methods for creating self-dual categories; and we identify conditions ensuring that they give models for linear logic.

3.1 Simple self-dualization

Suppose that we are given a category $\mathbf{C}$. There is a natural way of creating a category with duality: we consider $\mathbf{C}^{\mathsf{d}} := \mathbf{C} \times \mathbf{C}^{\mathsf{op}}$. This category is self-dual under 'swapping components': we have a functor $(-)^{\perp} \colon \mathbf{C}^{\mathsf{d}} \longrightarrow (\mathbf{C}^{\mathsf{d}})^{\mathsf{op}}$ with $(U,X)^{\perp} = (X,U)$, and with the obvious action on morphisms. The functor $(-)^{\perp}$ clearly is a self duality.

In general the duality on $\mathbf{C}^{\mathsf{d}}$ is not particularly noteworthy. However it is an important fact that if $\mathbf{C}$ carries enough structure then $\mathbf{C}^{\mathsf{d}}$ will be a model of classical linear logic. Since (in the presence of a terminal object) one can regard $\mathbf{C}^{\mathsf{d}}$ as a degenerate form of Chu's construction [6], the result for the multiplicatives and additives should be well known. It still seems worth pointing out just how little is needed to make this work. In particular this construction can be used in situations where the general Chu construction does not have good structure, such as for the category **Rel**. The exponentials are in any case quite subtle.

Multiplicative structure. We start with the multiplicatives.

Proposition 3.1 *If $\mathbf{C}$ is a symmetric monoidal closed category with finite products, then $\mathbf{C}^{\mathsf{d}}$ is $*$-autonomous. The structure is given as follows.*

(i) The duality (negation) is defined as above by $(U,X)^{\perp} = (X,U)$.

(ii) The tensor product of $A = (U,X)$ and $B = (V,Y)$ is given by

$$A \otimes B = (U \otimes V, U \multimap Y \times V \multimap X)\,;$$

the unit for the tensor product is $\mathbf{I} = (\mathbf{I}, \mathbf{1})$.

From negation and tensor product we get the linear function space. If $A = (U,X)$ and $B = (V,Y)$ then it is given by

$$A \multimap B = (A \otimes B^{\perp})^{\perp} = (U \multimap V \times Y \multimap X, U \otimes Y).$$

Additive structure. We next cover the additives.

Proposition 3.2 *If* $\mathbf{C}$ *has finite products and coproducts, then so does* $\mathbf{C}^{\mathsf{d}}$*. The products are given as follows.*

$$(U,X)\times(V,Y)=(U\times V,X+Y)\,;$$

the unit for the product is $\mathbf{1}=(\mathbf{1},\mathbf{0})$*. Coproducts are given by the obvious dual process or equivalently by applying* $(-)^{\perp}$*. So*

$$(U,X)+(V,Y)=(U+V,X\times Y)\,;$$

the unit for the product is $\mathbf{0}=(\mathbf{0},\mathbf{1})$*.*

This result is really a triviality. In fact, if $\mathbf{C}$ has limits of shape $\mathbf{J}$ and colimits of shape $\mathbf{J}^{\mathsf{op}}$ then $\mathbf{C}^{\mathsf{d}}$ has limits of shape $\mathbf{J}$ (and colimits of shape $\mathbf{J}^{\mathsf{op}}$).

Exponential structure. Finally we consider the exponentials. Some special cases of this are known, but the structure on $\mathbf{C}$ we assume to give the general case is quite subtle. Of course we shall assume that $\mathbf{C}$ has a linear exponential comonad; this handles the structure in the first coordinate of $\mathbf{C}^{\mathsf{d}}$ straightforwardly. It is the structure in the second coordinate which presents the challenge. We assume that $\mathbf{C}$ is equipped with a monad (M,η,μ), whose (free) algebras are naturally commutative monoids *with respect to product as monoidal structure*, and which has a generalized tensorial strength $\tau\colon M(-)\otimes !-\longrightarrow M(-\otimes !(-))$ satisfying conditions ensuring that !-coalgebras act on M-algebras. We say in this case that $\mathbf{C}$ has *well-adapted monoids.* The details will be made available in a companion paper [34].

Proposition 3.3 *Let* $\mathbf{C}$ *be a model for intuitionistic linear logic with well-adapted monoids. Then* $\mathbf{C}\times\mathbf{C}^{\mathsf{op}}$ *has a linear exponential comonad* $!(U,X)=(!U,!U\multimap M(X))$*.*

Putting the above propositions together we have the following.

Theorem 3.4 *Let* $\mathbf{C}$ *be a model for intuitionistic linear logic with finite coproducts and well-adapted monoids. Then* $\mathbf{C}^{\mathsf{d}}$ *is a model for classical linear logic.*

Examples 3.5 (1) Abramsky and Jagadeesan use this construction in [3]. Their underlying category of games is of the form $\mathbf{G}\times\mathbf{G}^{\mathsf{op}}$, where $\mathbf{G}$ is a simple category of games and where they restrict morphisms to winning strategies. Note however that we cannot give the same simple abstract account of their category of games and history-free (winning) strategies.

(2) More recently, Bellin has made use of this construction in his analysis of the meaning of the long trip condition. In [10] he considers $\mathbf{C}^{\mathsf{d}}$ where $\mathbf{C}$ is the free symmetric monoidal category with finite products.

(3) In our work on categories of abstract games (see [37] for an outline) we apply the construction to $\mathbf{Rel}$. Of course $\mathbf{Rel}$ is a degenerate model of classical linear logic and so it is in any case self-dual. However, the resulting $\mathbf{Rel}^{\mathsf{d}}$ is definitely non-degenerate.

(4) Interesting models can be obtained by iterating the construction $(-)^{\mathsf{d}}$. Even as simple a category as $(\mathbf{Set}^{\mathsf{d}})^{\mathsf{d}}$ admits a reading as a category of abstract games in which the four component sets making up an object are strategies for Player/Opponent, playing first/second.

Note that if $\mathbf{C}$ is an affine model for intuitionistic linear logic (with finite coproducts) then $\mathbf{C}^{\mathsf{d}}$ is a model for classical linear logic that validates MIX.

3.2 Dialectica categories

Simple self-dualization can be regarded as a special case of the Dialectica-style categories suggested by Girard and developed by de Paiva in her thesis [24] and elsewhere [23]. We briefly indicate the general set-up.

Suppose that $\mathbf{C}$ is a symmetric monoidal category, and suppose that it is equipped with a self-dual poset fibration $\mathbf{P} \longrightarrow \mathbf{C}$ (or a functor $\mathbf{C}^{\mathsf{op}} \longrightarrow \mathbf{Poset}$). This means that each $\mathbf{P}(C)$, $C \in \mathbf{C}$, is a poset $\mathbf{P}(C) = (\mathbf{P}(C), \vdash)$ equipped with a self-duality $(-)^{\perp}$, and that this structure is preserved by re-indexing. We use the obvious set theoretic notation to indicate reindexing. We define the *Girard category* $\mathbf{G} = \mathbf{G}(\mathbf{P} \longrightarrow \mathbf{C})$ as follows:

- **Objects.** The objects of $\mathbf{G}$ are pairs (U, X) of objects of $\mathbf{C}$ equipped with $\alpha \in \mathbf{P}(U \otimes X)$. We write such an object as $(U \overset{\alpha}{\longleftarrow\!\!\!\!\!+} X)$.
- **Maps.** The maps of $\mathbf{G}$ from $(U \overset{\alpha}{\longleftarrow\!\!\!\!\!+} X)$ to $(V \overset{\beta}{\longleftarrow\!\!\!\!\!+} Y)$ are pairs of maps $f\colon U \longrightarrow V$ and $F\colon Y \longrightarrow X$ in $\mathbf{C}$ such that
$$\alpha(u \otimes F(y)) \vdash \beta(f(u) \otimes y) \qquad \text{in } \mathbf{P}(U \otimes Y).$$

It is easy to see that $\mathbf{G}$ is a category and that it is equipped with a self-duality
$$(U \overset{\alpha}{\longleftarrow\!\!\!\!\!+} X)^{\perp} = (X \overset{\alpha^{\perp}}{\longleftarrow\!\!\!\!\!+} U).$$

Multiplicative structure. We start with the multiplicatives.

Proposition 3.6 *Suppose that $\mathbf{C}$ is a symmetric monoidal closed category with finite products and suppose that $\mathbf{P}$ is a $*$-autonomous poset fibration. ($\mathbf{P}$ already has a self-duality so it suffices that it have a suitable binary tensor product.) Then $\mathbf{G}$ is $*$-autonomous. The structure is given as follows.*

(i) The duality (negation) is defined as above by $(U \overset{\alpha}{\longleftarrow\!\!\!\!\!+} X)^{\perp} = X \overset{\alpha^{\perp}}{\longleftarrow\!\!\!\!\!+} U$.

(ii) The tensor product of $A = U \overset{\alpha}{\longleftarrow\!\!\!\!\!+} X$ and $B = V \overset{\beta}{\longleftarrow\!\!\!\!\!+} Y$ is given by
$$A \otimes B = (U \otimes V \overset{\vartheta}{\longleftarrow\!\!\!\!\!+} U \multimap Y \times V \multimap X)\,;$$
where $\vartheta(u \otimes v \otimes (\phi, \psi))$ is $\alpha(u \otimes \psi(v)) \otimes \beta(v \otimes \phi(u)) \in \mathbf{P}(U \otimes V \otimes (U \multimap Y \times V \multimap X))$. The unit for the tensor product is $\mathbf{I} = (\mathbf{I}, \mathbf{1})$ in the natural internal logic.

From negation and tensor product we get the linear function space. If $A = U \overset{\alpha}{\longleftarrow\!\!\!\!\!+} X$ and $B = V \overset{\beta}{\longleftarrow\!\!\!\!\!+} Y$ then it is given by
$$A \multimap B = \left(A \otimes B^{\perp}\right)^{\perp} = (U \multimap V \times Y \multimap X \overset{\omega}{\longleftarrow\!\!\!\!\!+} U \otimes Y)$$
where $\omega((f, F) \otimes u \otimes y) = \alpha(u \otimes F(y)) \multimap \beta(f(u) \otimes y)$ again in the internal logic. (Here $\multimap$ is the linear function space in the $*$-autonomous poset $\mathbf{P}((U \multimap V \times Y \multimap X) \otimes U \otimes Y)$.)

Additive structure. We next deal with the additives. Here we must require substantially more basic structure than in the case of simple dualization.

Proposition 3.7 *Suppose that* $\mathbf{C}$ *is monoidal and has finite products and coproducts where the tensor distributes over coproducts. Suppose also that the poset fibration* $\mathbf{P}$ *has finite meets and joins and that we have poset isomorphisms*

$$\mathbf{P}(\mathbf{0}) \cong \mathbf{1} \qquad \text{and} \qquad \mathbf{P}(C+D) \cong \mathbf{P}(C) \times \mathbf{P}(D)$$

natural in C *and* D. *Then the category* $\mathbf{G}$ *has finite products and coproducts. The products are given as*

$$(U \overset{\alpha}{\longleftarrow\!\!\!+\!\!\!-} X) \times (V \overset{\beta}{\longleftarrow\!\!\!+\!\!\!-} Y) = (U \times V \overset{\vartheta}{-\!\!\!+\!\!\!\longrightarrow} X+Y)$$

where $\vartheta \in \mathbf{P}((U \times V) \otimes (X+Y)) \cong \mathbf{P}((U \times V) \otimes X) \times \mathbf{P}((U \times V) \otimes Y)$ *is* $\alpha(u \otimes x) \wedge \beta(v \otimes y)$. *The unit for product is* $(\mathbf{1} \longleftarrow\!\!\!+\!\!\!- \mathbf{0})$ *with unique choice of relation in* $\mathbf{P}(\mathbf{0} \otimes \mathbf{1}) \cong \mathbf{P}(\mathbf{0}) \cong \mathbf{1}$.

Coproducts are given by the obvious dual process or equivalently by applying $(-)^{\perp}$. *So*

$$(U \overset{\alpha}{\longleftarrow\!\!\!+\!\!\!-} X) + (V \overset{\beta}{\longleftarrow\!\!\!+\!\!\!-} Y) = (U+V \overset{\omega}{\longleftarrow\!\!\!+\!\!\!-} X \times Y)$$

where $\omega \in \mathbf{P}((U+V) \otimes (X \times Y)) \cong \mathbf{P}(U \otimes (X \times Y)) \times \mathbf{P}(V \otimes (X \times Y))$ *is* $\alpha(u \otimes x) \vee \beta(v \otimes y)$. *The unit for the coproduct is* $(\mathbf{0} \longleftarrow\!\!\!+\!\!\!- \mathbf{1})$ *with unique choice of relation in* $\mathbf{P}(\mathbf{0} \otimes \mathbf{1}) \cong \mathbf{P}(\mathbf{0}) \cong \mathbf{1}$.

Exponential structure. The situation is similar to that for simple dualization. We assume again that $\mathbf{C}$ has well-adapted monoids and we assume that the action of the comonad M extends in a natural way to $\mathbf{P}$. We say then that the poset fibration $\mathbf{P} \longrightarrow \mathbf{C}$ has well-adapted monoids. We shall give details in a companion paper [34].

Proposition 3.8 *Let* $\mathbf{C}$ *be a model for intuitionistic linear logic and let* $\mathbf{P} \longrightarrow \mathbf{C}$ *be a* $*$*-autonomous poset fibration with well-adapted monoids. Then* $\mathbf{G}$ *has a linear exponential comonad.*

Putting the above propositions together we have the following.

Theorem 3.9 *Let* $\mathbf{C}$ *be a model for intuitionistic linear logic with finite coproducts and well-adapted monoids. Then* $\mathbf{G}$ *is a model for classical linear logic.*

Example. The simplest example of the situation we describe is the subset fibration over the category of sets. This has well-adapted monoids via the finite multiset monad (whose algebras are exactly the commutative monoids). The resulting model of classical linear logic has particular interest as if one applies the Girard translation to it one gets the Diller-Nahm variant of Gödel's Dialectica interpretation [27, 26].

3.3 Chu's construction

Simple self-dualization can also be thought of as a special case of Chu's construction [18, 8]. We briefly recall the essentials. Suppose that $K \in \mathbf{C}$ is an object in a symmetric monoidal category $\mathbf{C}$. The category $\mathbf{Chu}(\mathbf{C}, K)$ is defined as follows.

- **Objects** of $\mathbf{Chu}(\mathbf{C}, K)$ are pairs (U, X) of objects of $\mathbf{C}$ with a map $U \otimes X \xrightarrow{\alpha} K$.

- **Maps** from $U \otimes X \xrightarrow{\alpha} K$ to $V \otimes Y \xrightarrow{\beta} K$ in $\mathbf{Chu}(\mathbf{C}, K)$ are given by pairs of maps $f\colon U \longrightarrow V$, $F\colon Y \longrightarrow X$ in $\mathbf{C}$ such that

$$\alpha \cdot (\mathsf{id} \otimes F) = \beta \cdot (f \otimes \mathsf{id}).$$

(Often one writes this condition suggestively as $\alpha(u, F(y)) = \beta(f(u), y)$ or even (omitting the names of the structure maps) $\langle u, F(y)\rangle = \langle f(u), y\rangle$.)

It is easy to see that $\mathbf{Chu}(\mathbf{C}, K)$ is a category and that it is equipped with a self-duality

$$(U \otimes X \xrightarrow{\alpha} K)^{\perp} = (X \otimes U \xrightarrow{\alpha \cdot \sigma} K)$$

using the twist $\sigma\colon X \otimes U \longrightarrow U \otimes X$.

Multiplicative structure. We start with the multiplicatives.

Proposition 3.10 *Suppose that $\mathbf{C}$ is a symmetric monoidal closed category with finite limits. Then $\mathbf{Chu}(\mathbf{C}, K)$ is $*$-autonomous. The structure is given as follows.*

(i) The duality (negation) is defined as above by $(U \otimes X \xrightarrow{\alpha} K)^{\perp} = (X \otimes U \xrightarrow{\alpha \cdot \sigma} K)$.

(ii) The tensor product of $A = (U \otimes X \xrightarrow{\alpha} K)$ and $B = (V \otimes Y \xrightarrow{\beta} K)$ is given by

$$A \otimes B = ((U \otimes V) \otimes P \xrightarrow{\vartheta} K)\,;$$

where P lies in a pullback diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & V \multimap X \arrow[d] \\ U \multimap Y \arrow[r] & U \otimes V \multimap K \end{tikzcd}$$

where the map $U \multimap Y \longrightarrow U \otimes V \multimap K$, for example, is the transpose of β combined with the evaluation map. The map ϑ is the transpose of the map $P \longrightarrow U \otimes V \multimap K$ in the pullback diagram. The unit for the tensor product is $\mathbf{I} \otimes K \xrightarrow{\lambda} K$ given by the left identity for tensor.

As for the Dialectica categories it is routine to define the linear function space and check the required adjunctions.

Additive structure. We next deal with the additives.

Proposition 3.11 *If $\mathbf{C}$ has finite products and coproducts then so does $\mathbf{Chu}(\mathbf{C}, K)$. The products are given as*

$$(U \otimes X \xrightarrow{\alpha} K) \times (V \otimes Y \xrightarrow{\beta} K) = ((U \times V) \otimes (X + Y) \xrightarrow{\alpha \times \beta} K),$$

where since $(U \times V) \otimes (X + Y) \cong (U \times V) \otimes X + (U \times V) \otimes Y$, we can define $\alpha \times \beta$ to correspond to the map $(U \times V) \otimes X + (U \times V) \otimes Y \longrightarrow K$ having components $\alpha \cdot (\pi_1 \otimes \mathsf{id})$ and $\beta \cdot (\pi_2 \otimes \mathsf{id})$. The unit for the product is the unique map $\mathbf{1} \times \mathbf{0} \cong \mathbf{0} \longrightarrow K$.

