EXAMPLE SHEET 2: NONLINEAR WAVE EQUATIONS

Throughout this example sheet, we take $\Box = -\partial_t^2 + \Delta$.
Notice that the examples with * are comparatively advanced.

(1) Consider the equation

$$\eta^{-2}\partial_t^2\phi - 2\eta^{-2}\sum_{i=1}^{3}\partial_i\phi\partial_{ti}^2\phi + \eta^{-2}\sum_{i,j=1}^{3}\partial_i\phi\partial_j\phi\partial_{ij}^2\phi - \sum_{i=1}^{3}\partial_i^2\phi = 0$$

in $I \times \mathbb{R}^3$. Given smooth and compactly initial data such that $\|(\phi_0, \phi_1)\|_{W^{1,\infty}(\mathbb{R}^3)\times L^\infty(\mathbb{R}^3} \leq \epsilon$, show that if ϵ is sufficiently small, the equation has a unique local-in-time solution.

(2) (Improved breakdown criterion) We have shown in class that for a general class of nonlinear wave equation, if the maximal time of existence $T_* < \infty$, then

$$\lim_{t\to T_*}\sum_{|\alpha|\leq\lfloor\frac{n}{2}+1\rfloor}\|\partial^\alpha\phi\|_{L^\infty(\mathbb{R}^n)} \to \infty.$$

Now, assume[1] the following functional inequality: For $s \geq 0$, we have

$$\|fg\|_{H^s(\mathbb{R}^n)} \leq C(\|f\|_{H^s(\mathbb{R}^n)}\|g\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{H^s(\mathbb{R}^n)}\|f\|_{L^\infty(\mathbb{R}^n)})$$

for some $C > 0$.

Show that for the class of equations that we considered in the lectures, in fact if $T_* < \infty$, then

$$\lim_{t\to T_*}\sum_{|\alpha|\leq 1}\|\partial^\alpha\phi\|_{L^\infty(\mathbb{R}^n)} \to \infty.$$

(3) (Local existence with a class of Lipschitz data *) Consider the semilinear problem

$$\Box\phi = T^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi,$$

where $T^{\alpha\beta}$ are constants. Given functions $\phi_{0,1}, \phi_{0,2}, \phi_{1,1}, \phi_{1,2} \in C_c^\infty$ and consider data of the form

$$(\phi, \partial_t\phi)\restriction_{\{t=0\}} = (\phi_{0,1} + (x_1)_+\phi_{0,2}, \phi_{1,1} + 1_{\{x_1\geq 0\}}\phi_{1,2}),$$

where $y_+ = y$ if $y \geq 0$ and $y_+ = 0$ otherwise. Notice that for such initial data, the first derivative of ϕ in general has a jump discontinuity across the set $\{x_1 = 0\}$.

(a) Show that in $(1+1)$-dimensions, there exists

$$T = T(\|\partial\phi_{0,1}\|_{L^\infty}, \|(x_1)_+\partial\phi_{0,2}\|_{L^\infty}, \|\phi_{0,2}\|_{L^\infty}, \|\phi_{1,1}\|_{L^\infty}, \|\phi_{1,2}\|_{L^\infty}) > 0$$

such that a unique solution $(\phi, \partial_t\phi) \in L^\infty([0,T]; W^{1,\infty}(\mathbb{R})) \times L^\infty([0,T]; L^\infty(\mathbb{R}))$ exists. (Hint: Integrate the equation directly in a manner similar to the second proof of the global regularity for $1+1$ dimensional wave map in the lecture notes.)

(b) We now consider the higher dimensional case. We use the notation the $x = (x_1, x')$, where $x' \in \mathbb{R}^{n-1}$. First, we consider the linear inhomogeneous wave equation

$$\Box\phi = F.$$

Prove the following L^2 estimate for solutions:

$$\sup_{t\in[0,T]}\left(\sum_{|\alpha|\leq k}\|\partial_{t,x_1}\partial_{x'}^\alpha\phi\|_{L^2(\mathbb{R}^n)}(t) + \sum_{|\alpha|\leq k+1}\|\partial_{x'}^\alpha\phi\|_{L^2(\mathbb{R}^n)}(t)\right)$$

$$\leq C\left(\sum_{|\alpha|\leq k}\|\partial_{t,x_1}\partial_{x'}^\alpha\phi\|_{L^2(\mathbb{R}^n)}(0) + \sum_{|\alpha|\leq k+1}\|\partial_{x'}^\alpha\phi\|_{L^2(\mathbb{R}^n)}(0) + \int_0^T\sum_{|\alpha|\leq k}\|\partial_{x'}^\alpha F\|(t)dt\right),$$

[1] Notice that this is indeed true and can be proved by Littlewood-Paley theory. See for instance Lemma A.8 in *Nonlinear dispersive equations* by Tao for a proof.

for some constant $C > 0$ depending only on dimensions and k.

