EXAMPLE SHEET 3: NONLINEAR WAVE EQUATIONS

In this example sheet, $\Box = -\partial_t^2 + \Delta$.

(1) (Conformal energy) Let $\Box\phi = 0$ in $\mathbb{R} \times \mathbb{R}^3$ for some sufficiently regular ϕ.

(a) Show that

$$CE[\phi] := \int_{\mathbb{R}^3} \left(\frac{1}{2}(|x|^2 + t^2)((\partial_t\phi)^2 + \sum_{i=1}^{3}(\partial_i\phi)^2) + 2tr\partial_r\phi\partial_t\phi + 2t\phi\partial_t\phi - \phi^2 \right) dx$$

is independent of time. This is called the **conformal energy**.

(b) Show that this integral is in fact positive. (Hint: Add the integrand to $\frac{1}{2}\partial_i(\frac{x_i\phi^2(|x|^2+t^2)}{|x|^2})$ and complete the square.)

(2) (Integrated local energy decay *) In this problem, ϕ is a sufficiently smooth solution to $\Box\phi = 0$ with compactly supported initial data.

(a) Show that for some constant $C > 0$ independent of T, we have

$$\left| \int_0^T \int_{\mathbb{R}^3} \frac{-(\partial_t\phi)^2 + (\partial_r\phi)^2}{r} \, dx \, dt \right| \le C\|(\phi_0, \phi_1)\|_{\dot{H}^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)}.$$

(Hint: Start with integrating $\partial_r\phi\Box\phi$ and integrate by parts.)

(b) Using the previous part, show that

$$\left| \int_0^T \int_{\mathbb{R}^3} \frac{|\nabla\!\!\!/\phi|^2}{r} \, dx \, dt - \int_0^T \int_{\mathbb{R}^3} \frac{1}{2r}\Box\phi^2 \, dx \, dt \right| \le C\|(\phi_0, \phi_1)\|_{\dot{H}^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)}.$$

(c) Show that

$$\int_0^\infty \int_{\mathbb{R}^3} \frac{|\nabla\!\!\!/\phi|^2}{r} \, dx \, dt + \int_0^\infty |\phi|^2(0,t) dt \le C\|(\phi_0, \phi_1)\|_{\dot{H}^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)}.$$

(Hint: Recall the Hardy inequality in the first example sheet.)

(d) The previous estimate only controls the angular derivatives. We will derive a stronger estimate. First, show that

$$\begin{aligned} &|\int_0^\infty \int_{\mathbb{R}^3} \frac{f'(r)}{2}(\partial_r\phi)^2 + \frac{f(r)|\nabla\!\!\!/\phi|^2}{2r} - \frac{1}{8}\Delta(f'(r) + \frac{2f(r)}{r})|\phi|^2) \, dx \, dt| \\ \le &C\|(\phi_0, \phi_1)\|_{\dot{H}^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)} \end{aligned}$$

by integrating $f(r)\partial_r\phi\Box\phi$ where $f(r)$ is bounded and $f'(r) \le \frac{C}{1+r^2}$.

(e) By choosing $f(r) = \frac{\epsilon}{\epsilon+r}$, show that

$$|\int_0^\infty \int_{\mathbb{R}^3} \frac{\epsilon}{(\epsilon + r)^2}(\partial_r\phi)^2 \, dx \, dt| \le C\|(\phi_0, \phi_1)\|_{\dot{H}^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)},$$

where the constant is independent of ϵ. By choosing $\epsilon = 2^k$, show that

$$\int_0^\infty \int_{r\le 1} (\partial_r\phi)^2 + \int_0^\infty \int_{2^{k-1}\le r\le 2^k} \frac{(\partial_r\phi)^2}{r} \le C\|(\phi_0, \phi_1)\|_{\dot{H}^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)},$$

where the constant C is independent of k.

(f) Conclude that in fact for every $\delta > 0$, there exists $C_\delta > 0$ depending on δ such that

$$\int_0^\infty \int_{\mathbb{R}^3} (\frac{(\partial_r\phi)^2}{(1 + r)^{1+\delta}} + \frac{|\nabla\!\!\!/\phi|^2}{r}) \, dx \, dt \le C_\delta\|(\phi_0, \phi_1)\|_{\dot{H}^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)}.$$

(g) Is the estimate still true if $\delta = 0$?