Coproducts are given by the obvious dual process or equivalently by applying $(-)^{\perp}$. So

$$(U \otimes X \xrightarrow{\alpha} K) + (V \otimes Y \xrightarrow{\beta} K) = ((U + V) \otimes (X \times Y) \xrightarrow{\alpha+\beta} K),$$

where $\alpha+\beta$ corresponds to the map $U \otimes (X \times Y) + V \otimes (X \times Y) \longrightarrow K$ having components $\alpha \cdot (\mathsf{id} \otimes \pi_1)$ and $\beta \cdot (\mathsf{id} \otimes \pi_2)$. The unit for the coproduct is the unique map $\mathbf{0} \otimes \mathbf{1} \longrightarrow K$.

Exponential structure. As we observed in Section 2, if a symmetric monoidal closed or $*$-autonomous category admits free commutative comonoids then it certainly has a linear exponential comonad. In a couple of papers Barr ([8, 7]) has investigated the existence of exponentials of this form. In the most general setting the result is rather special.

Proposition 3.12 (Barr) *Let $\mathbf{C}$ be a complete and cocomplete cartesian closed category and K an internal cogenerator. Then the category of separated object in $\mathbf{Chu}(\mathbf{C}, K)$ has free commutative comonoids.*

In the context of accessible categories one has a much more general result.

Proposition 3.13 (Barr) *Let $\mathbf{C}$ be a locally presentable symmetric monoidal closed category and K an internal cogenerator. Then $\mathbf{Chu}(\mathbf{C}, K)$ has free commutative comonoids.*

(The notion of an internal cogenerator depends on a suitable factorization system, but we do not go into that.)

Both Barr's results lead to models of full classical linear logic; and the second result applies very widely. However there are important cases which it does not cover. One natural example is the category $\mathbf{Rel}$ of sets and relations with K set equal to $\mathbf{0}$. $\mathbf{Chu}(\mathbf{Rel}, \mathbf{0})$ is $\mathbf{Rel} \times \mathbf{Rel}^{\mathsf{op}}$ and our Section 3.1 applies to give a model of full classical linear logic. But of course $\mathbf{Rel}$ is not complete and $\mathbf{0}$ is not an internal cogenerator.

The methods we have used earlier in this section are algebraic in nature, and they can also be applied in the case of Chu constructions. The structure needed in addition to well-adapted monoids is that of a strength (with respect to product) for the monad functor M which is well-behaved with respect to the monoid operations. The details will be given in a companion paper [34].

Proposition 3.14 *Let $\mathbf{C}$ be a model for intuitionistic linear logic with well-adapted monoids and a suitable strength. Then $\mathbf{Chu}(\mathbf{C}, K)$ has a linear exponential comonad.*

Putting the above propositions together we have the following.

Theorem 3.15 *Let $\mathbf{C}$ be a model for intuitionistic linear logic with finite limits, finite coproducts and well-adapted monoids as well as a suitable strength. Then $\mathbf{Chu}(\mathbf{C}, K)$ is a model for classical linear logic.*

4 Glueing

4.1 Glueing for intuitionistic linear logic

Before starting our treatment of double glueing we recall some basic facts about the standard form of glueing for models of intuitionistic linear logic. With the possible exception of the material on the exponentials this is pretty much folklore; anyone familiar with glueing in categorical logic generally and for toposes in particular will find no surprises.

Suppose $L\colon \mathbf{C} \longrightarrow \mathbf{E}$ is a functor. The category $\mathbf{G} = \mathbf{G}(L)$ obtained by glueing along L is the comma category obtained from

$$\mathbf{E} \xrightarrow{\mathsf{id}} \mathbf{E} \xleftarrow{L} \mathbf{C}.$$

(So MacLane [45] would write this category $(\mathsf{id} \downarrow L)$.)

The $\mathbf{G}$ lies in a lax pullback diagram

$$\begin{tikzcd} \mathbf{G} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & \mathbf{E} \arrow[d, "\mathsf{id}"] \\ \mathbf{C} \arrow[r, "L"'] & \mathbf{E}. \end{tikzcd}$$

$\mathbf{G}$ can be described explicitly as follows.

- **Objects** $(R, U, (U \longrightarrow L(R))) = (U \longrightarrow L(R))$ of $\mathbf{G}$ consist of an object $R \in \mathbf{C}$, an object $U \in \mathbf{E}$ and a map $U \longrightarrow L(R)$ of $\mathbf{E}$.
- **Maps** from $(U \longrightarrow L(R))$ to $(V \longrightarrow L(S))$ in $\mathbf{G}$ are given by commuting diagrams in $\mathbf{E}$

$$\begin{tikzcd} U \arrow[r] \arrow[d, "\phi"'] & L(R) \arrow[d, "L(f)"] \\ V \arrow[r] & L(S), \end{tikzcd}$$

where $f\colon R \longrightarrow S$ in $\mathbf{C}$ and $\phi\colon U \longrightarrow V$ in $\mathbf{E}$.

Often we consider full subcategories of $\mathbf{G}$ which are obtained by restricting the structure maps $U \longrightarrow L(R)$ which occur. Typically we ask that $U \longrightarrow L(R)$ be a monomorphism (or regular monomorphism—there are many possible variations). We shall not introduce any special notation for such subcategories: we shall just refer to the glued category $\mathbf{G}$ and the context should make it clear which glued category is meant.

We start by supposing that $\mathbf{C}$ and $\mathbf{E}$ are symmetric monoidal and that L is a monoidal functor in the usual lax sense. We write the monoidal structure as

$$I \xrightarrow{m_{\mathbf{I}}} L(\mathbf{I}) \text{ and } L(R) \otimes L(S) \xrightarrow{m_{R,S}} L(R \otimes S).$$

Proposition 4.1 $\mathbf{G}$ *is a symmetric monoidal category where the unit is* $(\mathbf{I} \xrightarrow{m_{\mathbf{I}}} L(\mathbf{I}))$ *and the tensor of* $(U \longrightarrow L(R))$ *and* $(V \longrightarrow L(S))$ *is given by*

$$U \otimes V \longrightarrow L(R) \otimes L(S) \xrightarrow{m_{R,S}} L(R \otimes S).$$

The forgetful functors $\mathbf{G} \longrightarrow \mathbf{C}$ *and* $\mathbf{G} \longrightarrow \mathbf{E}$ *are strict monoidal functors.*

We get the basic facts about models for linear logic by adding to this structure.

Multiplicative structure. We first add linear function spaces. Note that if $\mathbf{C}$ and $\mathbf{E}$ are symmetric monoidal closed then from m we obtain a natural transformation

$$L(S \multimap T) \longrightarrow L(S) \multimap L(T),$$

and we use m to denote this, too.

Proposition 4.2 *Suppose* $\mathbf{C}$ *and* $\mathbf{E}$ *are symmetric monoidal closed and* $\mathbf{E}$ *has pullbacks. Then* $\mathbf{G}$ *is symmetric monoidal closed and* $\mathbf{G} \longrightarrow \mathbf{C}$ *is a strict map of symmetric monoidal closed categories.*

Proof. The function space $(V \longrightarrow L(S)) \multimap (W \longrightarrow L(T))$ is the map $(Q \longrightarrow L(S \multimap T))$ from the pullback

$$\begin{tikzcd}
Q \arrow[rrr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & & V \multimap W \arrow[d] \\
L(S \multimap T) \arrow[r, "m"'] & L(S) \multimap L(T) \arrow[rr] & & V \multimap L(T).
\end{tikzcd}$$

The general verifications are routine. $\square$

Additive structure. It turns out that there is no reason to treat products and coproducts separately, even though we are treating the intuitionistic case here.

Proposition 4.3 *(i) Suppose* $\mathbf{C}$ *has finite products and* $\mathbf{E}$ *has pullbacks. Then* $\mathbf{G}$ *has finite products and* $\mathbf{G} \longrightarrow \mathbf{C}$ *preserves them strictly.*

(ii) Suppose $\mathbf{C}$ *and* $\mathbf{E}$ *have finite coproducts. Then* $\mathbf{G}$ *has finite coproducts and both functors* $\mathbf{G} \longrightarrow \mathbf{C}$ *and* $\mathbf{G} \longrightarrow \mathbf{E}$ *preserve them strictly.*

Proof. (i) The terminal object is $L(\mathbf{1}) \xrightarrow{\mathsf{id}} L(\mathbf{1})$; and the product of $(U \longrightarrow L(R))$ and $(V \longrightarrow L(S))$ is the map $P \longrightarrow L(R \times S)$ in the double pullback diagram

$$\begin{tikzcd}
U \arrow[d] & P \arrow[l] \arrow[r] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d] \\
L(R) & L(R \times S) \arrow[l, "L(\pi_1)"] \arrow[r, "L(\pi_2)"'] & L(S).
\end{tikzcd}$$

Again the verifications are routine.

(ii) The initial object is the unique $(\mathbf{0} \longrightarrow L(\mathbf{0}))$; the coproduct of $(U \longrightarrow L(R))$ and $(V \longrightarrow L(S))$ is the obvious composite $(U + V \longrightarrow L(R) + L(S) \xrightarrow[{[L(\mathsf{inl}), L(\mathsf{inr})]}]{} L(R + S)$. $\square$

Exponential structure. We finally turn to the linear exponentials.

Proposition 4.4 *Suppose* $\mathbf{C}$ *and* $\mathbf{E}$ *have linear exponential comonads and* $L\colon \mathbf{C} \longrightarrow \mathbf{E}$ *is linearly distributive. Then* $\mathbf{G}$ *has a linear exponential comonad and* $\mathbf{G} \longrightarrow \mathbf{C}$ *preserves it strictly.*

Proof. In addition to the monoidal structure we have $\kappa\colon\, !L \longrightarrow L!$ satisfying the conditions of Definition 4. We set $!(U \longrightarrow L(R))$ to be

$$!U \longrightarrow\, !L(R) \xrightarrow{\kappa_R} L(!R).$$

The rest of the structure falls easily into place and the verifications are routine. $\square$

Examples and applications.

Examples 4.5 (1) **Logical relations.** Glueing is the abstract mathematical counterpart of the technique of logical relations. For example we may take $\mathbf{C}$ to be $\mathbf{Set} \times \mathbf{Set}$, $\mathbf{E}$ to be $\mathbf{Set}$ and $L\colon \mathbf{C} \longrightarrow \mathbf{E}$ given by $L(A, B) = A \times B$. Then the monomorphisms version of the glued category $\mathbf{G}$ is the simplest category of logical relations: its objects are relations between sets. Note in passing that the category $\mathbf{Chu}(\mathbf{G}, 2)$ (where 2 is the identity relation on the two element set) is the category of Chu logical relations of [25]. We give another identification of their category later.

(2) **Indecomposability.** Glueing was first introduced by Freyd to give neat proofs of projectivity and indecomposability results for toposes. One can readily adapt this argument. Let $\mathbf{C}$ be the free symmetric monoidal closed category with coproducts on a collection of objects. Let $\mathbf{G}$ be obtained by glueing along $\mathbf{C}(\mathbf{I}, -)\colon \mathbf{C} \longrightarrow \mathbf{Set}$. As $\mathbf{C}$ is free we have a structure preserving functor $\mathbf{C} \longrightarrow \mathbf{G}$ given by taking generators A to $(\mathbf{C}(\mathbf{I}, A) \xrightarrow{\mathsf{id}} \mathbf{C}(\mathbf{I}, A))$; and the composite $\mathbf{C} \longrightarrow \mathbf{G} \longrightarrow \mathbf{C}$ is the identity. A map $\mathbf{I} \longrightarrow R + S$ in $\mathbf{C}$ thus maps to

$$(\{\mathsf{id}_{\mathbf{I}}\} \longrightarrow \mathbf{C}(\mathbf{I}, \mathbf{I})) \longrightarrow (U + V \longrightarrow \mathbf{C}(\mathbf{I}, R + S))$$

in $\mathbf{G}$; $\mathsf{id}_{\mathbf{I}}$ maps to either U or V, and so $\mathbf{I}$ maps to one of R and S. Thus

$$\mathbf{C}(\mathbf{I}, R + S) \cong \mathbf{C}(\mathbf{I}, R) + \mathbf{C}(\mathbf{I}, S),$$

and $\mathbf{I}$ is indecomposable. This argument scales up to the free model for intuitionistic linear logic with coproducts.

(3) **Conservativity.** Lafont found a neat way to use glueing to give conservative extension results. His ideas apply here. Suppose $\mathbf{M}$ is a symmetric monoidal category and $\Phi\colon \mathbf{M} \longrightarrow \mathbf{C}$ is obtained by freely adding closed structure. Let $\mathbf{E}$ be $[\mathbf{M}^{\mathsf{op}}, \mathbf{Set}]$ which is symmetric monoidal closed with the Day tensor product, and let $L\colon \mathbf{C} \longrightarrow \mathbf{E}$ be given by $L(R) = \mathbf{C}(\Phi(-), R)$. Let $\mathbf{G}$ be the glued category. There is an obvious map $\mathbf{M} \longrightarrow \mathbf{G}$ taking A to $(\mathbf{M}(-, A) \xrightarrow{\Phi} \mathbf{C}(\Phi(-), \Phi(A)))$ which extends by freeness to one $\mathbf{C} \longrightarrow \mathbf{G}$. The composite $\mathbf{C} \longrightarrow \mathbf{G} \longrightarrow \mathbf{C}$ is the identity. If $A, B \in \mathbf{M}$, a map $\Phi(A) \longrightarrow \Phi(B)$ in $\mathbf{C}$ gives a map $(\mathbf{M}(-, A) \longrightarrow \mathbf{C}(\Phi(-), \Phi(A))) \longrightarrow (\mathbf{M}(-, B) \longrightarrow \mathbf{C}(\Phi(-), \Phi(B)))$ in $\mathbf{G}$ and then by a Yoneda argument we see that Φ is full and faithful. This principle scales up to a number of other free extensions.

4.2 Double glueing for intuitionistic linear logic

The general case

In this section we give a brief outline of a general construction without pausing to take care of detailed proofs.

Suppose first that $L\colon \mathbf{C} \longrightarrow \mathbf{E}$ and $K\colon \mathbf{C} \longrightarrow \mathbf{E}^*$ are functors from $\mathbf{C}$ to two categories $\mathbf{E}$ and $\mathbf{E}^*$. The *double glued category* $\mathbf{G} = \mathbf{G}(L, K)$ is the universal object lying in the lax limit diagram

$$\begin{tikzcd}
 & \mathbf{G} \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathbf{E} \arrow[r, phantom, "\Rightarrow", near end, shift left=6ex] & \mathbf{C} \arrow[l, "L"] \arrow[r, "K"'] \arrow[r, phantom, "\Rightarrow", near start, shift left=6ex] & \mathbf{E}^*.
\end{tikzcd}$$

(Note that the direction of the two cells means that we cannot write $\mathbf{G}$ as a comma category.) $\mathbf{G}$ can be described explicitly as follows.

- **Objects** $(R, U, X, (U \longrightarrow L(R)), (K(R) \longrightarrow X)) = (U \longrightarrow L(R), K(R) \longrightarrow X)$ of $\mathbf{G}$ consist of object $R \in \mathbf{C}$, $U \in \mathbf{E}$, $X \in \mathbf{E}^*$ and maps $U \longrightarrow L(R)$ in $\mathbf{E}$ as well as $K(R) \longrightarrow X$ in $\mathbf{E}^*$.

- **Maps** from $(U \longrightarrow L(R), K(R) \longrightarrow X)$ to $(V \longrightarrow L(S), K(S) \longrightarrow Y)$ in $\mathbf{G}$ are given by pairs of commuting diagrams

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "\phi"'] & L(R) \arrow[d, "L(f)"] \\
V \arrow[r] & L(S)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
K(R) \arrow[r] \arrow[d, "K(f)"'] & X \arrow[d, "\psi"] \\
K(S) \arrow[r] & Y
\end{tikzcd}$$

  in $\mathbf{E}$ and $\mathbf{E}^*$ respectively.

Again we are often interested in full subcategories obtained by restricting the structure maps $U \longrightarrow L(R)$ and $K(R) \longrightarrow X$ which occur. (Typically we require $U \longrightarrow L(R)$ monic and $K(R) \longrightarrow X$ epic.) Again we introduce no special notation for such full subcategories.

With a view to applications to linear logic we specialize by taking $\mathbf{E}^*$ to be the opposite $\mathbf{E}^* = \mathbf{E}^{\mathsf{op}}$ of the category $\mathbf{E}$. Thus the double glued category can be taken to consist of objects $R \in \mathbf{C}$, $U, X \in \mathbf{E}$ and maps $U \longrightarrow L(R)$, $X \longrightarrow K(R)$ in $\mathbf{E}$. (In our restricted categories we shall typically take both structure maps to be monic.)

In this situation there is not much difference between the structure required to make $\mathbf{G}$ symmetric monoidal and that to make it symmetric monoidal closed. So we turn at once to describing the multiplicative structure.

Multiplicative structure. We assume without further comment that $L\colon \mathbf{C} \longrightarrow \mathbf{E}$ is monoidal with structure $m_{\mathbf{I}}\colon \mathbf{I} \longrightarrow L(\mathbf{I})$ and $m_{R,S}\colon L(R) \otimes L(S) \longrightarrow L(R \otimes S)$. We describe what further we need to make $\mathbf{G}$ a model of the multiplicative fragment of intuitionistic linear logic. We shall certainly need the conditions from Propositions 4.2. In addition we need structure linking $L\colon \mathbf{C} \longrightarrow \mathbf{E}$ and $K\colon \mathbf{C} \longrightarrow \mathbf{E}^{\mathsf{op}}$. We assume that we have a contraction

$$k_{R,S}\colon L(R) \otimes K(R \otimes S) \longrightarrow K(S)$$

which is natural in S, dinatural in R and such that the following diagrams commute

$$\begin{tikzcd}
\mathbf{I} \otimes K(S) \arrow[r, "\lambda_S"] \arrow[d, "m_{\mathbf{I}} \otimes K(\lambda_S^{-1})"'] & K(S) \arrow[d, "\mathsf{id}"] \\
L(\mathbf{I}) \otimes K(\mathbf{I} \otimes S) \arrow[r, "k_{\mathbf{I},S}"'] & K(S)
\end{tikzcd}$$

$$\begin{tikzcd}
L(R) \otimes L(S) \otimes K(R \otimes S \otimes T) \arrow[r, "\sigma_{R \otimes S} \otimes \mathsf{id}"] \arrow[dd, "m_{R,S} \otimes \mathsf{id}"'] & L(S) \otimes L(R) \otimes K(R \otimes S \otimes T) \arrow[d, "\mathsf{id} \otimes k_{R,S \otimes T}"] \\
& L(S) \otimes K(S \otimes T) \arrow[d, "k_{S,T}"] \\
L(R \otimes S) \otimes K(R \otimes S \otimes T) \arrow[r, "k_{R \otimes S,T}"'] & K(T).
\end{tikzcd}$$

We note that using the closed structure in $\mathbf{E}$ we obtain from k two natural transformations

$$\begin{array}{rcl}
K(R \otimes S) & \longrightarrow & L(R) \multimap K(S) \\
L(R) & \longrightarrow & K(R \otimes S) \multimap K(S).
\end{array}$$

We can combine these with the closed structure in $\mathbf{C}$ to obtain natural transformations

$$\begin{array}{rcl}
K(T) & \longrightarrow & L(S \multimap T) \multimap K(S) \\
L(S \multimap T) & \longrightarrow & K(T) \multimap K(S)
\end{array}$$

respectively; and treating the original k similarly we obtain the natural transformation

$$L(S) \otimes K(T) \longrightarrow K(S \multimap T).$$

For our purposes these are all manifestations of the same structure and we let k denote any of these versions.

