EXAMPLE SHEET 1: NONLINEAR WAVE EQUATIONS

Throughout this example sheet, we take $\Box = -\partial_t^2 + \Delta$.

(1) (Solving Maxwell’s equations as wave equations) Given vector fields $E_0, B_0 : \mathbb{R}^3 \to \mathbb{R}^3$ which are divergence free. Show that the solution $(E, B)$ to

$$\Box E = 0, \quad \Box B = 0$$

with initial data

$$E|_{t=0} = E_0, \quad B|_{t=0} = B_0,$$

$$\partial_t E|_{t=0} = \nabla \times B_0, \quad \partial_t B|_{t=0} = -\nabla \times E_0$$

is indeed a solution to the Maxwell’s equations.

(2) (Compressible irrotational Euler equations) Recall from lecture that the for irrotational fluid flow, the Euler equations reduce to

$$\eta^{-2} \partial_t^2 \phi - 2n^{-2} \sum_{i=1}^3 \partial_i \phi \partial_i \phi + \eta^{-2} \sum_{i,j=1}^3 \partial_i \phi \partial_j \phi \partial_i \phi - \sum_{i=1}^3 \partial_i^2 \phi = 0.$$ 

Find the metric associated with the equation. When is it Lorentzian?

(3) Show that $\Box \phi = (\partial_t \phi)^2$ has a blow up solution. (Hint: consider initial data independent of $x$.) Now, can you construct a blow up solution with compactly supported initial data? (Hint: use finite speed of propagation.) Given $\epsilon > 0$, can you moreover construct a blow up solution with compactly supported initial data $(\phi_0, \phi_1)$ such that its $H^1 \times L^2$ norm $\leq \epsilon$? (Hint: use scaling.)

(4) (The wave map equation) Recall that the wave map equation is given by

$$\Box \phi = \phi(\partial_t \phi^i \partial_i \phi - \sum_{i=1}^n \partial_i \phi^i \partial_i \phi),$$

where $\phi : I \times \mathbb{R}^n \to S^m := \{ x \in \mathbb{R}^{m+1} : |x| = 1 \}$.

(a) Suppose that $|\phi_0|^2 = 1$ and $\phi_0^i \phi_1 = 0$. Then, if a solution $\phi$ exists in $I \times \mathbb{R}^n$, then $|\phi|^2 = 1$, i.e., $\phi$ is indeed a map to the sphere.

(b) Show that the inverse stereographic projection $\mathbb{R}^2 \to \mathbb{S}^2$ is a time-independent wave map.

(5) Given smooth and compactly supported initial data to the linear wave equation in any dimensions. Use the Fourier representation formula to show that the $H^k$ norm is uniformly bounded in time for any $k \geq 1$.

(6) (Decay for Klein-Gordon equation) Suppose $\phi$ satisfies the Klein-Gordon equation, i.e.,

$$\Box \phi - \phi = 0$$

on $\mathbb{R} \times \mathbb{R}^n$ with initial data given by

$$(\phi, \partial_t \phi)|_{t=0} = (\phi_0, \phi_1) \in C^\infty_c \times C^\infty_c.$$ 

(a) First, show using the Fourier transform that if the solution is sufficiently regular, it is given by the formula

$$\hat{\phi}(t, \xi) = \hat{\phi}_0(\xi) \cos(\sqrt{\frac{4\pi^2}{\xi^2}} + 1)t) + \frac{\hat{\phi}_1(\xi)}{\sqrt{\frac{4\pi^2}{\xi^2}} + 1} \sin(\sqrt{\frac{4\pi^2}{\xi^2}} + 1)t).$$

(b) For the $n = 1$ case, prove that $\sup_{t \geq 0} |\phi(t)| \leq \frac{C}{(1+t)^2}$, where $C = C(\phi_0, \phi_1) > 0$ is independent of $t$. Note that it suffices (why?) to show that

$$| \int e^{it\frac{2\pi^2x}{t} + \frac{4\pi^2}{\xi^2} + 1})\hat{\phi}_0(\xi)d\xi | \leq \frac{C}{t^2}$$

for $t$ large. Let

$$\varphi := \frac{2\pi x}{t} + \sqrt{\frac{4\pi^2}{\xi^2} + 1}.$$
Show that
\[ \frac{\partial}{\partial \xi} \varphi = 0 \implies \xi^2 = \frac{x^2}{4\pi^2(t^2 - x^2)}. \]

Show also that
\[ \left( \frac{\partial}{\partial \xi} \right)^2 \varphi \geq \frac{4\pi^2}{(4\pi^2|\xi|^2 + 1)^{\frac{3}{2}}}. \]

Now, split the integral into \( \int_{|\xi^2 - \frac{x^2}{(4\pi^2|\xi|^2 + 1)^{\frac{3}{2}}} \leq \delta} + \int_{|\xi^2 - \frac{x^2}{(4\pi^2|\xi|^2 + 1)^{\frac{3}{2}}} \geq \delta} \), estimate each part with the above observations and optimized in \( \delta \) to obtain the desired decay result.