(3) Prove the following corollary in the lecture notes: Let F be a smooth function supported in $B(0, t+1)$ for every $t \geq 0$ such that

$$\sum_{|\alpha| \leq 10} (1+T)^{-\frac{1}{10}} \int_0^T \|\Gamma^\alpha F\|_{L^2(\mathbb{R}^3)}(t) dt + \sup_{t,x} (1+t)^2 \log^2(2+t) |F(t,x)| \leq C'.$$

Suppose ϕ is a solution to $\Box\phi = F$ with compactly supported initial data, then

$$(1+t)\|\partial\phi\|_{L^\infty(\mathbb{R}^3)}(t) \leq C,$$

for some $C > 0$ depending on initial data and C'.

(4) Consider the system of equations

$$\begin{cases} \Box\phi = 0 \\ \Box\psi = (\partial_t \phi)^2. \end{cases}$$

Show that there exist smooth and compactly supported initial data such that the following bound does *not* hold for *any* constant $C > 0$:

$$\sup_x |\partial_t \psi|(t,x) \leq \frac{C}{1+t}.$$

(5) (Decay for linear Klein-Gordon equations *) We have seen in the first example sheet that solutions to the Klein-Gordon equation decays as $\frac{C}{(1+t)^{\frac{n}{2}}}$ using the Fourier transform. This can in fact also be shown by combining energy methods and ODE techniques. We look at the $n = 1$ case here. The equation can be written as

$$-\partial_t^2 \phi + \partial_x^2 \phi - \phi = 0$$

and suppose we have smooth initial data supported in $B(0, 1)$.

(a) Show that the H^1 norm of the solution is bounded in time. Conclude that in fact $\|\phi\|_{L^\infty(\mathbb{R})}$ is bounded uniformly in time.

(b) Define ρ and y by $x = \rho \sinh y$ and $t = \rho \cosh y$. Show that

$$\partial_\rho = \sinh(y)\partial_x + \cosh(y)\partial_t, \quad \partial_y = \rho\cosh(y)\partial_x + \rho\sinh(y)\partial_t = t\partial_x + x\partial_t.$$

By considering the equation satisfied by $\partial_y^k \phi$, conclude that $\|\partial_y^k \phi\|_{L^\infty(\mathbb{R})}$ is bounded uniformly in time.

(c) Rewrite the equation in the (ρ, y) coordinate. Show that the equation is equivalent to

$$-\rho^{-\frac{1}{2}} \partial_\rho^2 (\rho^{\frac{1}{2}} \phi) - \frac{1}{4\rho^2}\phi - \phi + \frac{1}{\rho^2}\partial_y^2 \phi = 0.$$

(d) For $\psi := \rho^{\frac{1}{2}} \phi$, show that

$$\partial_\rho^2 \psi + \psi = O(\frac{1}{\rho^{\frac{3}{2}}}).$$

Conclude that in fact $|\phi| \leq \frac{C}{(1+t)^{\frac{1}{2}}}$. (Hint: use finite speed of propagation.)

(6) (Schwarzschild solution) (This problem requires some background in differential geometry. It is not really a problem on nonlinear wave equations, but it derives certain results that were used in the proof of the stability of Minkowski space.) In lecture, we defined the Schwarzschild solution in local coordinates by

$$g = -\frac{r-2M}{r+2M} dt^2 + \frac{r+2M}{r-2M} dr^2 + (r+2M)^2 (d\theta^2 + \sin^2\theta d\phi^2).$$

(a) Fix $R \gg M$. Show that the curves given by $t = \pm(r - R + 4M \log \frac{r-2M}{R-2M})$ with fixed θ and ϕ are null geodesics, i.e., they are geodesics and the tangent vectors $\dot\gamma$ satisfy $g(\dot\gamma, \dot\gamma) = 0$ at every point along the curves.

(b) Show that this is isometric to the more familiar form

$$g = -(1 - \frac{2M'}{r}) dt^2 + (1 - \frac{2M'}{r})^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2).$$

(c) * Show that it is indeed a solution to the vacuum Einstein equations.

(7) As always, let me know about any mistakes in this example sheet!