Proposition 4.6 *Suppose* $\mathbf{C}$ *and* $\mathbf{E}$ *are symmetric monoidal closed,* $\mathbf{E}$ *with pullbacks, and* L, K *equipped with a contraction. Then* $\mathbf{G}$ *is symmetric monoidal closed and* $\mathbf{G} \longrightarrow \mathbf{C}$ *is a strict map of symmetric monoidal closed categories.*

Proof. Take $A = (U \longrightarrow L(R),\, X \longrightarrow K(R))$, $B = (V \longrightarrow L(S),\, Y \longrightarrow K(S))$ and $C = (W \longrightarrow L(T),\, Z \longrightarrow K(T))$ in $\mathbf{G}$. The tensor of A and B is

$$A \otimes B = (U \otimes V \longrightarrow L(R) \otimes L(S) \xrightarrow{m_{R,S}} L(R \otimes S),\, P \longrightarrow K(R \otimes S))$$

where $P \longrightarrow K(R \otimes S)$ lies in the double pullback diagram

$$\begin{tikzcd}
U \multimap Y \arrow[d] & & P \arrow[ll] \arrow[rr] \arrow[d] & & V \multimap X \arrow[d] \\
U \multimap K(S) & L(R) \multimap K(S) \arrow[l] & K(R \otimes S) \arrow[l, "k"'] \arrow[r, "k"] & L(S) \multimap K(R) \arrow[r] & V \multimap K(R).
\end{tikzcd}$$

The unit for the tensor is

$$\mathbf{I} = (\mathbf{I} \xrightarrow{m_{\mathbf{I}}} L(\mathbf{I}),\, K(\mathbf{I}) \xrightarrow{\mathsf{id}} K(\mathbf{I})).$$

The linear function space $B \multimap C$ is

$$B \multimap C = (Q \longrightarrow L(S \multimap T),\, V \otimes Z \longrightarrow L(S) \otimes K(T) \xrightarrow{k_{S,T}} K(S \multimap T)),$$

where $Q \longrightarrow L(S \multimap T)$ lies in the double pullback diagram

$$\begin{tikzcd}
V \multimap W \arrow[d] & & Q \arrow[ll] \arrow[rr] \arrow[d] & & Z \multimap Y \arrow[d] \\
V \multimap L(T) & L(S) \multimap L(T) \arrow[l] & L(S \multimap T) \arrow[l, "m"'] \arrow[r, "k"] & K(T) \multimap K(S) \arrow[r] & Z \multimap K(S).
\end{tikzcd}$$

One can check that indeed $\mathbf{G}(R \otimes S, T) \cong \mathbf{G}(R, S \multimap T)$ naturally in R, S, and T. □

Remark. The tensor in $\mathbf{G}$ does not require the closed structure in $\mathbf{C}$.

Additive structure. The conditions giving additive structure are simple but maybe not completely obvious. Again there is no reason to treat products and coproducts separately.

Proposition 4.7 *Suppose* $\mathbf{E}$ *has pullbacks and finite coproducts.*

(i) If $\mathbf{C}$ *has finite products then so has* $\mathbf{G}$ *and* $\mathbf{G} \longrightarrow \mathbf{C}$ *preserves them strictly.*

(ii) If $\mathbf{C}$ *has finite coproducts then so has* $\mathbf{G}$ *and* $\mathbf{G} \longrightarrow \mathbf{C}$ *preserves them strictly.*

Proof. (i) The terminal object is

$$(L(\mathbf{1}) \xrightarrow[\mathsf{id}]{} L(\mathbf{1}),\, \mathbf{0} \longrightarrow K(\mathbf{1}));$$

the product of $A = (U \longrightarrow L(R),\, X \longrightarrow K(R))$ and $B = (V \longrightarrow L(S),\, Y \longrightarrow K(S))$ is

$$A \times B = (P \longrightarrow L(R \times S),\, X + Y \longrightarrow K(R) + K(S) \xrightarrow[K[\pi_1, \pi_2]]{} K(R \times S)),$$

where $P \longrightarrow L(R \times S)$ lies in the double pullback

$$\begin{tikzcd} U & P \arrow[l] \arrow[r] \arrow[d] & V \arrow[d] \\ L(R) \arrow[from=u, d] & L(R \times S) \arrow[l] \arrow[r] & L(S). \end{tikzcd}$$

(ii) The initial object is

$$(\mathbf{0} \longrightarrow L(\mathbf{0}),\, K(\mathbf{0}) \xrightarrow{\mathsf{id}} K(\mathbf{0}));$$

the coproduct of $A = (U \longrightarrow L(R),\, X \longrightarrow K(R))$ and $B = (V \longrightarrow L(S),\, Y \longrightarrow K(S))$ is

$$A + B = (U + V \longrightarrow L(R) + L(S) \longrightarrow L(R + S),\, Q \longrightarrow K(R + S))$$

where $Q \longrightarrow K(R + S)$ lies in the double pullback

$$\begin{tikzcd} X \arrow[d] & Q \arrow[l] \arrow[r] \arrow[d] & Y \arrow[d] \\ K(R) & K(R + S) \arrow[l] \arrow[r] & K(S). \end{tikzcd}$$

□

Exponential structure. To obtain additive structure for double glueing we took that for simple glueing and sorted out what happens in the second component. There is a very simple way to do this for the exponentials, and that is basically to ignore the second component. There are more general possibilities, but it does not seem worth describing an abstract framework: that just amounts to analysing the requirements. We give an example in the more concrete setting of the next section, and for the moment restrict ourselves to the *crude exponential.* We assume as background enough structure to ensure that $\mathbf{G}$ is symmetric monoidal; with that in place we have the following.

Proposition 4.8 *Suppose* $\mathbf{C}$ *and* $\mathbf{E}$ *have linear exponential comonads and* $L \colon \mathbf{C} \longrightarrow \mathbf{E}$ *is linearly distributive. Then* $\mathbf{G}$ *has a linear exponential comonad and* $\mathbf{G} \longrightarrow \mathbf{C}$ *preserves it strictly.*

Proof. We define the exponential of $A = (U \longrightarrow L(R),\, X \longrightarrow K(R))$ to be

$$!A = (!U \longrightarrow !L(R) \xrightarrow{\kappa_R} L(!R),\, K(!R) \xrightarrow{\mathsf{id}} K!(R)).$$

The rest of the structure is easy to construct: the first component works as in the proof of Proposition 4.4 and there is only one sensible choice for the second. □

Theorem 4.9 *Suppose that* $\mathbf{C}$ *is a model for intuitionistic linear logic and that* $\mathbf{E}$ *is symmetric monoidal closed, has pullbacks and finite coproducts as well as a linear exponential comonad. Further suppose that* L *is linearly distributive and that* L *and* K *are linked via a contraction. Then* $\mathbf{G}$ *is a model for intuitionistic linear logic. The forgetful functor* $\mathbf{G}(\mathbf{C}) \longrightarrow \mathbf{C}$ *is a strict functor of such models.*

Double glueing along hom-functors

In this section we describe in greater detail a useful simple special case of the double glueing construction, namely that of glueing along hom-functors to the category of sets. We specialize further by considering only the case where the structure maps are monomorphisms (that is, injections).

Suppose that $\mathbf{C}$ is a symmetric monoidal closed category, and let J be any object in $\mathbf{C}$. We consider glueing for the functors $\mathbf{C}(\mathbf{I}, -)\colon \mathbf{C} \longrightarrow \mathbf{Set}$ and $\mathbf{C}(-, J)\colon \mathbf{C} \longrightarrow \mathbf{Set}^{\mathsf{op}}$. We describe the glued category $\mathbf{G}_J(\mathbf{C})$ explicitly as follows:

- **Objects** $A = (R, U, X)$ of $\mathbf{G}_J(\mathbf{C})$ are given by an object R of $\mathbf{C}$ together with sets

$$U \subseteq \mathbf{C}(\mathbf{I}, R) \text{ and } X \subseteq \mathbf{C}(R, J).$$

- **Maps** in $\mathbf{G}_J(\mathbf{C})$ from (R, U, X) to (S, V, Y) are given by maps $f\colon R \longrightarrow S$ in $\mathbf{C}$ such that:
  - for all $\mathbf{I} \xrightarrow{u} R$ in U, $\mathbf{I} \xrightarrow{u} R \xrightarrow{f} S$ is in V and
  - for all $S \xrightarrow{y} J$ in Y, $R \xrightarrow{f} S \xrightarrow{y} J$ is in X.

We need a notation for generalized composition. Given $h\colon R \otimes S \longrightarrow J$ and $v\colon \mathbf{I} \longrightarrow S$, we define $\langle v|h\rangle_S\colon R \longrightarrow J$ to be

$$R \cong R \otimes \mathbf{I} \xrightarrow{\mathsf{id}_R \otimes v} R \otimes S \xrightarrow{h} J.$$

We can think of this as the result of cutting on the formula S. Provided with some $u\colon \mathbf{I} \longrightarrow R$ we can similarly define $\langle u|h\rangle_R\colon S \longrightarrow J$, this time cutting on R. (If $h\colon R \otimes R \longrightarrow J$, then this notation is ambiguous; there are two composites which have the same name. We shall draw attention to the only occasion where this case arises.)

Multiplicative structure. We give rather more detail than we did for the general case described in Proposition 4.6.

Proposition 4.10 *If $\mathbf{C}$ is a symmetric monoidal closed category then so is $\mathbf{G}_J(\mathbf{C})$, and the forgetful functor $\mathbf{G}_J(\mathbf{C}) \longrightarrow \mathbf{C}$ preserves the structure. The structure is defined as follows.*

- *The tensor unit is $\mathbf{I} = (\mathbf{I}, \{\mathsf{id}_\mathbf{I}\}, \mathbf{C}(\mathbf{I}, J))$.*
- *The tensor product of $A = (R, U, X)$ and $B = (S, V, Y)$ is*

$$\begin{aligned} & A \otimes B = (R \otimes S, U \otimes V, Z) \\ \textit{where} \quad & U \otimes V = \{\mathbf{I} \cong \mathbf{I} \otimes \mathbf{I} \xrightarrow{u \otimes v} R \otimes S \mid u \in U, v \in V\} \\ \textit{and} \quad & Z = \{R \otimes S \xrightarrow{z} J \mid \forall \mathbf{I} \xrightarrow{u} R \textit{ in } U.\ \langle u|z\rangle_R\colon S \longrightarrow J \in Y \\ & \qquad \forall \mathbf{I} \xrightarrow{v} S \textit{ in } V.\ \langle v|z\rangle_S\colon R \longrightarrow J \in X\}. \end{aligned}$$

- *The linear function space*

$$A \multimap B = (R \multimap S, W, U \multimap Y)$$

$$\textit{where} \quad W = \{\mathbf{I} \xrightarrow{w} R \multimap S \mid \forall \mathbf{I} \xrightarrow{u} R \textit{ in } U.\ \mathbf{I} \xrightarrow{u} R \xrightarrow{\hat{w}} S \in V$$
$$\forall S \xrightarrow{y} J \textit{ in } Y.\ R \xrightarrow{\hat{w}} S \xrightarrow{y} J \in X\}$$

represents the hom-set $\mathbf{G}_J(\mathbf{C})(A, B)$ and

$$U \multimap Y = \{u \multimap y \colon R \multimap S \longrightarrow \mathbf{I} \multimap J \cong J \mid u \in U, y \in Y\}.$$

(In the definition of W, $\hat{w} \colon R \longrightarrow S$ is the transpose of $w \colon \mathbf{I} \longrightarrow R \multimap S$.)

Proof. Let us show first of all that if $f \colon A \longrightarrow A'$ and $g \colon B \longrightarrow B'$ are two morphisms in $\mathbf{G}(\mathbf{C})$, then their tensor product, taken in $\mathbf{C}$, is a morphism $A \otimes A' \longrightarrow B \otimes B'$ in $\mathbf{G}(\mathbf{C})$. Obviously, $f \otimes g$ provides a function $U \otimes V \longrightarrow U' \otimes V'$ as desired. So let us assume we have $z' \colon R' \otimes S' \longrightarrow J$ such that for all $u' \in U'$, $\langle u'|z'\rangle_{R'} \in Y'$ and such that for all $v' \in V'$, $\langle v'|z'\rangle'_S \in X'$. Then for all $u \in U$,

$$\langle u|z' \cdot (f \otimes g)\rangle_R = \langle f \cdot u|z'\rangle_{R'} \cdot g.$$

Since $f \cdot u \in U'$, $\langle f \cdot u|z'\rangle_{R'} \in Y'$, and so $\langle f \cdot u|z'\rangle_{R'} \cdot g \in Y$ as g is a morphism in $\mathbf{G}(\mathbf{C})$. Similarly we can show that for all $v \in V$, $\langle v|z' \cdot (f \otimes g)\rangle \in X$. Thus $z' \cdot (f \otimes g)$ is in Z as desired. As $\otimes$ is lifted from $\mathbf{C}$, it is a symmetric and associative monoidal product; it is not difficult to see that $\mathbf{I}$ lifts from $\mathbf{C}$ to the unit given. The main issue is to show that the given linear function space is a closed structure for this tensor product. Morphisms

$$(R, U, X) \otimes (S, V, Y) \longrightarrow (T, W, Z)$$

are maps $f \colon R \otimes S \longrightarrow T$ in $\mathbf{C}$ such that

- for all $u \colon \mathbf{I} \longrightarrow R$ and all $v \colon \mathbf{I} \longrightarrow S$,

$$\mathbf{I} \cong \mathbf{I} \otimes \mathbf{I} \xrightarrow{u \otimes v} R \otimes S \xrightarrow{f} T \text{ in } W \qquad \text{and}$$

- for all $z \colon T \longrightarrow Z$, $R \otimes S \xrightarrow{f} T \xrightarrow{z} J$ satisfies
  - for all $u \colon \mathbf{I} \longrightarrow R$, $z \cdot \langle u|f\rangle_R = \langle u|z \cdot f\rangle_R \in Y$ and
  - for all $v \colon \mathbf{I} \longrightarrow S$, $z \cdot \langle v|f\rangle_S = \langle v|z \cdot f\rangle_S \in X$.

This is equivalent to satisfying the more symmetric conditions

- for all $u \colon \mathbf{I} \longrightarrow R$ in U and all $v \colon \mathbf{I} \longrightarrow S$ in V,

$$\mathbf{I} \xrightarrow{v} S \cong \mathbf{I} \otimes S \xrightarrow{u \otimes \mathsf{id}_S} R \otimes S \xrightarrow{f} T \text{ in } W \text{ and}$$

- for all $u \colon \mathbf{I} \longrightarrow R$ in U and all $z \colon T \longrightarrow J$ in Z,

$$S \cong \mathbf{I} \otimes S \xrightarrow{u \otimes \mathsf{id}_S} R \otimes S \xrightarrow{f} T \xrightarrow{z} J \text{ in } Y \text{ and}$$

- for all $v\colon \mathbf{I} \longrightarrow S$ in V and all $z\colon T \longrightarrow J$ in Z,

$$R \cong R \otimes \mathbf{I} \xrightarrow{\mathsf{id}_R \otimes v} R \otimes S \xrightarrow{f} T \xrightarrow{z} J \text{ in } X.$$

It is a simple matter to wrap these up differently and show that they are equivalent to the transpose $R \longrightarrow S \multimap T$ being a morphism

$$(R, U, X) \longrightarrow (S, V, Y) \multimap (T, W, Z).$$

□

Additive structure. Products and coproducts lift readily from $\mathbf{C}$ to $\mathbf{G}_J(\mathbf{C})$. The only point to note is that since our structure maps are injections, a union replaces the coproduct of Proposition 4.7.

Proposition 4.11 *(i) If $\mathbf{C}$ has finite products then so has $\mathbf{G}_J(\mathbf{C})$, and $\mathbf{G}_J(\mathbf{C}) \longrightarrow \mathbf{C}$ preserves them. They are given by*

$$\begin{aligned} & A \times B = (R \times S, U \times V, X \oplus Y) \\ \textit{where} \quad & U \times V = \{\langle u, v\rangle \colon \mathbf{I} \longrightarrow R \times S \mid u \in U, v \in V\} \\ \textit{and} \quad & X \oplus Y = \{R \times S \xrightarrow{\pi_1} R \xrightarrow{x} J \mid x \in X\} \cup \{R \times S \xrightarrow{\pi_2} S \xrightarrow{y} J \mid y \in Y\}. \end{aligned}$$

The terminal object is $(\mathbf{1}, \mathbf{C}(\mathbf{I}, \mathbf{1}), \emptyset)$.