(c) Prove the following functional inequality which controls the L^∞ norm :

$$\sup_{t\in[0,T]}\sum_{|\alpha|\leq k}\|\partial_{x'}^\alpha\phi\|_{L^\infty(\mathbb{R}^n)}(t)$$
$$\leq C\sup_{t\in[0,T]}\left(\sum_{|\alpha|\leq k+\frac{n}{2}+1}\|\partial_{t,x_1}\partial_{x'}^\alpha\phi\|_{L^2(\mathbb{R}^n)}(t)+\sum_{|\alpha|\leq k+\frac{n}{2}+2}\|\partial_{x'}^\alpha\phi\|_{L^2(\mathbb{R}^n)}(t)\right)$$

for some C depending on n and k. (Hint: This should be thought of as a slightly refined version of Sobolev embedding. Use a combination of the one-dimensional Sobolev embedding and the $(n-1)$-dimensional Sobolev embedding.)

(d) Then prove the following L^∞ estimate for solutions to

$$\Box\phi = F :$$

$$\sup_{t\in[0,T]}\sum_{|\alpha|\leq k-1}\|\partial_{t,x_1}\partial_{x'}^\alpha\phi\|_{L^\infty(\mathbb{R}^n)}(t)$$
$$\leq C\sum_{|\alpha|\leq k-1}\|\partial_{t,x_1}\partial_{x'}^\alpha\phi\|_{L^\infty(\mathbb{R}^n)}(0)+\int_0^T(\sum_{|\alpha|\leq k+\frac{n}{2}}\|\partial_{t,x_1}\partial_{x'}^\alpha F\|_{L^2(\mathbb{R}^n)}(t)+\sum_{|\alpha|\leq k+\frac{n}{2}+1}\|\partial_{x'}^\alpha F\|_{L^2(\mathbb{R}^n)}(t))dt$$
$$+C\int_0^T\left(\sum_{|\alpha|\leq k+\frac{n}{2}+2}\|\partial_{x_1}\partial_{x'}^\alpha\phi\|_{L^2(\mathbb{R}^n)}(t)+\sum_{|\alpha|\leq k+\frac{n}{2}+3}\|\partial_{x'}^\alpha\phi\|_{L^2(\mathbb{R}^n)}(t)\right)dt.$$

(Hint: Imitate the proof in the $(1+1)$-dimensional case and use the previous part.)

(e) Combine the estimates above and perform an iteration in an appropriate function space to conclude that the exists a unique solution in $[0,T]\times\mathbb{R}^n$ for T suitably small.

(4) (Nirenberg's example) Consider the equation

$$\Box\phi = (\partial_t\phi)^2 - \sum_{i=1}^3(\partial_i\phi)^2$$

in $I\times\mathbb{R}^3$. Show that there exists smooth and compactly supported initial data such that the solution blows up in finite time. (Hint: Consider the change of variable $\psi = e^\phi - 1$. When is this change of variables invertible?) On the other hand, given smooth and compactly supported functions (f,g), show that there exists ϵ_0 (depending on f and g) such that if the initial data satisfy

$$(\phi,\partial_t\phi)\restriction_{\{t=0\}} = (\epsilon f, \epsilon g)$$

then the solution is global in time whenever $\epsilon\leq\epsilon_0$.

(5) As a consequence of the general local existence theorem, we know that the equation (in $I\times\mathbb{R}^3$)

$$\Box\phi = (\partial_t\phi)^2 - \sum_{i=1}^3(\partial_i\phi)^2$$

has a local solution in $[0,T]$ when the initial data $(\phi,\partial_t\phi)\restriction_{\{t=0\}}\in H^5(\mathbb{R}^3)\times H^4(\mathbb{R}^3)$. Moreover, the time of existence T depends only of the $H^5(\mathbb{R}^3)\times H^4(\mathbb{R}^3)$ norm of $(\phi,\partial_t\phi)\restriction_{\{t=0\}}$.

(a) Show, using the result in the previous problem, that the same result does not hold if we only assume that the initial data satisfy $(\phi,\partial_t\phi)\restriction_{\{t=0\}}\in H^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$.

(b) Suppose that the data are in $H^s(\mathbb{R}^3)\times H^{s-1}(\mathbb{R}^3)$. Find the smallest s_0 such that if $s > s_0$, then there exists a unique local solution $(\phi,\partial_t\phi)\in L^\infty([0,T];H^s(\mathbb{R}^3))\times L^\infty([0,T];H^{s-1}(\mathbb{R}^3))$ for some $T>0$ depending only on the $H^s(\mathbb{R}^3)\times H^{s-1}(\mathbb{R}^3)$ norm of the data.

(6) Consider the equation

$$\Box\phi = |\phi|^2\phi$$

in $\mathbb{R}\times\mathbb{R}^3$. We have shown in class that finite energy smooth data give rise to global-in-time finite energy smooth solutions. Show that smooth initial data, i.e., $(\phi,\partial_t\phi)\restriction_{\{t=0\}} = (\phi_0,\phi_1)\in C^\infty\times C^\infty$ (without assumption on the finiteness of the initial energy!) give rise to global smooth solutions.