(c) Finally, can you prove that
\[ |\varphi| \leq \frac{C}{(1 + t)^\frac{n}{2}} \]
holds for some \( C = C(\phi_0, \phi_1) > 0 \) for arbitrary dimensions?

(7) (Precise constant for the forward fundamental solution) Recall from the lectures that the forward fundamental solution to the wave equation in \( \mathbb{R} \times \mathbb{R}^n \) is given by
\[ E_+ = -c_n 1_{\{t \geq 0\}} \chi_+^{-\frac{n-1}{2}}(t^2 - |x|^2). \]

We now show that the precise constant \( c_n \) is given by \( c_n = \frac{2^{\frac{1-n}{2}}}{\pi} \). Let \( r = |x|, u = t - r, v = t + r \). Let \( f \) and \( g \) be rotationally symmetric, smooth, compactly supported functions in \( \mathbb{R} \times \mathbb{R}^n \), i.e., \( f, g \) are functions of \( t \) and \( r \) alone. Begin by proving the identity
\[ < \Box f, g > = \omega_{n-1} \int \int (\partial_u f \partial_v g + \partial_u g \partial_v f) \left( \frac{v - u}{2} \right)^{-\frac{n-1}{2}} \ du \ dv, \]
where
\[ \omega_{n-1} = \text{Area of } n - 1 \text{ dimensional sphere.} \]

Apply this to \( f = E_+ \) and \( g = 1_{\{t + r \leq 1\}} \) (Why is this allowed?). Show therefore that
\[ c_n^{-1} = 2^{-n+1}(n - 1)\omega_{n-1}(\chi_+^{-\frac{n-1}{2}} * \chi_+^{-\frac{n-1}{2}})(1). \]

Now conclude using the following facts:
\[ \chi_+^a * \chi_+^b = \chi_+^{a+b+1}, \quad \chi_+^a(1) = \frac{1}{\Gamma(a + 1)}, \]
\[ \omega_{n-1} = \text{Area of } n - 1 \text{ dimensional sphere} = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}, \]
\[ \Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n + 1}{2}\right) = 2^{-n+1}\sqrt{\pi}(n - 1)! \]

(I learnt this proof from P. Isett and S.-J. Oh.)

(8) We used in class the forward fundamental solution to solve the homogeneous wave equation
\[ \Box \varphi = 0. \]

In fact, this can also be used to solve the equation \( \Box \varphi = F \) for given \( F \in C^\infty \). Show that if \( E_+ \) is the forward fundamental solutions, then
\[ \varphi = -E_+ * (F1_{\{t \geq 0\}}) \]
is a solution to \( \Box \varphi = F \) with zero data, i.e., \( (\varphi, \partial_t \varphi) \mid_{\{t = 0\}} = (0, 0) \).

(9) Show that the bound \( \sup_{x,t} |\partial\varphi| \leq C \) for the solutions to the wave equation in \( \mathbb{R} \times \mathbb{R} \) with smooth and compactly supported data is sharp in the following sense: Suppose the initial data are given by \( (\varphi, \partial_t \varphi) \mid_{\{t = 0\}} = (\varphi_0, \phi_1) \in C^\infty \times C^\infty_c \). Then if there exists a non-increasing function \( A(t) \) such that \( \lim_{t \to \infty} A(t) = 0 \) and
\[ (\sup_x |\partial\varphi|)(t) \leq A(t), \]
then \( \phi_0 = \phi_1 = 0 \).
(10) Consider the linear wave equation in \( \mathbb{R} \times \mathbb{R}^3 \). First, show that spherically symmetric (smooth, compactly supported) data lead to spherically symmetric solutions. Moreover, the wave equation becomes \( \partial_t \partial_s (r \phi) = 0 \). Then, use this to construct solutions such that \[ (\sup_x |\phi|)(t) \geq \frac{C}{1 + t} \]
for some \( C > 0 \).

(11) Construct solutions to the linear wave equation in \( \mathbb{R} \times \mathbb{R}^3 \) with smooth (and necessarily non-compactly supported!) initial data satisfying \( \lim_{|x| \to \infty} |\phi_0(x)| + |\phi_1(x)| = 0 \) such that
\[ (\sup_x |\phi|)(t) \to \infty \]
as \( t \to \infty \).

(12) Show using the fundamental solution that that solutions to the linear wave equation in \( (1 + 1) \)-dimensions, i.e., \( \mathbb{R} \times \mathbb{R} \) satisfy
\[ \sup_{t,x} |\partial \phi| \leq C \sup_x |\partial \phi|(0, x) \]
for some \( C > 0 \) independent of \( \phi \). Show that the corresponding estimate fails in \( \mathbb{R} \times \mathbb{R}^3 \) (in fact all dimensions \( n \geq 2! \)), i.e.,
\[ \sup_{\{(\phi_0, \phi_1) : \sup_x |\phi(0, x)| = 1\}} (\sup_{t,x} |\partial \phi|) = \infty. \]
This shows that unlike the \( L^2 \) norm for the derivatives of \( \phi \), which is propagated by the equation in all dimensions, the \( L^\infty \) norm for the derivative of \( \phi \) is not propagated.