(ii) If $\mathbf{C}$ has finite coproducts then so has $\mathbf{G}_J(\mathbf{C})$, and $\mathbf{G}_J(\mathbf{C}) \longrightarrow \mathbf{C}$ preserves them. They are given by

$$\begin{aligned} & A + B = (R + S, U \oplus V, X + Y) \\ \textit{where} \quad & X + Y = \{[x, y] \colon R + S \longrightarrow J \mid x \in X, y \in Y\} \\ \textit{and} \quad & U \oplus V = \{\mathbf{I} \xrightarrow{u} R \xrightarrow{\mathsf{inl}} R + S \mid u \in U\} \cup \{\mathbf{I} \xrightarrow{v} S \xrightarrow{\mathsf{inr}} R + S \mid v \in V\} \end{aligned}$$

The unit for the coproduct is $(\mathbf{0}, \emptyset, \mathbf{C}(\mathbf{0}, J))$.

Proof. The projections in $\mathbf{C}$ are morphisms in $\mathbf{G}_J(\mathbf{C})$; and pairing two $\mathbf{G}_J(\mathbf{C})$ morphisms in $\mathbf{C}$ results in a morphism in $\mathbf{G}_J(\mathbf{C})$. Coproducts behave dually. □

Exponential structure. Let $\mathbf{C}$ be symmetric monoidal as well as equipped with a linear exponential comonad $(!, \varepsilon, \delta)$ with associated natural transformations $e\colon ! \longrightarrow \mathbf{I}$ and $d\colon !(-) \longrightarrow !(-)\otimes!(-)$. The functor $\mathbf{C}(\mathbf{I}, -)\colon \mathbf{C} \longrightarrow \mathbf{Set}$ is monoidal; and as the category $\mathbf{Set}$ trivially is a model for intuitionistic linear logic (with the identity comonad as the exponential), we can ask for a natural transformation $\kappa\colon \mathbf{C}(\mathbf{I}, -) \longrightarrow \mathbf{C}(\mathbf{I}, !(-))$ making $\mathbf{C}(\mathbf{I}, -)$ linearly distributive. This amounts to the following:

- κ is well-behaved with respect to the comonad structure:

$$\begin{tikzcd}
& \mathbf{C}(\mathbf{I}, R) \arrow[r, "\kappa_R"] \arrow[dl, "\mathrm{id}_{\mathbf{C}(\mathbf{I},R)}"'] \arrow[d, "\kappa_R"] & \mathbf{C}(\mathbf{I}, !R) \arrow[d, "\kappa_{!R}"] \\
\mathbf{C}(\mathbf{I}, R) & \mathbf{C}(\mathbf{I}, !R) \arrow[l, "\mathbf{C}(\mathbf{I},\varepsilon_R)"] \arrow[r, "\mathbf{C}(\mathbf{I},\delta_R)"'] & \mathbf{C}(\mathbf{I}, !!R).
\end{tikzcd}$$

- κ_R respects the comonoid structure:

$$\begin{tikzcd}
\mathbf{1} \arrow[d, "\ulcorner \mathsf{id}_{\mathbf{I}} \urcorner"] & \mathbf{C}(\mathbf{I}, R) \arrow[l] \arrow[r, "\Delta_{\mathbf{C}(\mathbf{I},R)}"] \arrow[d, "\kappa_R"] & \mathbf{C}(\mathbf{I}, R) \times \mathbf{C}(\mathbf{I}, R) \arrow[r, "\kappa_R \times \kappa_R"] & \mathbf{C}(\mathbf{I}, !R) \times \mathbf{C}(\mathbf{I}, !R) \arrow[d, "\otimes"] \\
\mathbf{C}(\mathbf{I}, \mathbf{I}) & \mathbf{C}(\mathbf{I}, !R) \arrow[l, "\mathbf{C}(\mathbf{I},e_R)"] \arrow[rr, "\mathbf{C}(\mathbf{I},d_R)"'] & & \mathbf{C}(\mathbf{I}, !R \otimes !R).
\end{tikzcd}$$

- κ is monoidal

$$\begin{tikzcd}
\mathbf{C}(\mathbf{I}, \mathbf{I}) \arrow[r, "\kappa_{\mathbf{I}}"] & \mathbf{C}(\mathbf{I}, !\mathbf{I}) & \mathbf{C}(\mathbf{I}, R) \times \mathbf{C}(\mathbf{I}, S) \arrow[rr, "\otimes"] \arrow[d, "\kappa_R \times \kappa_S"'] & & \mathbf{C}(\mathbf{I}, R \otimes S) \arrow[d, "\kappa_{R \otimes S}"] \\
= & & & & \\
\mathbf{C}(\mathbf{I}, \mathbf{I}) \arrow[r, "\mathbf{C}(\mathbf{I},m_{\mathbf{I}})"] & \mathbf{C}(\mathbf{I}, !\mathbf{I}) & \mathbf{C}(\mathbf{I}, !R) \times \mathbf{C}(\mathbf{I}, !S) \arrow[r, "\otimes"'] & \mathbf{C}(\mathbf{I}, !R \otimes !S) \arrow[r, "\mathbf{C}(\mathbf{I},m_{R,S})"'] & \mathbf{C}(\mathbf{I}, !(R \otimes S)).
\end{tikzcd}$$

Given such a structure we observed in Section 4.1 that we can define a crude exponential on $\mathbf{G}_J(\mathbf{C})$. Now, however, we can also do something more subtle.

Proposition 4.12 *Let $\mathbf{C}$ be a model for intuitionistic linear logic with a linear distribution κ as above. Set $\kappa_R[U] = \{\kappa_R(u) \mid u \in U\}$.*

(i) We can define a linear exponential comonad on $\mathbf{G}_J(\mathbf{C})$ by

$$!(R, U, X) = (!R, \kappa_R[U], \mathbf{C}(!R, J)),$$

and $\mathbf{G}_J(\mathbf{C}) \longrightarrow \mathbf{C}$ preserves the structure.

(ii) We can define a linear exponential comonad on $\mathbf{G}_J(\mathbf{C})$ by

$$!(R, U, X) = (!R, \kappa_R[U], ?X),$$

where $?X$ is the smallest subset of $\mathbf{C}(!R, J)$

- *containing $\{x \cdot \varepsilon_R \mid x \in X\}$,*
- *containing $\{\chi \cdot e_R \mid \chi \colon \mathbf{I} \longrightarrow J\}$,*
- *and such that whenever for some $h \colon !R \otimes !R \longrightarrow J$, for all $u \in U$ both composites $\langle \kappa_R(u) | h \rangle_{!R}$ are in $?X$, then $h \cdot d_R \colon !R \longrightarrow J$ is in $?X$.*

Again $\mathbf{G}_J(\mathbf{C}) \longrightarrow \mathbf{C}$ *preserves the structure.*

Proof. See Appendix A. $\square$

Theorem 4.13 *Let* $\mathbf{C}$ *be a model for intuitionistic linear logic equipped with a linear distribution* κ *as above. Then* $\mathbf{G}_J(\mathbf{C})$ *is a model for intuitionistic linear logic, and* $\mathbf{G}_J(\mathbf{C}) \longrightarrow \mathbf{C}$ *preserves all the structure.*

Application

Coindecomposability. We can turn Freyd's argument (see (2) in Examples 4.5) around in the double glueing context. Let $\mathbf{C}$ be the free symmetric monoidal closed category with products. Set $J = \mathbf{I}$ and let $\mathbf{G}$ be the result of glueing along $\mathbf{C}(\mathbf{I}, -)\colon \mathbf{C} \longrightarrow \mathbf{Set}$, $\mathbf{C}(-, \mathbf{I})\colon \mathbf{C} \longrightarrow \mathbf{Set}^{\mathsf{op}}$. For this argument we do not restrict to monomorphisms. Once again we have $\mathbf{C} \longrightarrow \mathbf{G} \longrightarrow \mathbf{C}$ the identity. Now Freyd's considerations applied to the last component of the structure in $\mathbf{G}$ gives

$$\mathbf{C}(R \times S, \mathbf{I}) \cong \mathbf{C}(R, \mathbf{I}) + \mathbf{C}(S, \mathbf{I})$$

and $\mathbf{I}$ is coindecomposable.

4.3 Double glueing for classical linear logic

The general case

In this section we take a symmetric form of the general construction of Section 4.2 and apply it to get models for classical linear logic. We take as ever $L\colon \mathbf{C} \longrightarrow \mathbf{E}$ a monoidal functor, but now we assume that $\mathbf{C}$ has a self-duality $(-)^{\perp}\colon \mathbf{C} \longrightarrow \mathbf{C}^{\mathsf{op}}$ and we set K to be $K = L^{\mathsf{op}} \cdot (-)^{\perp}\colon \mathbf{C} \longrightarrow \mathbf{E}^{\mathsf{op}}$: that is, K is the composite

$$\mathbf{C} \xrightarrow{(-)^{\perp}} \mathbf{C}^{\mathsf{op}} \xrightarrow{L^{\mathsf{op}}} \mathbf{E}^{\mathsf{op}}.$$

It follows at once that $\mathbf{G}$ has a self-duality

$$(U \longrightarrow L(R),\, X \longrightarrow L(R^{\mathsf{op}}))^{\perp} = (X \longrightarrow L(R^{\mathsf{op}}),\, U \longrightarrow L(R)).$$

Multiplicative structure. Suppose that $\mathbf{C}$ is $*$-autonomous. Then there is an obvious choice of contraction linking L and K. We take the composite

$$L(R) \otimes K(R \otimes S) \cong L(R) \otimes L(R \multimap S^{\perp}) \xrightarrow{m} L(R \otimes (R \multimap S^{\perp})) \xrightarrow{L(\mathsf{ev})} L(S^{\perp}) = K(S).$$

Proposition 4.14 *Suppose* $\mathbf{C}$ *is* $*$*-autonomous and* $\mathbf{E}$ *symmetric monoidal closed with pullbacks. Then* $\mathbf{G}$ *is* $*$*-autonomous and* $\mathbf{G} \longrightarrow \mathbf{C}$ *is a strict map of* $*$*-autonomous categories.*

Proof. By Proposition 4.6 one has only to check that $A \multimap B \cong (A \otimes B^{\perp})^{\perp}$. naturally. This is routine. $\square$

Additive structure. The results do not depend on the symmetric situation as we have seen in Proposition 4.7. We restate the results here for completeness' sake.

Proposition 4.15 *Suppose* $\mathbf{E}$ *has pullbacks and finite coproducts. If* $\mathbf{C}$ *has finite products then so does* $\mathbf{G}$ *and* $\mathbf{G} \longrightarrow \mathbf{C}$ *preserves them strictly. If* $\mathbf{C}$ *has finite coproducts then so does* $\mathbf{G}$ *and the forgetful functor* $\mathbf{G} \longrightarrow \mathbf{C}$ *preserves them strictly.*

Exponential structure. As things stand we cannot do better than we did in the intuitionistic case, Proposition 4.8.

Theorem 4.16 *Assume that* $\mathbf{C}$ *is a model for classical linear logic and that* $\mathbf{E}$ *is symmetric monoidal closed, has pullbacks, coproducts, and a linear exponential comonad. Further assume that* L *is linearly distributive. Then* $\mathbf{G}$ *is a model for classical linear logic and the forgetful functor* $\mathbf{G}(\mathbf{C}) \longrightarrow \mathbf{C}$ *preserves all the structure.*

Double glueing along hom-functors—the classical case

Now we give details for the case of double glueing along hom-functors. Again we specialize by considering only the case when the structure maps are monomorphisms (injections).

Take $\mathbf{C}$ to be a category with an involution $(-)^\perp$. (A self-duality whose square is isomorphic to the identity functor will do—there is less book-keeping if we assume the square is the identity.) To get the symmetry from the beginning of Section 4.3 we are forced to set $J = \mathbf{I}^\perp = \perp$. So objects of the glued category $\mathbf{G}(\mathbf{C})$ are given by objects R of $\mathbf{C}$ together with $U \subseteq \mathbf{C}(\mathbf{I}, R)$ and $X \subseteq \mathbf{C}(R, \perp) \cong \mathbf{C}(\mathbf{I}, R^\perp)$. Making the identification of $\mathbf{C}(R, \perp)$ with $\mathbf{C}(\mathbf{I}, R^\perp)$ enables us to write the involution as

$$(R, U, X)^\perp = (R^\perp, X, U).$$

We make such identifications without further comment below.

Multiplicative structure. We expand slightly on the treatment in the previous section.

Proposition 4.17 *If* $\mathbf{C}$ *is* $*$*-autonomous then so is* $\mathbf{G}(\mathbf{C})$*, and the forgetful functor to* $\mathbf{C}$ *preserves the* $*$*-autonomous structure, which is as follows.*

- *The involution* $(R, U, X)^\perp = (R^\perp, X, U)$.
- *The tensor unit* $\mathbf{I} = (\mathbf{I}, \{\mathsf{id}_\mathbf{I}\}, \mathbf{C}(\mathbf{I}, \perp))$.
- *the tensor product of* $A = (R, U, X)$ *and* $B = (S, V, Y)$ *is*

$$A \otimes B = (R \otimes S, U \otimes V, \mathbf{G}(\mathbf{C})(A, B^\perp))$$

where $U \otimes V = \{\mathbf{I} \cong \mathbf{I} \otimes \mathbf{I} \xrightarrow{u \otimes v} R \otimes S \mid u \in U, v \in V\}$

and $\mathbf{G}(\mathbf{C})(A, B^\perp) = \{R \otimes S \xrightarrow{z} \perp \mid \forall \mathbf{I} \xrightarrow{u} R \in U.\ \langle u|z\rangle_R \colon S \longrightarrow \perp \in Y$
$\forall \mathbf{I} \xrightarrow{v} S \in V.\ \langle v|z\rangle_S \colon R \longrightarrow \perp \in X\}.$

Up to natural identification the last component is the set of maps in $\mathbf{G}(\mathbf{C})$ *from* (R, U, X) *to* $(S, V, Y)^\perp$*, hence the notation.*

Proof. All we need to show is that $A \multimap B \cong (A \otimes B^\perp)^\perp$. But the last component of $A \otimes B^\perp$ is obviously $\mathbf{G}(\mathbf{C})(A, B)$, and

$$\{R \multimap S \xrightarrow{u \multimap y} \mathbf{I} \multimap \perp \cong \perp \mid u \in U, y \in Y\}$$

can clearly be identified with

$$\{\perp^\perp \cong \mathbf{I} \cong \mathbf{I} \otimes \mathbf{I} \xrightarrow{(u \multimap y^\perp)^\perp = u \otimes y} (R \otimes S^\perp)^\perp \cong R \otimes S \mid u \in U, y \in Y\}.$$

□

Additive structure. We have already seen that products and coproducts lift readily from $\mathbf{C}$ to $\mathbf{G}_J(\mathbf{C})$. We restate the result from Proposition 4.11 for $\mathbf{G}(\mathbf{C})$ for completeness' sake.

Proposition 4.18 *If $\mathbf{C}$ has finite products, then so has $\mathbf{G}(\mathbf{C})$, and $\mathbf{G}(\mathbf{C}) \longrightarrow \mathbf{C}$ preserves them. If $\mathbf{C}$ has finite coproducts then so has $\mathbf{G}(\mathbf{C})$ and $\mathbf{G}(\mathbf{C}) \longrightarrow \mathbf{C}$ preseves them.*

Exponential structure. The extensive discussion in Section 4.2 needs no modification to deal with the symmetric structure.

Theorem 4.19 *If $\mathbf{C}$ is a model for classical linear logic, equipped with a linear distribution κ then so is $\mathbf{G}(\mathbf{C})$, and the forgetful functor $\mathbf{G}(\mathbf{C}) \longrightarrow \mathbf{C}$ preserves all the structure.*

Examples and applications

Examples 4.20 (1) **Logical relations.** Double glueing provides an appropriate form of logical relations for $*$-autonomous categories (compare (1) in Examples 4.5). For example we may take $\mathbf{C}$ to be **Rel** and consider $L\colon \mathbf{Rel} \longrightarrow \mathbf{Set}$ given by $L(R) = \mathbf{Rel}(\mathbf{I}, R)$, or equivalently L is the power set functor. Then (with the monomorphism restriction) the double glued category $\mathbf{G}$ is Loader's category **LLP** [43, 44] of linear logical predicates (see the examples in Section 2). Loader gives an account of two exponential structures in [44] but these do not constitute linear exponential comonads in our sense. Our Proposition 4.12 does provide linear exponential comonads. We could 'cut down' the first to give a substitute for Loader's 'continuous exponential'; we have not considered a substitute for his 'stable exponential'.

(2) **Pre-phase semantics.** A commutative monoid M can be regarded as a symmetric monoidal closed (indeed compact closed) category $\mathbf{M}$ on a single object, say $\mathbf{I}$: one sets $\mathbf{M}(\mathbf{I}, \mathbf{I}) = M$ and composition is multiplication. We consider the glued category $\mathbf{G}(\mathbf{M})$. By Proposition 4.17 it is $*$-autonomous. As it stands there are no additives or exponentials. For phase semantics we shall in any case restrict to maps in the fibre over $\mathbf{I}$ of $\mathbf{G}(\mathbf{M}) \longrightarrow \mathbf{M}$ (that is, maps whose image is the identity e). That gives us the poset $P(\mathbf{M}) \times P(\mathbf{M})^{\mathsf{op}}$ which has additive as well as multiplicative structure: The only exponential is trivial. We can view this category as a precursor to Girard's phase spaces; sensible exponentials arise for the tight orthogonality category (see Examples 5.15 (1)).

(3) **Process realizability.** Double glueing is behind this formulation by Abramsky [1]. Let $\mathbf{C}$ be the category whose objects are sets (of names) and whose maps from A to B are suitable equivalence classes of suitable processes with names from $A + B$. This is compact closed (duality is given by interchanging names and conames) and the corresponding double glued category is a category of process realizability. This accounts for multiplicative structure, but (weak) additive and exponential structure is derived from enrichment of $\mathbf{C}$ which we do not consider. (To get the final category indicated by Abramsky one also takes partial equivalence relations to enforce good structure.) Abramsky has his data coded in one object and we have not investigated the resulting superficial mismatches between what he gives and what we sketch here.