(7) (Constraint equations) This problem requires some familiarity with differential geometry. In the lecture, we showed that given initial data set $(\mathbb{R}^n, \hat{g}_{ij}, \hat{k}_{ij})$ for the Einstein vacuum equations, the tensors Ric_{0i} and $(Ric - \frac{1}{2}gR)_{00}$ are determined by the data. Show that in fact the vanishing of these curvature components is equivalent to

$$\hat{\nabla}_i \hat{k}^i{}_j - \hat{\nabla}_j \hat{k}^i{}_i = 0,$$
$$\hat{R}(\hat{g}) + (\hat{k}^i{}_i)^2 - \hat{k}^i{}_j \hat{k}^j{}_i = 0,$$

where $\hat{\nabla}$ is the Levi-Civita connection associated to $\hat{g}$ and $\hat{R}(\hat{g})$ is the scalar curvature associated to $\hat{g}$.

(8) (Maxwell's equations via electromagnetic potnetial) When studying the Einstein vacuum equations, we show that the metric only satisfies a system of nonlinear wave equations after choosing an appropriate gauge (coordinates). Moreover, the gauge condition is propagated by the equation. In this problem, we see a similar idea at work in a linear setting - for the Maxwell's equations. Let $A : I \times \mathbb{R}^3 \to \mathbb{R}^{3+1}$. We write the components of A as (A_0, A_1, A_2, A_3). We say A is a electromagnetic potential if

$$E_i = \partial_i A_0 - \partial_t A_i, \quad B_i = \sum_{j,k=1}^{3} \in_{ijk} \partial_j A_k.$$

(a) Suppose E and B satisfy the Maxwell's equations, show that A satisfies the following second order equation:
$$\Box A_\mu - m^{\nu\sigma} \partial^2_{\mu\nu} A_\sigma = 0.$$
(In particular, the components of A do not satisfy wave equations.)

(b) Let $\tilde{A}_\mu = A_\mu + \partial_\mu \phi$. Show that the electomagnetic fields $\tilde{E}$ and $\tilde{B}$ associated to $\tilde{A}$ coincide with E and B.

(c) Show that if A satisfies the Coulomb gauge condition $\sum_{i=1}^{3} \partial_i A_i = 0$, then A_0 satisfies an elliptic equation and A_i satisfy wave equations for $i = 1, 2, 3$ with derivatives of A_0 as a source.
$$0 = \Box A_i - m^{\nu\sigma} \partial^2_{i\nu} A_\sigma = \Box A_i + \partial^2_{it} A_0$$

(d) Show that if the Coulomb gauge condition is satisfied initially, then it is propagated by the evolution equations $\Box A_i = 0$.

(e) Prove using the above considerations that given sufficiently regular initial data (E^0, B^0) such that E^0 and B^0 are divergence-free, then there exists a unique solution to the Maxwell's equations such that $(E, B) \in L^\infty([0, T]; L^2(\mathbb{R}^3))$ for every $T > 0$.

(9) (Global regularity for the subcritical/critical defocusing nonlinear wave equation *) In this problem, we discuss the global regularity for the problem

$$\Box \phi = |\phi|^{p-1} \phi$$

in $\mathbb{I} \times \mathbb{R}^3$, where $p \le 5$, i.e., when the problem is subcritical or critical. (Recall that we discussed in class the $p = 3$ case. This problem goes beyong that discussion.) A key estimate that is useful to establish the global regularity is the Strichartz estimates. It is a fundamental estimate for the wave equation which unfortunately we do not have time to cover in class. Nevertheless, it reads as follows: Let ϕ be the solution to the inhomogeneous linear wave equation

$$\Box \phi = F$$

with initial data

$$(\phi, \partial_t \phi) \restriction_{\{t=0\}} = (\phi_0, \phi_1).$$

Then, for every q, r satisfying

$$\frac{1}{2} = \frac{1}{q} + \frac{3}{r}, \quad \frac{1}{q} \le \frac{1}{2} - \frac{1}{r}, \quad 2 \le q, r \le \infty, \quad r \ne \infty,$$

there exists $C = C(q, r) > 0$ such that the following estimate holds

$$\|\phi\|_{L^q([0,\infty);L^r(\mathbb{R}^3))} \le C(\|(\phi_0, \phi_1)\|_{\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)} + \|F\|_{L^{q'}([0,\infty);L^{r'}(\mathbb{R}^3))}).$$

Here q' and r' are the conjugate exponents of q and r defined by $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$. Assume this estimate for the rest of this problem.

(a) Show that for $p < 5$, if $(\phi_0, \phi_1) \in C_c^\infty \times C_c^\infty$, then the solution exists for all time and remains smooth. (Hint: Prove that the time of existence depends only on the $\dot{H}^1 \times L^2$ norm of (ϕ_0, ϕ_1). To this end, first show using a bootstrap argument that $\|\phi\|_{L^q([0,T];L^r(\mathbb{R}^3))}$ is finite for some well-chosen q and r.)

(b) Show that for $p = 5$, there exists $\epsilon_0 > 0$ such that if $(\phi_0, \phi_1) \in C_c^\infty \times C_c^\infty$ and

$$\|(\phi_0, \phi_1)\|_{\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)} \leq \epsilon_0,$$

then there exists a global-in-time smooth solution.

(10) As always, please look for mistakes in this example sheet and let me know about them immediately!