EXAMPLE SHEET 2: LINEAR ANALYSIS

(1) Let \( p \in [1, \infty) \) and \( x \in \ell_p \). Show that there is an \( f \in \ell_p^* \) such that \( f(x) = \|x\|_{\ell_p} \) and \( \|f\|_{\ell_p^*} \leq 1 \).

(2) Let \( X \) be a normed vector space over \( \mathbb{F} \) and \( f : X \to \mathbb{F} \) be a linear map. Prove that \( f \) is continuous if and only if \( ker(f) \) is closed.

(3) Show that \( \hat{L}_p([0,1]) := \{ f : [0,1] \to \mathbb{R} \text{ continuous} \} \) with the norm given by
\[
\|f\|_{\hat{L}_p([0,1])} := \left( \int_0^1 |f|^p dx \right)^{\frac{1}{p}}
\]
is not complete.

(4) Let \( V \) be the space of polynomials on \( \mathbb{R} \). Does there exists a norm \( \| \cdot \| \) on \( V \) such that \((V, \| \cdot \|)\) is complete?

(5) Let \( \{f_n\}_{n=1}^\infty \) be a sequence of continuous functions \( f_n : [0,1] \to \mathbb{R} \). If for every \( t \in [0,1] \), \( \sup_n |f_n(t)| \) is finite, show that there is an interval \( [a,b] \) with \( a < b \) such that \( \sup_n \sup_{t \in [a,b]} |f_n(t)| < \infty \). (Osgood Theorem)

(6) Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function such that for every \( x > 0 \), we have \( f(nx) \to 0 \) as \( n \to \infty \). Show that \( f(x) \to 0 \) as \( x \to \infty \).

(7) Show that the set of all rational numbers \( \mathbb{Q} \) is not a \( G_\delta \) set, i.e., it is not a countable intersection of open subsets of \( \mathbb{R} \).

(8) Does there exist a function \( f : [0,1] \to \mathbb{R} \) which is continuous at every rational number and discontinuous at every irrational number? (Hint: You may find the previous problem useful.)

(9) In this problem, we study the Fourier series and its convergence. Define the operator \( \hat{c}_0 : C(S^1) \to \hat{c}_0 \) so that for every \( k \in \mathbb{Z} \),
\[
\hat{f}(k) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt.
\]
Define also \( S_n : C(S^1) \to C(S^1) \) so that \( S_n(f) \) is the \( n \)-th partial sum of the Fourier series of \( f \) given by the following formula:
\[
S_n(f)(x) = \sum_{k=-n}^{n} \hat{f}(k) e^{ikx}.
\]
Here \( \hat{c}_0 \) is defined\(^1\) as \( \hat{c}_0 := \{ g : \mathbb{Z} \to \mathbb{C} : |g(n)| \to 0 \text{ as } n \to \pm \infty \} \) endowed with the sup norm, i.e.,
\[
\|g\| := \sup_{n \in \mathbb{Z}} |\hat{g}(n)|.
\]
A basic question that we investigate here is whether \( S_n(f) \) converges as \( n \to \infty \).

\(^1\)Notice that this is slightly different from the usual \( c_0 \)!
(a) Show that the image of \( \hat{\cdot} \) indeed lies in \( \tilde{c}_0 \). In other words, show that for every \( f \in C(S^1) \), we have \(|\hat{f}(k)| \to 0\). (You can use that continuously differentiable functions are dense in \( C(S^1) \). This will be proven later in the course.)

(b) Prove the following formula:

\[
S_n(f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)D_n(x - t)dt,
\]

where \( D_n \) is defined by

\[
D_n(t) := \sum_{k=-n}^{n} e^{ikt} = \frac{\sin((n + \frac{1}{2})t)}{\sin(\frac{t}{2})}.
\]

(c) Let \( \phi_n \in C(S^1)^* \) be defined as \( \phi_n(f) := S_n(f)(0) \). Show that for every \( n \), \( \|\phi_n\|_{C(S^1)^*} < \infty \), but \( \sup_n \|\phi_n\|_{C(S^1)^*} = \infty \).

(d) Deduce that there exists a function \( f \in C(S^1) \) whose Fourier series diverges at 0, i.e., \( S_n(f)(0) \) does not have a finite limit as \( n \to \infty \).

(10) This problem, which continues the discussions on Fourier series, is intended for students who have learnt measure theory.

(a) Show that we can in fact define \( \hat{\cdot} : L^1(S^1) \to \tilde{c}_0 \), i.e., for every \( f \in L^1(S^1) \), we have \(|\hat{f}(k)| \to 0\) as \( k \to \pm \infty \). (Here, \( L^1(S^1) \) is defined to be the space of (equivalent classes\(^2\) of) Lebesgue measurable functions with \( \int_{-\pi}^{\pi} |f(t)|dt < \infty \), with the norm \( \|f\|_{L^1(S^1)} := \int_{-\pi}^{\pi} |f|(t)dt \). This makes \( L^1(S^1) \) into a Banach space.)

(b) Show that the map \( \hat{\cdot} \) is bounded and injective.

(c) On the other hand, prove that \( \hat{\cdot} \) is not surjective. (Hint: Prove that \( D_n \) defined in the previous problem has the property that \( \|D_n\|_{L^1} \to \infty \) as \( n \to \infty \) but \( \|\hat{D_n}\|_{\tilde{c}_0} = 1 \).)

(11) Let \( f : [0, 1] \to \mathbb{R} \) be a pointwise limit of a sequence of continuous functions. Show that \( f \) has a point of continuity.

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\(^2\)where two functions are equivalent if they agree except on a measure 0 set.