(4) **Conservativity.** Lafont's argument (see Example (3) in 4.5) adapts easily to the double glueing context. Suppose $\mathbf{M}$ is a symmetric monoidal closed category and $\Phi\colon \mathbf{M} \longrightarrow \mathbf{C}$ the result of freely adding $*$-autonomous structure on top of that. Let $\mathbf{E}$ be $[\mathbf{M}^{\mathsf{op}}, \mathbf{Set}]$ and $\mathbf{G}$ the result of double glueing along the map $L\colon \mathbf{C} \longrightarrow \mathbf{E}$ with $L(R) = \mathbf{C}(\Phi(-), R)$. Then arguing as before we can deduce that $\Phi\colon \mathbf{M} \longrightarrow \mathbf{C}$ is full and faithful. (This argument was noticed independently by Hasegawa.) There are a number of related versions.

5 Orthogonality categories

In the previous section we showed how to obtain models for linear logic by means of double glueing. A number of interesting models for linear logic are subcategories of glued categories and we now describe a general technique for carving out such subcategories. We shall treat the cases of categories for intuitionistic and classical linear logic in parallel.

5.1 Orthogonality of maps

We concentrate attention on categories obtained by glueing along hom-functors. We assume that we start with a symmetric monoidal category $\mathbf{C}$: for $J \in \mathbf{C}$ we have the glued category $\mathbf{G}_J(\mathbf{C})$. An orthogonality on $\mathbf{C}$ is then an indexed family of relations $\perp_R$ between maps $u\colon \mathbf{I} \longrightarrow R$ and maps $x\colon R \longrightarrow J$

$$\mathbf{I} \xrightarrow{u} R \perp_R R \xrightarrow{x} J$$

satisfying the conditions in the following definition.

Definition 6 *Let $\mathbf{C}$ be a symmetric monoidal closed category. An* orthogonality *on $\mathbf{C}$ is a family of relations $\perp_R$ between maps $\mathbf{I} \longrightarrow R$ and $R \longrightarrow J$ satisfying the following:*

(i) (Isomorphism) If $f\colon R \longrightarrow S$ is an isomorphism, then for all $u\colon \mathbf{I} \longrightarrow R$ and all $x\colon R \longrightarrow J$, we have

$$u \perp_R x \qquad \text{if and only if} \qquad f \cdot u \perp_S x \cdot f^{-1};$$

that is, orthogonality is invariant under isomorphism.

(ii) (Tensor) Given $\mathbf{I} \xrightarrow{u} R$, $\mathbf{I} \xrightarrow{v} S$ and $R \otimes S \xrightarrow{h} J$, then

$$u \perp_R \langle v|h\rangle_S \text{ and } v \perp_S \langle u|h\rangle_R \qquad \text{imply} \qquad u \otimes v \perp_{R\otimes S} h.$$

(iii) *(Implication) Given* $\mathbf{I} \xrightarrow{u} R$, $S \xrightarrow{y} J$ *and* $R \xrightarrow{f} S$, *then*

$$u \perp_R y \cdot f \text{ and } f \cdot u \perp_S y \qquad \text{imply} \qquad \check{f} \perp_{R \multimap S} u \multimap y,$$

where $\check{f} \colon \mathbf{I} \longrightarrow R \multimap S$ *is the transpose of* f *and* $u \multimap y \colon R \multimap S \longrightarrow \mathbf{I} \multimap J \cong J$.

(iv) *(Identity) For all* $\mathbf{I} \xrightarrow{u} R$ *and all* $R \xrightarrow{x} J$,

$$u \perp_R x \qquad \text{implies} \qquad \mathsf{id}_{\mathbf{I}} \perp_{\mathbf{I}} x \cdot u = \langle u|x \rangle_{\mathbf{I}}.$$

Given an orthogonality $\perp$, we define for $U \subseteq \mathbf{C}(\mathbf{I}, R)$ its orthogonal $U^\circ \subseteq \mathbf{C}(R, J)$ by

$$U^\circ = \{x \colon R \longrightarrow J \mid \forall u \in U.\, u \perp_R x\}.$$

Similarly for $X \subseteq \mathbf{C}(R, J)$ we define its orthogonal $X^\circ \subseteq \mathbf{C}(\mathbf{I}, R)$ by

$$X^\circ = \{u \colon \mathbf{I} \longrightarrow R \mid \forall x \in X.\, u \perp_R x\}.$$

Obviously this gives rise to a Galois connection. Hence, for example, if $U = X^\circ$ then $U^{\circ\circ} = X^{\circ\circ\circ} = X^\circ = U$. We call sets U, X with $U = U^{\circ\circ}$, $X = X^{\circ\circ}$ *closed.*

We shall also make use of the following natural notation. If $U \subseteq \mathbf{C}(\mathbf{I}, R)$, $V \subseteq \mathbf{C}(\mathbf{I}, S)$, $Y \subseteq \mathbf{C}(S, J)$ then we set

$$\begin{aligned} U \otimes V &= \{u \otimes v \mid u \in U, v \in V\} \subseteq \mathbf{C}(\mathbf{I}, R \otimes S) \\ U \multimap Y &= \{u \multimap y \mid u \in U, y \in Y\} \subseteq \mathbf{C}(R \multimap S, J). \end{aligned}$$

When we deal with classical linear logic we glue along the pair of dual functors $\mathbf{C}(\mathbf{I}, -)$ and $\mathbf{C}(-, \perp)$. Then to get good behaviour of our orthogonality categories we shall need a further condition on the orthogonality.

Definition 7 *An orthogonality on a $*$-autonomous category is* symmetric *just when it satisfies the following condition:*

(v) *(Symmetry) For all* $\mathbf{I} \xrightarrow{u} R$ *and all* $R \xrightarrow{x} \perp$,

$$u \perp_R x \qquad \text{if and only if} \qquad x^\perp \perp_{R^\perp} u^\perp.$$

Remark. (a) The symmetry condition (v) together with the tensor condition (ii) implies the implication condition (iii).

(b) Symmetry enables us to regard the orthogonality $\perp$ in lots of different ways. For example, we can consider $u \colon \mathbf{I} \longrightarrow R$ orthogonal to $x^\perp \colon \mathbf{I} \longrightarrow R^\perp$ without ambiguity.

(c) Together conditions (ii) and (v) mean that $u \perp_R x$ implies $u \otimes x^\perp \perp_{R \otimes R^\perp} \mathsf{ev}_R$. (Here $\mathsf{ev}_R \colon R \otimes R^\perp \longrightarrow \perp$ corresponds to the evaluation map $R \otimes (R \multimap \perp) \longrightarrow \perp$.)

An orthogonality naturally gives rise to the following two full subcategories of the glued category.

Definition 8 *The* slack (orthogonality) category $\mathbf{S}_J(\mathbf{C})$ *is the full subcategory of* $\mathbf{G}_J(\mathbf{C})$ *on those objects* (R, U, X) *such that for all* $u \in U$ *and for all* $x \in X$ *we have* $u \perp_R x$*; in other words, such that* $U \subseteq X^\circ$ *and* $X \subseteq U^\circ$*. We use* $\mathbf{S}(\mathbf{C})$ *to denote* $\mathbf{S}_\perp(\mathbf{C})$*.*

The tight (orthogonality) subcategory $\mathbf{T}_J(\mathbf{C})$ *is the full subcategory of* $\mathbf{G}_J(\mathbf{C})$ *on those* (R, U, X) *for which* $U = X^\circ$ *and* $X = U^\circ$*. We use* $\mathbf{T}(\mathbf{C})$ *to denote* $\mathbf{T}_\perp(\mathbf{C})$*.*

For any object (R, U, X) of the slack category, $(R, U^{\circ\circ}, X)$ and $(R, U, X^{\circ\circ})$ are further objects of $\mathbf{S}_J(\mathbf{C})$. If $U \subseteq \mathbf{C}(\mathbf{I}, R)$ is closed, then (R, U, U°) is an object of the tight subcategory; and if $X \subseteq \mathbf{C}(R, J)$ is closed then (U, X°, X) is an object of the tight subcategory.

Examples 5.1 (1) **Trivial orthogonalities.** The *full orthogonality* is defined by

$$u \perp_R x \qquad \text{for all } u\colon \mathbf{I} \longrightarrow R,\ x\colon R \longrightarrow J.$$

The *empty orthogonality* is defined by

$$u \perp_R x \qquad \text{for no } u\colon \mathbf{I} \longrightarrow R,\ x\colon R \longrightarrow J.$$

We call these the trivial orthogonalities: they are of rather limited interest.

(2) **Focussed orthogonalities.** Suppose that $F \subseteq \mathbf{C}(\mathbf{I}, J)$ is any set. Then

$$\mathbf{I} \xrightarrow{u} R \perp_R R \xrightarrow{x} J \qquad \text{if and only if} \qquad x \cdot u \in F$$

defines an orthogonality on $\mathbf{C}$. The conditions are automatic and easy to check. Moreover in case $\mathbf{C}$ is $*$-autonomous and J is $\perp$, such an orthogonality is automatically symmetric. We say that an orthogonality determined by an $F \subseteq \mathbf{C}(\mathbf{I}, J)$ is *focussed* with *focus* F. Note that the focus is automatically

$$F = \{f\colon \mathbf{I} \longrightarrow J \mid \mathsf{id}_\mathbf{I} \perp_\mathbf{I} f\}.$$

We give a few representative examples.

(i) If we take $F = \mathbf{C}(\mathbf{I}, J)$ we get the full orthogonality; and if we take $F = \emptyset \subseteq \mathbf{C}(\mathbf{I}, J)$ we get the empty orthogonality. So the trivial orthogonalities are focussed.

(ii) Recall from Examples 4.20 (2) the compact closed category $\mathbf{M}$ corresponding to a monoid M. Any $F \subseteq M$ then gives rise to an orthogonality on $\mathbf{M}$. We briefly explain how this gives rise to the phase semantics of [29, 30] in Examples 5.15 (1).

(iii) Scott domains and linear maps. Consider the category **LinDom** of Scott domains and linear maps. The tensor unit $\mathbf{I}$ is the two element lattice, and $\mathbf{LinDom}(\mathbf{I}, \mathbf{I})$ has two elements, $\lambda x.x$ and $\lambda x.\perp$. Hence there are two non-trivial subsets $\{\lambda x.\perp\}$ and $\{\lambda x.x\}$ of $\mathbf{LinDom}(\mathbf{I}, \mathbf{I})$. For any domain A, $\mathbf{LinDom}(\mathbf{I}, A)$ can be identified with the elements $a \in A$, while $\mathbf{LinDom}(A, \mathbf{I})$ can be identified with the linear open subsets $x \subseteq A$. Then our two non-trivial focussed orthogonalities are

$$a \perp_A x \qquad \text{if and only if} \qquad a \in x$$

and

$$a \perp_A x \qquad \text{if and only if} \qquad a \notin x,$$

respectively.

(iv) SETS AND RELATIONS. Consider the category **Rel** of sets and relations. The tensor unit $\mathbf{I}$ is the one point set and $\mathbf{Rel}(\mathbf{I},\mathbf{I})$ has two members, the true and false relations $\mathbf{I} \longrightarrow\!\!\!\!\!\!+\;\; \mathbf{I}$. So again there are two non-trivial subsets $\{\mathsf{true}\}$ and $\{\mathsf{false}\}$ of $\mathbf{Rel}(\mathbf{I},\mathbf{I})$. For any set R, $\mathbf{Rel}(\mathbf{I},R)$ and $\mathbf{Rel}(R,\mathbf{I})$ can both be identified with the powerset $\mathsf{P}(R)$. We again get two non-trivial focussed orthogonalities

$$u \perp_R x \qquad \text{if and only if} \qquad u \cap x \neq \emptyset$$

and

$$u \perp_R x \qquad \text{if and only if} \qquad u \cap x = \emptyset,$$

respectively.

(v) ORTHOGONALITIES ON PRODUCT CATEGORIES. Suppose $\mathbf{D}$ is a symmetric monoidal closed category. Let $\mathbf{C} = \mathbf{D}^2$ be its square. Fix an object of form $(J,J) \in \mathbf{C}$. Then for $(R,S) \in \mathbf{C}$ we have maps $(u,v)\colon (\mathbf{I},\mathbf{I}) \longrightarrow (R,S)$, $(x,y)\colon (R,S) \longrightarrow (J,J)$. We can define the *equality orthogonality* by

$$(u,v) \perp_{(R,S)} (x,y) \qquad \text{if and only if} \qquad x \cdot u = y \cdot v.$$

This is a focussed orthogonality whose focus is the equality relation in the hom-set $\mathbf{C}((\mathbf{I},\mathbf{I})(J,J)) = \mathbf{D}(\mathbf{I},J)^2$. Of course other relations on maps also give orthogonalities.

(3) **Orthogonalities in compact closed categories.** Suppose $\perp$ is a precise symmetric orthogonality on a $*$-autonomous category $\mathbf{C}$ in the sense of Definition 10 (given in Section 5.3). Take $u\colon \mathbf{I} \longrightarrow R$, $x\colon R \longrightarrow \perp$ and consider $\mathsf{ev}\colon R \otimes R^\perp \longrightarrow \perp$. We have

$$\begin{array}{lll} u \otimes x^\perp \perp_{R\otimes R^\perp} \mathsf{ev} & \text{if and only if} & u \perp_R \langle x^\perp|\mathsf{ev}\rangle_{R^\perp} \text{ and } x^\perp \perp_{R^\perp} \langle u|\mathsf{ev}\rangle_R \\ & \text{if and only if} & x^\perp \perp_{R^\perp} u^\perp \text{ and } u \perp_R x \\ & \text{if and only if} & u \perp_R x. \end{array}$$

So $\perp$ is determined by the sets $\{f\colon \mathbf{I} \longrightarrow R \otimes R^\perp \mid f \perp_{R\otimes R^\perp} \mathsf{ev}\}$. Now suppose $\mathbf{C}$ is compact closed. We write $(-)^*$ for $(-)^\perp$ to emphasize this assumption. Then $f\colon \mathbf{I} \longrightarrow R \otimes R^*$ corresponds to $\tilde{f}\colon R \longrightarrow R$. So a precise symmetric orthogonality on a compact closed category $\mathbf{C}$ is determined by a family $F_R \subseteq \mathsf{End}_{\mathbf{C}}(R)$. The natural conditions on F_R are

- invariance under isomorphism;
- $f \in F_{R\otimes S}$ if and only if $\mathsf{tr}_S(f) \in F_R$ and $\mathsf{tr}_R(f) \in F_S$;
- $f \in F_R$ implies $\mathsf{tr}_R(f) \in F_{\mathbf{I}}$

where tr is the usual trace operator on a compact closed category. (The orthogonality is focussed just when the last is an equivalence.)

We give two instructive examples in case of the compact closed category **Rel**. (Note that for $R \in \mathbf{Rel}$ we can identify $\mathsf{End}(R)$ with $\mathsf{P}()(R \times R)$, the subsets of $R \times R$.) We write $|a|$ for the cardinality of a set a.

(i) The *partial orthogonality* is

$$u \perp_R x \qquad \text{if and only if} \qquad |u \cap x| \leq 1.$$

Here $F_R = \{f \subseteq R \times R \mid |f \cap \Delta_R| \leq 1\}$.

(ii) The *total orthogonality* is

$$u \perp_R x \qquad \text{if and only if} \qquad |u \cap x| = 1.$$

Here $F_R = \{f \subseteq R \times R \mid |f \cap \Delta_R| = 1\}$.

We observe that neither of these two orthogonalities is focussed.

Note that a precise symmetric orthogonality on a compact closed category gives for each R, S a relation $\perp$ between maps $f\colon R \longrightarrow S$ and $g\colon S \longrightarrow R$. We set

$$\begin{array}{lll} f \perp g & \text{if and only if} & (\check{f}\colon \mathbf{I} \longrightarrow R^* \otimes S) \perp_{R^* \otimes S} (\hat{g}\colon R^* \otimes S \longrightarrow \perp) \\ & \text{if and only if} & \sigma \cdot (f \otimes g)\colon R \otimes S \longrightarrow R \otimes S \in F_{R \otimes S}. \end{array}$$

(4) **Orthogonalities on traced monoidal categories.** Let $\mathbf{D}$ be a traced symmetric monoidal category and $\mathbf{C}$ the compact closed category it generates. (The situation is the basis of approaches to the Geometry of Interaction.) Objects of $\mathbf{C}$ are pairs (U, X) of $\mathbf{D}$ and maps $(U, X) \longrightarrow (V, Y)$ in $\mathbf{C}$ are maps $U \otimes Y \longrightarrow V \otimes X$ in $\mathbf{D}$. Composition is given by tensor and trace. Now maps $(\mathbf{I}, \mathbf{I}) \longrightarrow (U, X)$ and $(U, X) \longrightarrow (\mathbf{I}, \mathbf{I})$ in $\mathbf{C}$ correspond to maps $X \longrightarrow U$ and $U \longrightarrow X$ in $\mathbf{D}$. A precise symmetric orthogonality (again in the sense of Definition 10 in Section 5.3) on $\mathbf{C}$ corresponds to a precise orthogonality on the traced monoidal category $\mathbf{D}$ in the obvious sense. This is given either by a family $F_U \subseteq \mathsf{End}_{\mathbf{D}}(U)$ satisfying the conditions of example (3) above, or else by a suitable relation between maps $f\colon U \longrightarrow V$ and $g\colon V \longrightarrow U$ in $\mathbf{D}$.

We give some examples from the Geometry of Interaction, though we do not analyse the categories they give rise to in this paper.

(i) Take for $\mathbf{D}$ the free symmetric monoidal category on an object: $\mathbf{D}$ is equivalent to the direct sum of the symmetric groups and to the category of finite sets and bijections. For ϕ, ψ in S_n, the cyclic group on n symbols, *Girard's orthogonality* is

$$\phi \perp \psi \qquad \text{if and only if} \qquad \phi \cdot \psi \text{ is an } n\text{-cycle.}$$

(ii) Take for $\mathbf{D}$ the category whose objects are finite sets with maps $A \longrightarrow B$ being partitions of $A + B$. ($\mathbf{D}$ is equivalent to the free symmetric monoidal category generated by a relational Frobenius object (see [36]). $\mathbf{D}$ is in fact already compact closed. For maps $p\colon A \longrightarrow B$ and $q\colon B \longrightarrow A$ in $\mathbf{D}$, that is partitions p and q of $A + B$, the *Danos-Regnier orthogonality* [22] is

$$p \perp q \text{ if and only if the graph induced by } p \text{ and } q \text{ is connected and acyclic.}$$

(iii) Let $\mathbf{D}$ be the traced monoidal category of finite sets and relations with $+$ as tensor product. Given relations $u\colon A \longrightarrow B$ and $x\colon B \longrightarrow A$ we can set

$$\begin{array}{lll} u \perp x & \text{if and only if} & u \cdot x \text{ is nilpotent} \\ & \text{if and only if} & x \cdot u \text{ is nilpotent} \\ & \text{if and only if} & \begin{pmatrix} 0 & u \\ x & 0 \end{pmatrix} \text{ is nilpotent.} \end{array}$$

5.2 Slack orthogonality subcategories

The basic facts about the structure of the slack orthogonality categories are unproblematic.

Multiplicative structure. We first consider the case of intuitionistic linear logic.

Proposition 5.2 *If* $\mathbf{C}$ *is symmetric monoidal closed with an orthogonality then* $\mathbf{S}_J(\mathbf{C})$ *is also symmetric monoidal closed: it is closed under linear function space and tensor in* $\mathbf{S}_J(\mathbf{C})$ *and has the tensor unit* $(\mathbf{I}, \{\mathsf{id}_\mathbf{I}\}, \{\mathsf{id}_\mathbf{I}\}^\circ)$.

Proof. We refer to the tensor and linear function spaces for $\mathbf{G}_J(\mathbf{C})$ (see Proposition 4.10). A map $f\colon R \longrightarrow S$ is a morphism from (R, U, X) to (S, V, Y) in $\mathbf{S}_J(\mathbf{C})$ if and only if for all $u \in U$, $f \cdot u \in V \subseteq Y^\circ$ and for all $y \in Y$, $y \cdot f \in X \subseteq U^\circ$. This implies $\check{f} \perp_{R \multimap S} u \multimap y$ for all $u \in U, y \in Y$. Therefore $\check{f} \in (U \multimap Y)^\circ$. This shows closure under linear function space. Closure under tensor product works similarly: A morphism $z\colon R \otimes S \longrightarrow J$ is an element of the last component of $(R, U, X) \otimes (S, V, Y)$ if and only if for all $u \in U$, $\langle u|z\rangle_R \in Y \subseteq V^\circ$, and for all $v \in V$, $\langle v|z\rangle_S \in X \subseteq U^\circ$ which implies that $u \otimes v \perp_{R \otimes S} z$ for all $u \in U, v \in V$. It is straightforward to check that $(\mathbf{I}, \{\mathsf{id}_\mathbf{I}\}, \{\mathsf{id}_\mathbf{I}\}^\circ)$ is the tensor unit. □

The corresponding result for classical linear logic is now easy.

Proposition 5.3 *If* $\mathbf{C}$ *is* $*$*-autonomous with a symmetric orthogonality then the slack category* $\mathbf{S}(\mathbf{C})$ *is* $*$*-autonomous; it is closed under negation and tensor in* $\mathbf{G}(\mathbf{C})$ *and has the tensor unit* $(\mathbf{I}, \{\mathsf{id}_\mathbf{I}\}, \{\mathsf{id}_\mathbf{I}\}^\circ)$.

To get additive and exponential structure on $\mathbf{S}_J(\mathbf{C})$ and $\mathbf{S}(\mathbf{C})$ we need to know something about the relevant structure maps in $\mathbf{C}$.

Definition 9 *Suppose that* $R, S \in \mathbf{C}$ *and that* $U \subseteq \mathbf{C}(\mathbf{I}, R)$ *and* $Y \subseteq \mathbf{C}(S, J)$. *We say that* $f\colon R \longrightarrow S$ *is* positive *(with respect to* U *and* Y*) just when*

$$f \cdot u \perp_S y \qquad \textit{implies} \qquad u \perp_R y \cdot f \qquad \textit{for all } u \in U, y \in Y,$$

and negative *(with respect to* U *and* Y*) just when*

$$u \perp_R y \cdot f \qquad \textit{implies} \qquad f \cdot u \perp_S y \qquad \textit{for all } u \in U, y \in Y.$$

If $f\colon R \longrightarrow S$ *is positive and negative (with respect to* U *and* Y*) we say that it is* focussed *(with respect to* U *and* Y*).*

We use these properties in two cases.

(i) In case $U = \mathbf{C}(\mathbf{I}, R)$ and $Y = \mathbf{C}(S, J)$, when we say that $f\colon R \longrightarrow S$ is *positive* or *negative* outright.

(ii) In case $f\colon !R \longrightarrow S$, $U = \kappa_R[U]$, $Y = \mathbf{C}(S, J)$ when we say that f is *positive for* κ. (We shall not use negative in this context.)

Note that if $\perp$ is a focussed orthogonality then all maps are positive and negative. Indeed we have

$$f \cdot u \perp_S y \qquad \text{if and only if} \qquad y \cdot f \cdot u \in F \qquad \text{if and only if} \qquad u \perp_R y \cdot f.$$

Additive structure. There is no virtue in treating the intuitionistic and classical cases separately.

Proposition 5.4 *(i) Suppose* $\mathbf{C}$ *has finite products and the projections are positive. Then* $\mathbf{S}_J(\mathbf{C})$ *is closed under products in* $\mathbf{G}_J(\mathbf{C})$*; so in particular* $\mathbf{S}_J(\mathbf{C})$ *has finite products and* $\mathbf{S}_J(\mathbf{C}) \longrightarrow \mathbf{C}$ *preserves them strictly.*

(ii) Suppose $\mathbf{C}$ *has finite coproducts and inclusions are negative. Then* $\mathbf{S}_J(\mathbf{C})$ *is closed under coproducts in* $\mathbf{G}_J(\mathbf{C})$*; so in particular* $\mathbf{S}_J(\mathbf{C})$ *has finite coproducts and* $\mathbf{S}_J(\mathbf{C}) \longrightarrow \mathbf{C}$ *preserves them strictly.*

Proof. Recall the description of products and coproducts in Proposition 4.11. To show that $\mathbf{S}_J(\mathbf{C})$ is closed under products we have to establish that if $u \in U$, $v \in V$ and $x \in X$ then $\langle u,v\rangle \perp_{R\times S} x \cdot \pi_1$ (and symmetrically for $y \in Y$). This follows immediately from $\pi_1\langle u,v\rangle = u \perp_R x$ since π_1 is positive. The remainder of the proof is similar in style. $\Box$

Exponential structure. Again there is no difference between the intuitionistic and the classical cases. But because the unit of $\mathbf{S}_J(\mathbf{C})$ is not that of $\mathbf{G}_J(\mathbf{C})$ there is a little more to do.

Proposition 5.5 *Suppose that the structure maps* ε*,* e *and* d *are positive for* κ*. We can define an exponential comonad on* $\mathbf{S}_J(\mathbf{C})$ *by*

$$!(R,U,X) = (!R, \kappa_R[U], ?X),$$

where $?X$ *is defined as in* Proposition 4.12*, but the second clause is replaced by*

$$\{\chi \cdot e_R \mid \mathsf{id}_{\mathbf{I}} \perp_{\mathbf{I}} \chi\} \subseteq ?X.$$

Proof. For the proposed $!(R,U,X)$ to be an object in this subcategory, we have to show that all elements of $\kappa_R[U]$ are perpendicular to all elements of $?X$. We run through the inductive argument. Let $u \in U$.

- For all $x \in X$, $\kappa_R(u)$ is orthogonal to $x \cdot \varepsilon_R$ since $u = \varepsilon_R \cdot \kappa_R(u)$ is perpendicular to x and ε_R is positive for κ.
- Let $\chi\colon \mathbf{I} \longrightarrow J$ be orthogonal to $\mathsf{id}_{\mathbf{I}}$. That implies that for all $u \in U$, χ is perpendicular to $\mathsf{id}_{\mathbf{I}} = e_R \cdot \kappa_R(u)$, and since e_R is positive for κ, that implies $\kappa_R(u)$ orthogonal to $\chi \cdot e_R$.
- Finally, assume that we have $h\colon !R \otimes !R \longrightarrow J$ such that all for all $u \in U$, $\kappa_R(u)$ is perpendicular to both composites $\langle \kappa_R(u) | h\rangle$. But then $d_R \cdot \kappa_R(u) = \kappa_R(u) \otimes \kappa_R(u)$ is perpendicular to h by condition (ii). Since d_R is positive for κ, that implies $\kappa_R(u)$ orthogonal to $h \cdot d_R$, and we are done.

The functoriality of ! and the structure maps for the linear exponential comonad are dealt with just as in Proposition 4.12. (The minor change to the definition of $?X$ makes no serious difference.) $\Box$

Theorem 5.6 *(i) Assume that* $\mathbf{G}_J(\mathbf{C})$ *is obtained from a model for intuitionistic linear logic* $\mathbf{C}$ *as in* Theorem 4.13. *If* $\mathbf{C}$ *has an orthogonality such that the projection maps are positive and the structure maps* ε*,* d *and* e *are positive for* κ *then* $\mathbf{S}_J(\mathbf{C})$ *is a model for intuitionistic logic.*

(ii) Suppose $\mathbf{G}(\mathbf{C})$ *is obtained from a* $*$*-autonomous* $\mathbf{C}$ *as in* Theorem 4.19. *If the projections are positive, injections are negative, and the structure maps* ε*,* d *and* e *are positive for* κ *then the* $\mathbf{S}(\mathbf{C})$ *is a model for classical linear logic.*

Examples 5.7 (1) We just remark on $\mathbf{S}_J(\mathbf{C})$ for the trivial orthogonalities. For the full orthogonality, $\mathbf{S}_J(\mathbf{C})$ is simply $\mathbf{G}_J(\mathbf{C})$. For the empty orthogonality $\mathbf{S}_J(\mathbf{C})$ consists of objects of the form $(R, U, \emptyset)$ and $(R, \emptyset, U)$: oddly enough this does give models for linear logic.

(2) We consider the category $\mathbf{S}(\mathbf{Rel})$ for the partial orthogonality on $\mathbf{Rel}$: the objects are sets R such that if $u \in U$, $x \in X$ then $|u \cap x| \leq 1$. One can check that the conditions of Theorem 5.6 (ii) are valid so that $\mathbf{S}(\mathbf{Rel})$ is a model of classical linear logic. We can embed the category $\mathbf{QDom}$ of qualitative domains in $\mathbf{S}(\mathbf{Rel})$ as follows: Recall (see the examples on stable domain theory in Section 2) that a qualitative domain is a set R equipped with a suitable domain U of subsets of R. We map $(R, U) \in \mathbf{QDom}$ to (R, U, U°) in $\mathbf{S}(\mathbf{Rel})$. This gives rise to a full and faithful functor $\mathbf{QDom} \longrightarrow \mathbf{S}(\mathbf{Rel})$ preserving tensor. However we can consider the full subcategory of $\mathbf{S}(\mathbf{Rel})$ on objects of the form (R, U, U°). This is a model for intuitionistic linear logic and the embedding of $\mathbf{QDom}$ in it preserves multiplicative and additive structure. For an indication of the complexity of the situation for exponentials see the discussion for coherence spaces in Examples 5.15 (3).

(3) We consider the category $\mathbf{S}(\mathbf{Chu}^2)$ for the equality orthogonality (see Examples 5.1 (2)(v)) on $\mathbf{Chu}^2$. The orthogonality is focussed so the conditions of Theorem 5.6 (ii) automatically apply so that $\mathbf{S}(\mathbf{Chu}^2)$ is a model of classical linear logic. We identify $\mathbf{S}(\mathbf{Chu}^2)$ with the category of Chu logical relations (see [25]) as follows. Take a pair of objects $\alpha\colon U \times X \longrightarrow K$, $\beta\colon V \times Y \longrightarrow K$ in $\mathbf{Chu} = \mathbf{Chu}(\mathbf{Set}, \mathbf{K})$. Then $\mathbf{Chu}^2((\mathbf{I}, \mathbf{I}), (\alpha, \beta)) \cong U \times V$ and $\mathbf{Chu}^2((\alpha, \beta), (\perp, \perp)) \cong X \times Y$ and $\mathbf{Chu}^2((\mathbf{I}, \mathbf{I}), (\perp, \perp)) \cong K \times K$. Under this identification $(x \cdot u, y \cdot v) = (\alpha(u, x), \beta(v, y)) \in K \times K$. So an object of $\mathbf{S}(\mathbf{Chu}^2)$ is a pair of objects $(\alpha, \beta) \in \mathbf{Chu}^2$ equipped with relations $\sim_0 \subseteq U \times V$, $\sim_1 \subseteq X \times Y$ such that $u \sim_0 v$ and $x \sim_1 y$ implies $\alpha(u, x) = \beta(v, y)$; so it is just a Chu logical relation. The rest of the identification is routine. (Note that in Examples 4.5 (1) we already identified Chu logical relations with the result of performing the Chu construction on a simple category of logical relations. The equivalence of these two approaches can be described more generally.)

5.3 Tight orthogonality categories

It seems that in general the tight categories $\mathbf{T}_J(\mathbf{C})$ and $\mathbf{T}(\mathbf{C})$ do not have good multiplicative structure. However in naturally occurring orthogonalities the (Tensor) and (Implication) conditions from Definition 6 are in fact equivalences. In these circumstances there is something to be said at once about the tight categories.

Definition 10 *Let* $\mathbf{C}$ *be a symmetric monoidal closed category. An orthogonality* $\perp$ *is* precise *just when the following hold.*

- *(Precise tensor) Given* $u\colon \mathbf{I} \longrightarrow R$, $v\colon \mathbf{I} \longrightarrow S$ *and* $h\colon R \otimes S \longrightarrow J$ *then*

$$u \perp_R \langle v|h\rangle_S \text{ and } v \perp_S \langle u|h\rangle_R \qquad \text{if and only if} \qquad u \otimes v \perp_{R\otimes S} h.$$

- *(Precise implication) Given* $u\colon \mathbf{I} \longrightarrow R$, $y\colon S \longrightarrow J$ *and* $f\colon R \longrightarrow S$ *then*

$$u \perp_R y \cdot f \text{ and } f \cdot u \perp_S y \qquad \text{if and only if} \qquad \check{f} \perp_{R \multimap S} u \multimap y.$$

Note that if $\perp$ is a symmetric orthogonality on a $*$-autonomous category $\mathbf{C}$ then the two conditions are equivalent.

All our examples of orthogonalities are precise. Observe in particular that any focussed orthogonality is automatically precise.

For the remainder of this section assume that all orthogonalities are precise. Suppose that $A = (R, U, X)$ and $B = (S, V, Y)$ are objects of the tight category. Then $f\colon R \longrightarrow S$ is a map from A to B if and only if $\check{f} \perp_{R\multimap S} u \multimap y$ for all $u \in U$ and $y \in Y$, that is $\check{f} \in (U \multimap Y)^\circ$. We can generalize from this observation in the following way. In case $\mathbf{C}$ is symmetric monoidal closed (respectively $*$-autonomous) we define collections of multimaps (respectively polymaps) thus.

- Suppose $\mathbf{C}$ is symmetric monoidal closed; let $A_1 = (R_1, U_1, X_1), \ldots , A_n = (R_n, U_n, X_n)$ and $B = (S, V, Y)$ be objects of $\mathbf{T}_J = \mathbf{T}_J(\mathbf{C})$. Then the collection $\mathbf{T}_J(A_1, \ldots , A_n, B)$ of multimaps from $A_1, \ldots , A_n$ to B is

$$\mathbf{T}_J(A_1, \ldots , A_n) = \{f\colon R_1 \otimes \cdots \otimes R_n \longrightarrow S \mid \check{f} \in (U_1 \otimes \cdots \otimes U_n \multimap Y)^\circ\}.$$

- Suppose $\mathbf{C}$ is $*$-autonomous and that $\perp$ is symmetric. Now let $A_1 = (R_1, U_1, X_1)$, $\ldots$, $A_n = (R_n, U_n, X_n)$ and $B_1 = (S_1, V_1, Y_1)$, $\ldots$, $B_m = (S_m, V_m, Y_m)$ be in $\mathbf{T} = \mathbf{T}(\mathbf{C})$. Then the collection $\mathbf{T}(A_1, \ldots , A_n; B_1, \ldots , B_m)$ of polymaps from $A_1, \ldots , A_n$ to $B_1, \ldots , B_m$ is

$$\begin{aligned}&\mathbf{T}(A_1, \ldots , A_n; B_1, \ldots , B_m)\\ &\quad = \{f\colon R_1 \otimes \cdots \otimes R_n \longrightarrow S_1 \parr \cdots \parr S_n \mid f \in (U_1 \otimes \cdots \otimes U_n \otimes Y_1 \otimes \cdots \otimes Y_n)^\circ\}.\end{aligned}$$

  (Here we exploit the flexible meaning of $\perp$ in the symmetric case.)

Now the value of a precise orthogonality is just this: that in each case we can define a good composition using exactly the generalized composition for which we have a standard notation. Thus for example if $f \in \mathbf{T}(A_1, A_2; B, C)$ and $g \in \mathbf{T}(C, D; E_1, E_2)$ then

$$\langle f|g\rangle_C \in \mathbf{T}(A_1, A_2, D; B, E_1, E_2).$$

Obviously we have identities for this associative composition and so we get the following.

Proposition 5.8 *(i) Suppose* $\perp$ *is a precise orthogonality on a symmetric monoidal closed category* $\mathbf{C}$*. Then* $\mathbf{T}_J$ *is a multicategory.*

(ii) Suppose $\perp$ *is a precise symmetric orthogonality on a* $*$*-autonomous category* $\mathbf{C}$*. Then* $\mathbf{T}$ *is a* $*$*-polycategory.*

For more information about multicategories see [41, 42] and for polycategories see [50, 20]. $*$-polycategories are explained in [32].

A multicategory in which the multimaps are fully representable can be regarded as a symmetric monoidal closed category; and a $*$-polycategory in which the polymaps are fully representable can be regarded as a $*$-autonomous category. (Implicitly there is a choice of structure.) It follows that the issue of the multiplicative structure of $\mathbf{T}_J(\mathbf{C})$ and $\mathbf{T}(\mathbf{C})$ is one of representability. Representability requires a further condition on the orthogonality.

Definition 11 *A precise orthogonality $\perp$ is* stable *just when for all $U \subseteq \mathbf{C}(\mathbf{I}, R)$, $V \subseteq \mathbf{C}(\mathbf{I}, S)$ and $Y \subseteq \mathbf{C}(S, J)$*

- *(Stable tensor) $(U^{\circ\circ} \otimes V^{\circ\circ})^{\circ} = (U^{\circ\circ} \otimes V)^{\circ} = (U \otimes V^{\circ\circ})^{\circ}$;*
- *(Stable implication) $(U^{\circ\circ} \multimap Y^{\circ\circ})^{\circ} = (U \multimap Y^{\circ\circ})^{\circ} = (U^{\circ\circ} \multimap Y)^{\circ}$.*

This condition turns out to be too strong to capture some of the examples we have in mind. We will therefore introduce a weaker, although somewhat less intuitive notion. First of all we establish that same is indeed entailed by stability.

Lemma 5.9 *Suppose $\perp$ is a stable orthogonality. Then for closed sets $U_1 \subseteq \mathbf{C}(\mathbf{I}, R_1)$, ... , $U_n \subseteq \mathbf{C}(\mathbf{I}, R_n)$, $U \subseteq \mathbf{C}(\mathbf{I}, R)$, $Y \subseteq \mathbf{C}(S, J)$ we have*

$$\begin{aligned}
((U_1 \otimes U_2)^{\circ\circ} \otimes \cdots \otimes U_n \multimap Y)^{\circ} &= (U_1 \otimes \cdots U_n \multimap Y)^{\circ}, \\
(\{\mathsf{id}_{\mathbf{I}}\}^{\circ\circ} \otimes U_1 \otimes \cdots \otimes U_n \multimap Y)^{\circ} &= (U_1 \otimes \cdots \otimes U_n \multimap Y)^{\circ}, \\
(U_1 \otimes \cdots \otimes U_n \multimap (U \multimap Y)^{\circ\circ})^{\circ} &= (U_1 \otimes \cdots \otimes U_n \multimap (U \multimap Y))^{\circ}.
\end{aligned}$$

Definition 12 *A precise orthogonality is* self-stable *if it satisfies the conditions given in Lemma 5.9.*

A stable orthogonality is self-stable. Note that if $\perp$ is a symmetric orthogonality on a $*$-autonomous category then these two conditions are equivalent.

Multiplicative structure. We first consider the case of intuitionistic linear logic.

Proposition 5.10 *Suppose $\mathbf{C}$ is a symmetric monoidal closed category with a self-stable orthogonality. Then $\mathbf{T}_J(\mathbf{C})$ is symmetric monoidal closed. The tensor product of $A = (R, U, X)$ and $B = (S, V, Y)$ is given by*

$$A \otimes B = (R \otimes S, (U \otimes V)^{\circ\circ}, (U \otimes V)^{\circ}),$$

and the tensor unit is $(\mathbf{I}, \{\mathsf{id}_{\mathbf{I}}\}^{\circ\circ}, \{\mathsf{id}_{\mathbf{I}}\}^{\circ})$; the linear function space of A and B is

$$A \multimap B = (R \multimap S, (U \multimap Y)^{\circ}, (U \multimap Y)^{\circ\circ}).$$

Aside. It is worth noting before we begin the proof that as $\perp$ is precise, $(U \otimes V)^{\circ}$ is the final component of $A \otimes B$ in $\mathbf{G}_J(\mathbf{C})$, and $(U \multimap V)^{\circ}$ is the middle component of $A \multimap B$ in $\mathbf{G}_J(\mathbf{C})$.

Proof. By our earlier discussion it suffices to check that $A \otimes B$ and $A \multimap B$ fully represent multimaps in the obvious sense; this is essentially the content of the definition of self-stable.

We omit the details but give one key observation. Suppose $A = (R, U, X)$, $B = (S, V, Y)$ and $C = (T, W, Z)$ are objects of $\mathbf{T}_J(\mathbf{C})$. By the observation above $f \colon R \otimes S \longrightarrow T$ is a map $A \otimes B \longrightarrow C$ just when $\check{f} \in ((U \otimes V)^{\circ\circ} \multimap Z)^\circ$; as Z is closed that is equivalent to $\check{f} \in (U \otimes V \multimap Z)^\circ$; that is equivalent to $\check{f} \in (U \multimap (V \multimap Z))^\circ$ and as U is closed to $\check{f} \in (U \multimap (V \multimap Z)^{\circ\circ})^\circ$; but that is exactly the condition for $\check{f}$ to be a map $A \longrightarrow B \multimap C$. □

The corresponding result for classical linear logic is now easy.

Proposition 5.11 *Suppose* $\mathbf{C}$ *is* $*$*-autonomous with a symmetric and self-stable orthogonality. Then* $\mathbf{T}(\mathbf{C})$ *is* $*$*-autonomous and* $\mathbf{T}(\mathbf{C}) \longrightarrow \mathbf{C}$ *preserves the structure.* $\mathbf{T}(\mathbf{C})$ *is closed under negation in* $\mathbf{G}(\mathbf{C})$*; and the tensor product of* $A = (R, U, X)$ *and* $B = (S, V, Y)$ *is*

$$A \otimes B = (R \otimes S, (U \otimes V)^{\circ\circ}, (U \otimes V)^\circ),$$

and the tensor unit is $(\mathbf{I}, \{\mathsf{id}_\mathbf{I}\}^{\circ\circ}, \{\mathsf{id}_\mathbf{I}\}^\circ)$.

Additive structure. We refer the reader to the notation used for additives in glued categories. As for the slack categories there is no difference between the intuitionistic and classical cases.

Proposition 5.12 *Let* $\mathbf{C}$ *be a symmetric monoidal closed category with a stable orthogonality.*

(i) Suppose $\mathbf{C}$ *has finite products and the projections are focussed. The* $\mathbf{T}_J(\mathbf{C})$ *has finite products and* $\mathbf{T}_J(\mathbf{C}) \longrightarrow \mathbf{C}$ *preserves them. The product of* $A = (R, U, X)$ *and* $B = (S, V, Y)$ *is*

$$A \times B = (R \times S, U \times V, (U \times V)^\circ)$$

and the terminal object is $(\mathbf{1}, \mathbf{C}(\mathbf{I}, \mathbf{1}), \mathbf{C}(\mathbf{I}, \mathbf{1})^\circ)$.

(ii) Suppose $\mathbf{C}$ *has finite coproducts and the injections are focussed. Then* $\mathbf{T}_J(\mathbf{C})$ *has finite coproducts and* $\mathbf{T}_J(\mathbf{C}) \longrightarrow \mathbf{C}$ *preserves them. The coproduct of* $A = (R, U, X)$ *and* $B = (S, V, Y)$ *is*

$$A + B = (R + S, (X + Y)^\circ, X + Y),$$

and the initial object is $(\mathbf{0}, \mathbf{C}(\mathbf{0}, J)^\circ, \mathbf{C}(\mathbf{0}, J))$.

Proof. We just treat (i) since (ii) is similar. We first check that $U \times V = (X \oplus Y)^\circ$ so that $A \times B$ is indeed in $\mathbf{T}_J(\mathbf{C})$. Since the projections are positive we know by considerations for the slack category that $U \times V \subseteq (X \oplus Y)^\circ$. Now take $\psi \in (X \oplus Y)^\circ$. As for all $x \in X$, $\psi \perp_{R \times S} x \cdot \pi_1$ and π_1 is negative we get $\pi_1 \cdot \psi \perp_R x$ for all $x \in X$ and so $\pi_1 \cdot \psi \in U$. Similarly $\pi_2 \cdot \psi \in V$ so $\psi \in U \times V$.

Now take $C = (T, W, Z)$ in $\mathbf{T}_J(\mathbf{C})$. As $\perp$ is precise we have that $\langle f, g \rangle \colon T \longrightarrow R \times S$ is a map $C \longrightarrow A \times B$ in $\mathbf{T}_J(\mathbf{C})$ just if $\langle f, g \rangle \in (W \multimap (U \times V)^\circ)^\circ = (W \multimap (X \oplus Y)^{\circ\circ})^\circ$. As $\perp$ is stable this holds just if $\langle f, g \rangle \in (W \multimap (X \oplus Y))^\circ$. Again as $\perp$ is precise this is equivalent to

$$\langle f, g \rangle \cdot w \perp_{R \times S} x \cdot \pi_1, \qquad \langle f, g \rangle \cdot w \perp_{R \times S} y \cdot \pi_2, \qquad w \perp_T x \cdot f, \qquad w \perp_T y \cdot g$$

for all $w \in W$, $x \in X$ and $y \in Y$. But now as π_1 and π_2 are focussed this is equivalent to

$$f \cdot w \perp_R x, \qquad w \perp_T x \cdot f; \qquad g \cdot w \perp_S y, \qquad w \perp_T y \cdot g,$$

which is exactly the condition that $f\colon C \longrightarrow A$ and $g\colon C \longrightarrow B$ in $\mathbf{T}_J(\mathbf{C})$. □

Note that we make rather limited use of the full stability assumption in the proof.

Exponential structure. As before we treat the intuitionistic and classical cases together.

Proposition 5.13 *Let $\perp$ be a stable orthogonality in a symmetric monoidal closed category $\mathbf{C}$. Let $\mathbf{C}$ have a linear exponential comonad with linear distribution κ on $\mathbf{C}(\mathbf{I},-)$. Suppose that all structure maps ε, δ, e, d, and all maps of the form $!f$ are positive for κ. Then we can define a linear exponential comonad on $\mathbf{T}_J(\mathbf{C})$ by*

$$!(R, U, X) = (!R, (\kappa_R[U])^{\circ\circ}, (\kappa_R[U])^{\circ}).$$

Furthermore the functor $\mathbf{T}_J(\mathbf{C}) \longrightarrow \mathbf{C}$ preserves the exponential structure.

Proof. See Appendix A. □

Theorem 5.14 *Let $\perp$ be a stable orthogonality in a model for intuitionistic linear logic $\mathbf{C}$. Suppose the linear exponential comonad is equipped with a linear distribution κ on $\mathbf{C}(\mathbf{I},-)$. Suppose that all structure maps ε, δ, e, d, and all maps of the form $!f$ are positive for κ and that the projections are focussed.*

(i) Then $\mathbf{T}_J(\mathbf{C})$ is a model for intuitionistic linear logic.

(ii) If in addition $\mathbf{C}$ is a model for classical linear logic, the orthogonality is symmetric and the injections are focussed then $\mathbf{T}(\mathbf{C})$ is a model for classical linear logic.

Examples of tight orthogonality categories

Examples 5.15 (1) Let M be a commutative monoid giving the one object category $\mathbf{M}$ and let $F \subseteq \mathbf{M}(\mathbf{I},\mathbf{I})$. Consider the category $\mathbf{T}(\mathbf{M})$ for the orthogonality on $\mathbf{M}$ induced by F. By Proposition 5.11 $\mathbf{T}(\mathbf{M})$ is $*$-autonomous. Restricting maps to the fibre over I in $\mathbf{T}(\mathbf{M}) \longrightarrow \mathbf{M}$ as described in Examples 5.1 (2)(ii) gives us a poset of closed subsets of M in the sense of phase semantics [29, 30]. Then we can easily get additives and can take for example the simple exponential in [30]. Alternatively we can take the free commutative comonoid comonad (which necessarily exists in this context). Generally $\mathbf{M}$ does not have a linear exponential comonad and we cannot apply Proposition 5.13.

(2) Consider the category $\mathbf{T_t}(\mathbf{Rel})$ for the total orthogonality. One can show directly that the total orthogonality is stable. One can check further that the projections in $\mathbf{Rel}$ are focussed and that the maps referred to in Proposition 5.13 are positive. Thus the conditions of Theorem 5.14 (ii) are satisfied and $\mathbf{T_t}(\mathbf{Rel})$ is a model for classical linear logic. $\mathbf{T_t}(\mathbf{Rel})$ is almost exactly Loader's category **Tot** of totality spaces [43, 44]. (There is a slight mismatch. $\mathbf{T_t}(\mathbf{Rel})$ contains objects of the form $(R, \mathsf{P}()(R), \emptyset)$ and $(R, \emptyset, \mathsf{P}()(R))$ which do not appear in **Tot**. There is a standard technique which removes them but we will not go into it here.)

We comment on the structure given by general theory. The multiplicative structure in **Tot** corresponds exactly to that in $\mathbf{T_t}(\mathbf{Rel})$. Since it is given by universal properties, the additive structure corresponds once we have fixed the mismatch mentioned above. Similarly the exponentials on $\mathbf{T_t}(\mathbf{Rel})$ given by Proposition 5.13 induces an exponential on **Tot**. This certainly differs from the one briefly described by Loader in [44]; we have not checked that the latter is a linear exponential comonad in our sense.

(3) Consider the category $\mathbf{T_p}(\mathbf{Rel})$ for the partial orthogonality. This orthogonality is not stable, but it is self-stable. (In fact it satisfies the stability conditions for U, V, Y containing all singletons.) It follows from Proposition 5.11 that $\mathbf{T_p}(\mathbf{Rel})$ is $*$-autonomous. Moreover it is clear that $X \oplus Y$ is closed whenever X and Y are so the proof of Proposition 5.12 goes through and $\mathbf{T_p}(\mathbf{Rel})$ has the standard additive structure. However failure of stability means that we cannot apply Proposition 5.13 to get exponential structure. Still $\mathbf{T_p}(\mathbf{Rel})$ is exactly (isomorphic to) the category **Coh** of coherence spaces. Given a coherence space $(R, \frown)$, the corresponding object (R, U, X) in $\mathbf{T_p}(\mathbf{Rel})$ has U the collection of cliques and X the collection of co-cliques (independent sets) in $(R, \frown)$. The multiplicative structure in $\mathbf{T_p}(\mathbf{Rel})$ and **Coh** correspond exactly as does the additive structure (since it is given by universal properties).

Turning now to exponential structure, it is known that **Coh** cannot have an exponential preserved by the forgetful functor $\mathbf{Coh} \longrightarrow \mathbf{Rel}$. (We are grateful to Thomas Ehrhard and Laurent Regnier for a discussion of this point.) On the other hand **Coh** does have linear exponential comonads. First there is a domain-theoretic power set exponential on **Coh** (see [29]). (Its existence is surprisingly delicate from the abstract point of view.) Then there is the larger (more intensional) multiset exponential on **Coh**. See [30] for a discussion of both these. The existence of the multiset exponential can be explained on an abstract level using the linear exponential comonad as for $\mathbf{T_t}(\mathbf{Rel})$ and making do with the self-stability of the partial orthogonality, but one needs further structure on the category $\mathbf{C}$. One defines $!(R, U, X)$ to have underlying object

$$\dagger R = \bigwedge \{ M \rightarrowtail\, !R \mid \kappa_R(u) \text{ factors through } M \text{ for all } u \in U \}.$$

The proof of Proposition 5.13 can be modified to deal with this situation but we do not give the details here.

A Miscellaneous proofs

Proof of Proposition 4.12. We prove (i) and (ii) in parallel. The only difference is in the second component: where in the first case, there is nothing much to prove, the second case is somewhat more subtle.

The requirement that the forgetful functor $\mathbf{G}_J(\mathbf{C}) \longrightarrow \mathbf{C}$ preserve the structure tells us what the various constituents must be. All we need to prove is that all the structure maps are actually morphisms (and so natural transformations) in $\mathbf{G}_J(\mathbf{C})$.

For functoriality of !, we have to prove that if

$$A = (R, U, X) \xrightarrow{f} (S, V, Y) = B,$$

then $!f$ is a $\mathbf{G}_J(\mathbf{C})$-morphism $A \longrightarrow B$. But for $u \in U$,

$$!f \cdot \kappa_R(u) = \kappa_S(f \cdot u)$$

by naturality of κ, and $f \cdot u \in V$ since f is a $\mathbf{G}_J(\mathbf{C})$-morphism. Therefore, $\{\kappa_R(u) \mid u \in U\}$ is mapped to $\{\kappa_S(v) \mid v \in V\}$. That deals with the first component. In the first case (i), there is nothing to be shown for the second component. In the second, (ii), we need to check inductively that if $g\colon\, !S \longrightarrow J$ is in $?Y$ then $g\cdot !f$ is in $?X$. The two base cases follow from the simple equations. Assume that $y \in Y$, then $y \cdot \varepsilon_S \cdot !f = y \cdot f \cdot \varepsilon_R$, which is in $?X$ since $y \cdot f \in X$. If $\chi\colon \mathbf{I} \longrightarrow J$, then $\chi \cdot e_S \cdot f = \chi \cdot f \cdot e_R$, which is in $?X$. For the induction step, assume that we have

$$h\colon\, !S\otimes !S \longrightarrow J$$

such that for all $v \in V$, both composites $\langle \kappa_S(v)|h\rangle_S$ are in $?Y$, and so $h \cdot d_S \in ?Y$. We wish to show that $h \cdot d_S \cdot !f \in ?X$. We may assume inductively that $\langle \kappa_S(v)|h\rangle_S \cdot !f \in ?X$ for all $v \in V$. Now suppose $u \in U$. Then $f \cdot u \in V$, so for both composites we get

$$\langle \kappa_R(u)|h \cdot (!f\otimes !f)\rangle_R = \langle !f \cdot \kappa_R(u)|h\rangle_S \cdot !f = \langle \kappa_S(f \cdot u)|h\rangle_S \cdot !f \in ?X.$$

Hence $h \cdot d_S \cdot !f = h \cdot (!f\otimes !f) \cdot d_R$ is an element of $?X$ as required.

Much of the rest of the structure is straightforward. The counit ε for the comonad and discard e for the comonoid are morphisms by the the diagrams linking them with κ together with the base clauses of the definition of $?X$. The third clause, together with the connection between d and κ, ensures that the duplication d for the comonoid is a morphism. The argument for the comultiplication δ of the comonad is a similar induction to that for $!f$ which we presented in some detail above.

That leaves the monoidal structure. The nullary component, $m_{\mathbf{I}}$, is clearly a $\mathbf{G}_J(\mathbf{C})$-morphism by the equation linking it with κ (the last component is trivially well-behaved). It remains to prove that the binary component of the monoidal structure is well-behaved. Assume we are given (R, U, X) and (S, V, Y) in $\mathbf{G}_J(\mathbf{C})$. In case (i) we have only to check that composing with $m_{R,S}$ maps $\{\kappa_R(u) \otimes \kappa_S(v) \mid u \in U,\, v \in V\}$ to $\{\kappa_{R\otimes S}(u \otimes v) \mid u \in U,\, v \in V\}$; but this follows at once since $\kappa_{R\otimes S}(u \otimes v) = m_{R,S}(\kappa_R(u) \otimes \kappa_S(v))$. In case (ii), there is more to do: we have to establish the well-definedness of

$$\begin{tikzcd}
(!R\otimes !S, \arrow[d, "m_{R,S}"'] & \kappa_R[U] \otimes \kappa_S[V], \arrow[d, "m_{R,S}\cdot -"] & Z_!) \\
(!(R \otimes S), & \kappa_{R\otimes S}[U \otimes V], & ?Z)\,. \arrow[u, "-\cdot m_{R,S}"']
\end{tikzcd}$$

Here

$$\begin{aligned}
Z_! = \{!R\otimes !S \xrightarrow{f} J \mid & \forall \mathbf{I} \xrightarrow{u} R \text{ in } U.\ (\kappa_R(u)|f)_R\colon\, !S \longrightarrow J \in ?Y \\
& \forall \mathbf{I} \xrightarrow{v} S \text{ in } V.\ (\kappa_S(v)|f)_S\colon\, !R \longrightarrow J \in ?X\},
\end{aligned}$$

$$\begin{aligned}
Z = \{R \otimes S \xrightarrow{z} J \mid & \forall \mathbf{I} \xrightarrow{u} R \text{ in } U.\ (u|z)_R\colon S \longrightarrow J \in Y \\
& \forall \mathbf{I} \xrightarrow{v} S \text{ in } V.\ (v|z)_S\colon R \longrightarrow J \in X\}.
\end{aligned}$$

We prove by induction that $- \cdot m_{R,S}$ maps $?Z$ to $Z_!$. The base clauses are easy.

- Take $z\colon R \otimes S \longrightarrow J$ in Z. Now $z \cdot \varepsilon_{R\otimes S} \cdot m_{R,S} = z \cdot \varepsilon_R \otimes \varepsilon_S$. For $u \in U$ we have that $\langle \kappa_R(u)|z \cdot \varepsilon_R \otimes \varepsilon_S\rangle_{!R} = \langle u|z\rangle_R \cdot \varepsilon_S$ and since $\langle u|z\rangle_R \in Y$, this is in $?Y$; similarly for $v \in V$ we have $\langle \kappa_S(v)|z \cdot \varepsilon_R \otimes \varepsilon_S\rangle_{!S}$ in $?X$. This shows $z \cdot \varepsilon_{R\otimes S} \cdot m_{R,S} \in Z_!$ as required.

- Take $\chi\colon \mathbf{I} \longrightarrow J$. Now $\chi \cdot e_{R\otimes S} \cdot m_{R,S} = \chi \cdot e_R \otimes e_S$. For $u \in U$ we have $\langle \kappa_R(u) | \chi \cdot e_R \otimes e_S \rangle_{!R} = \chi \cdot e_S \in ?Y$; similarly for $v \in V$, $\langle \kappa_S(v) | \chi \cdot e_R \otimes e_S \rangle_{!S} \in ?X$. Thus $\chi \cdot e_{R\otimes S} \cdot m_{R,S} \in Z_!$, as required.

The induction clause takes a little more work.

- Suppose $h\colon !(R\otimes S) \otimes !(R \otimes S) \longrightarrow J$ is such that for all $u \in U$ and $v \in V$ each $\langle \kappa_{R\otimes S}(u \otimes v) | h \rangle_{!(R\otimes S)}$ is in in $?Z$ (so that $h \cdot d_{R\otimes S} \in ?Z$). We assume inductively that all $\langle \kappa_{R\otimes S}(u\otimes v)|h\rangle_{!(R\otimes S)} \cdot m_{R,S}$ are in $Z_!$ and wish to show that $h \cdot d_{R\otimes S} \cdot m_{R,S}$ in $Z_!$. For that it suffices to show that for $u \in U$, $\langle \kappa_R(u) | h \cdot d_{R\otimes S} \cdot m_{R,S} \rangle_{!R}$ is in $?Y$ (and $\langle \kappa_S(v) | h \cdot d_{R\otimes S} \cdot m_{R,S} \rangle_{!S} \in ?X$ for $v \in V$, but that will follow similarly). Now $\langle \kappa_R(u) | h \cdot d_{R\otimes S} \cdot m_{R,S}\rangle_{!R} = h \cdot (\kappa_R(u) \otimes \mathsf{id}_{!S} \otimes \kappa_R(u) \otimes \mathsf{id}_{!S}) \cdot d_{R\otimes S}$, so it suffices to show for $v \in V$ that each $\langle \kappa_S(v) | h \cdot (\kappa_R(u) \otimes \mathsf{id}_{!S} \otimes \kappa_R(u) \otimes \mathsf{id}_{!S})\rangle_{!S}$ is in $?Y$. But $\langle \kappa_S(v) | h \cdot (\kappa_R(u) \otimes \mathsf{id}_{!S} \otimes \kappa_R(u) \otimes \mathsf{id}_{!S})\rangle_{!S} = \langle \kappa_R(u) | \langle \kappa_{R\otimes S}(u\otimes v)|h\rangle_{!(R\otimes S)} \cdot m_{R,S}\rangle_{!R}$ and $\langle \kappa_{R\otimes S}(u \otimes V)|h\rangle_{!(R\otimes S)} \cdot m_{R,S}$ is in $Z_!$, so by the definition of $Z_!$ we are done. □

Proof of Proposition 5.13. Apart from m the structure maps are all of the form $!R \longrightarrow S$ for some R and S. For $f\colon !R \longrightarrow S$ to be a morphism $!A \longrightarrow B$ in $\mathbf{T}_J(\mathbf{C})$ we need to show that $\check{f} \in (\kappa_R[U]^{\circ\circ} \multimap Y)^\circ = (\kappa_R[U] \multimap Y)^\circ$. As $\perp$ is precise this is equivalent to $f \cdot \kappa_R(u) \perp_S y$ and $\kappa_R[u] \perp_{!R} y \cdot f$ for all $u \in U$, $y \in Y$. For all structure maps the first follows as κ is a linear distributivity; the second then is a consequence by positivity of the structure map. The case of $m_{\mathbf{I}}$ is easy. For $m_{R,S}$ we have to show that

$$\begin{array}{ccc}
(!R\otimes !S, & (\kappa_R[U]^{\circ\circ} \otimes \kappa_S[V]^{\circ\circ})^{\circ\circ}, & (\kappa_R[U]^{\circ\circ} \otimes \kappa_S[V]^{\circ\circ})^{\circ}) \\
\Big\downarrow{\scriptstyle m_{R,S}} & \Big\downarrow{\scriptstyle m_{R,S}\cdot -} & \Big\uparrow{\scriptstyle -\cdot m_{R,S}} \\
(!(R\otimes S), & (\kappa_{R\otimes S}[(U\otimes V)^{\circ\circ}])^{\circ\circ}, & (\kappa_{R\otimes S}[(U\otimes V)^{\circ\circ}])^{\circ})
\end{array}$$

is a map, that is we have to show that

$$\begin{aligned}
\check{m}_{R,S} &\in ((\kappa_R[U]^{\circ\circ} \otimes \kappa_S[V]^{\circ\circ})^{\circ\circ} \multimap \kappa_{R\otimes S}[(U\otimes V)^{\circ\circ}]^\circ)^\circ \\
&= (\kappa_R[U] \otimes \kappa_S[V]^{\circ\circ} \multimap \kappa_{R\otimes S}[(U\otimes V)^{\circ\circ}]^\circ)^\circ \\
&\cong (\kappa_R[U] \multimap (\kappa_S[V]^{\circ\circ} \multimap \kappa_{R\otimes S}[(U\otimes V)^{\circ\circ}]^\circ))^\circ.
\end{aligned}$$

This is equivalent to $\langle m_{R,S} | \kappa_R(u)\rangle_{!S\multimap !(R\otimes S)} \perp_{!(R\otimes S)} \vartheta$ and $\kappa_R(u) \perp_{!R} \langle \vartheta | m_{R,S}\rangle_{!S}$ for all $u \in U$, $\vartheta \in \kappa_S[V]^{\circ\circ} \multimap \kappa_{R\otimes S}[(U\otimes V)^{\circ\circ}]^\circ$ (where we take some liberty in using the notation $m_{R,S}$ here). The second now follows from the first by positivity of m. Hence it is enough to show that $\langle m_{R,S}|\kappa_R(u)\rangle_{!R} \in (\kappa_S[V]^{\circ\circ} \multimap \kappa_{R\otimes S}[(U\otimes V)^{\circ\circ}]^\circ)^\circ = (\kappa_S[V] \multimap \kappa_{R\otimes S}[(U\otimes V)^{\circ\circ}]^\circ)^\circ$. This is equivalent to

$$\begin{aligned}
& m_{R,S} \cdot (\kappa_R(u) \otimes \kappa_S(v)) = \langle \kappa_S(v) | \langle m_{R,S}|\kappa_R(u)\rangle_{!R}\rangle_{!S} \perp_{!R\otimes !S} \vartheta \\
\text{and}\quad & \kappa_S(v) \perp_{!S} \langle \langle m_{R,S}|\kappa_R(u)\rangle_{!R} | \vartheta\rangle_{!S}
\end{aligned}$$

for all $v \in V$, $\vartheta \in \kappa_{R\otimes S}[(U\otimes V)^{\circ\circ}]^\circ$. Again the second follows by positivity of m, and the first is true since $m_{R,S} \cdot (\kappa_R(u) \otimes \kappa_S(v)) = \kappa_{R\otimes S}(u\otimes v) \in \kappa[U\otimes V]$ and we are done. □

References

[1] S. Abramsky. Process realizability. Available at the following url: http://web.comlab.ox.ac.uk/oucl/work/samson.abramsky/pr209.ps.gz.

[2] S. Abramsky. Semantics of interaction: an introduction to game semantics. In Pitts and Dybjer [47], pages 1–31.

[3] S. Abramsky and R. Jagadeesan. Games and full completeness for multiplicative linear logic. *J. Symbolic Logic*, 59:543–574, 1994.

[4] S. Abramsky, R. Jagadeesan, and P. Malacaria. Games and full abstraction for PCF. *Information and Computation*. To appear.

[5] S. Abramsky, R. Jagadeesan, and P. Malacaria. Full abstraction for PCF (extended abstract). In *Theoretical Aspects of Computer Software: TACS '94*, volume 789 of *Lecture Notes in Computer Science*, pages 1–15, 1994.

[6] M. Barr. *$*$-Autonomous categories*, volume 752 of *Lecture Notes in Mathematics*. Springer-Verlag, 1979.

[7] M. Barr. Accessible categories and models of linear logic. *Journal of Pure and Applied Algebra*, 69, 1990.

[8] M. Barr. $*$-Autonomous categories and linear logic. *Mathematical Structures in Computer Science*, 1(2):159–178, July 1991.

[9] J.M. Beck. *Triples, Algebras, and Cohomology*. PhD thesis, Columbia University, 1967.

[10] G. Bellin. Chu's Construction: A Proof-theoretic Approach. Submitted for publication.

[11] N. Benton, G. Bierman, V. de Paiva, and M. Hyland. Linear lambda-calculus and categorical models revisited. In *Proceedings of the Sixth Workshop on Computer Science Logic*, volume 702 of *Lecture Notes in Computer Science*, pages 61–84. Springer-Verlag, 1993.

[12] N. Benton, G. Bierman, V. de Paiva, and M. Hyland. A term calculus for intuitionistic linear logic. In M. Bezem and J.F. Groote, editors, *Proceedings of the International Conference on Typed Lambda Calculi and Applications, TLCA'93*, volume 664 of *Lecture Notes in Computer Science*, pages 75–90. Springer-Verlag, 1993.

[13] G. Berry. Stable models of typed λ-calculi. In *Proc. of the 5th ICALP*, volume 62 of *Lecture Notes in Computer Science*, pages 72–89. Springer-Verlag, 1978.

[14] G. Berry. *Modèles Complèment Adéquats et Stable des Lambda-calculs typès*. PhD thesis, Université Paris VII, 1979.

[15] G.M. Bierman. On intuitionistic linear logic. Technical Report 346, University of Cambridge Computer Laboratory, August 1994.

[16] G. M. Bierman. What is a categorical model of intuitionistic linear logic? In *Proceedings of the Second International Conference on Typed Lambda Calculus*, volume 902 of *Lecture Notes in Computer Science*, pages 73–93, 1995.

[17] R. Blackwell, G.M. Kelly, and J. Power. Two-dimensional monad theory. *Journal of Pure and Applied Algebra*, 59:1–41, 1989.

[18] P.-H. Chu. $*$-*Autonomous categories*, chapter Constructing $*$-autonomous categories. Volume 752 of *Lecture Notes in Mathematics* [6], 1979. Appendix.

[19] J.R.B. Cockett and R.A.G. Seely. Proof theory for full intuitionistic logic, bilinear logic and mixcategories. *Theory and Application of Categories*, 3:85–131, 1997.

[20] J.R.B. Cockett and R.A.G. Seely. Weakly distributive categories. *Journal of Pure and Applied Algebra*, 114:133–173, 1997.

[21] P.-L. Curien. *Categorical Combinators, Sequential Algorithms and Functional Programming*. Birkhäuser, 2nd edn. edition, 1993.

[22] V. Danos and L. Regnier. The structure of the multiplicatives. *Arch. Math. Logic*, 28:181–203, 1989.

[23] V.C.V. de Paiva. A dialectica-like model of linear logic. In D.H. Pitt, D.E. Rydeheard, P. Dybjer, A.M. Pitts, and A. Poigné, editors, *Category Theory and Computer Science*, volume 389 of *Lecture Notes in Computer Science*, pages 341–356. Springer-Verlag, 1989.

[24] V.C.V. de Paiva. The dialectica categories. Technical Report 213, University of Cambridge Computer Laboratory, January 1991.

[25] H. Devarajan, D. Hughes, G. Plotkin, and V. Pratt. Full completeness of the multiplicative linear logic of chu spaces. In *Proc. Logic in Computer Science 1999*. IEEE Computer Society Press, 1999.

[26] J. Diller. Functional interpretation of Heyting's arithmetic in all finite types. *Nieuw Archief voor Wiskunde*, 27:70–97, 1979.

[27] J. Diller and W. Nahm. Eine Variante zur Dialectica-Interpretation der Heyting-Arithmetik endlicher Typen. *Archiv fuer Mathematische Logik und Grundlagenforschung*, 16:49–66, 1974.

[28] J.-Y. Girard. The system F of variable types fifteen years later. *Theoretical Computer Science*, 45:159–192, 1986.

[29] J.-Y. Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

[30] J.-Y. Girard. Linear logic: its syntax and semantics. In Girard et al. [31].

[31] J.-Y. Girard, Y. Lafont, and L. Regnier, editors. *Advances in Linear Logic*, volume 222 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1995.

[32] J.M.E. Hyland. Proof theory in the abstract. Submitted for publication.

[33] J.M.E. Hyland and C.-H.L. Ong. On full abstraction for PCF: I, II and III. *Information and Computation*. To appear.

[34] J.M.E. Hyland and A. Schalk. Linear exponential comonads for self-dualized categories. Unpublished manuscript, 2000.

[35] M. Hyland. Game semantics. In Pitts and Dybjer [47], pages 131–194.

[36] M. Hyland and J. Power. The structure of categories of wirings. Submitted for publication.

[37] M. Hyland and A. Schalk. Abstract games for linear logic. extended abstract. In Martin Hofmann, Giuseppe Rosolini, and Dusko Pavlovic, editors, *Proceedings of CTCS '99, Conference on Category Theory and Computer Science*, volume 29 of *Electronic Notes on Theoretical Computer Science*, 1999.

[38] A. Joyal and M. Tierney. An extension of the Galois theory of Grothendieck. *Memoirs of the A.M.S.*, 309, 1984.

[39] G.M. Kelly and M.L. Laplaza. Coherence for compact closed categories. *Journal of Pure and Applied Algebra*, 19:193–213, 1980.

[40] G.M. Kelly and J. Power. Adjunctions whose counits are coequalizers, and presentations of finitary enriched monads. *Journal of Pure and Applied Algebra*, 89:163–179, 1993.

[41] J. Lambek. Multicategories revisited. In J. Gray and A. Scedrov, editors, *Categories in Computer Science and Logic*, volume 92 of *Contemporary Mathematics*, pages 217–239. American Mathematical Society, 1989.

[42] J. Lambek. Bilinear logic. In Girard et al. [31], pages 43–59.

[43] R. Loader. Linear logic, totality, and full completeness. In *Proc. Logic in Computer Science 1994*. IEEE Computer Science Press, 1994.

[44] R. Loader. *Models of lambda calculi and linear logic: Structural, equational and proof-theoretic characterisations.* PhD thesis, St. Hugh's College, Oxford, Michaelmas 1994.

[45] S. MacLane. *Categories for the Working Mathematician.* Springer-Verlag, 1971.

[46] H. Nickau. Hereditarily sequential functionals. In *Proceedings of the Symposium of Logical Foundations of Computer Science*, volume 813 of *Lecture Notes in Computer Science*, pages 253–264. Springer-Verlag, 1994.

[47] A.M. Pitts and P. Dybjer, editors. *Semantics and Logics of Computation.* Cambridge University Press, 1997.

[48] R. Street. The formal theory of monads. *Journal of Pure and Applied Algebra*, 2:149–168, 1972.

[49] M.E. Sweedler. *Hopf Algebras.* W.A. Benjamin, 1969.

[50] M.E. Szabo. Polycategories. *Comm. Alg.*, 3:663–689, 1975.

[51] A.M. Tan. *Full completeness for models of linear logic.* PhD thesis, University of Cambridge, 1997.