(13) Show that the solutions to the linear wave equation in any dimensions with initial data \( (\phi_0, \phi_1) \in H^1 \times L^2 \) satisfy the estimate
\[ \sum_{|\alpha| = 2} \|\partial_{xt}^\alpha \|_{L^2(\mathbb{R}^n)} \leq C \|\phi_0, \phi_1\|_{H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} \]
for any interval \( I \) and for some \( C > 0 \). (Hint: integrate by parts.) (I learnt this from Exercise 2.25 in Nonlinear dispersive equations by T. Tao.)

(14) (Finite speed of propagation for non-constant coefficient wave equations) Consider the equation
\[ -\partial_t^2 \phi + \partial_i ((h^{-1})^{ij} \sqrt{h} \partial_j \phi) = 0, \]
where \( h_{ij} \) is a positive definite matrix such that
\[ \sum_{i,j=1}^n |h_{ij} - \delta_{ij}| \leq \frac{1}{2}. \]
Fix \( x_0 \). Suppose there is a smooth solution to
\[ \sum_{i,j=1}^n (h^{-1})^{ij} \partial_i q \partial_j q = 1, \quad q(x_0) = 0 \]
in \( B(x_0, R) \) such that \( q > 0 \) in \( B(x_0, R) \setminus \{x_0\} \). For every \( r < R \), define the set
\[ S := \{(t, x) : q(x) < r - t, \ 0 \leq t \leq r\}. \]
Show that if the initial data \( (\phi_0, \phi_1) = (0, 0) \) in \( \{x : q(x) \leq r\} \), then the solution \( \phi = 0 \) in \( S \). Show that the same conclusion holds for solutions to
\[ -\partial_t^2 \phi + \partial_i ((h^{-1})^{ij} \sqrt{h} \partial_j \phi) = (\partial_t \phi)^2 \]
assuming that the solutions are smooth. (Notice that geometrically, \( S \) is the past of the point \( (t = r, x_0) \).)

(15) Consider the equation
\[ \Box \phi - \partial_t \phi = 0 \]
on \( \mathbb{R} \times \mathbb{R}^n \) with finite energy data \( (\phi_0, \phi_1) \in H^1 \times L^2 \). Show that \( \int_{\mathbb{R}^n} (\partial_t \phi)^2 \) decays exponentially in time.
(16) In this problem, we consider the equation

\[ \sum_{\mu,\nu=0}^{n} \frac{1}{\sqrt{-g}} \partial_\mu ((g^{-1})^{\mu\nu} \sqrt{-g} \partial_\nu \phi) + \sum_{\mu=0}^{n} b^\mu \partial_\mu \phi + d\phi = F, \]

(i.e., it has an extra zeroth order term compared to what is done in lecture) in (3+1)-dimensions. In the formula above, we have also used \( g \) to denote the determinant of \( g \). (This convention will also be used throughout the example sheet.)

(a) In \( \mathbb{R}^3 \), prove the Hardy inequality

\[ \| \phi \|_{L^2(\mathbb{R}^3)} \leq C \| \partial_r \phi \|_{L^2(\mathbb{R}^3)} \]

for \( \phi \) smooth and compactly supported where \( C > 0 \) is independent of \( \phi \) (or its support). (Hint: Use the fact that \( 0 \leq \int_0^\infty (\partial_r \phi + \beta \phi)^2 r^2 dr \) for all functions \( \beta \). Choose \( \beta(r) \) and integrate by parts.)

(b) Consider now the equation

\[ \sum_{\mu,\nu=0}^{n} \frac{1}{\sqrt{-g}} \partial_\mu ((g^{-1})^{\mu\nu} \sqrt{-g} \partial_\nu \phi) + \sum_{\mu=0}^{n} b^\mu \partial_\mu \phi + d\phi = F \]

in (3+1)-dimensions with finite energy data \( (\phi_0, \phi_1) \in H^1 \times L^2 \). Use the Hardy inequality to show that as long as

\[ \sum_{\mu,\nu} |(g^{-1})^{\mu\nu} \sqrt{-g} - m^{\mu\nu}| + |\sqrt{-g} - 1| \leq \frac{1}{2}, \]

and

\[ \|(1 + |x|)d\|_{L^1([0,T];L^\infty(\mathbb{R}^3))} + \| \partial (g^{-1}\sqrt{-g}) \|_{L^1([0,T];L^\infty(\mathbb{R}^3))} \]

\[ + \| b \|_{L^1([0,T];L^\infty(\mathbb{R}^3))} = C_1 < \infty \]

and

\[ \| F \|_{L^1([0,T];L^2(\mathbb{R}^3))} = C_2 < \infty, \]

the solution \( (\phi, \partial_t \phi) \in L^\infty((0,T);H^1(\mathbb{R}^3)) \times L^\infty((0,T);L^2(\mathbb{R}^3)) \). Moreover, prove that the \( L^\infty((0,T);H^1(\mathbb{R}^3)) \times L^\infty((0,T);L^2(\mathbb{R}^3)) \) norm is bounded by a constant depending only on \( C_1, C_2 \) and the \( H^1 \times L^2 \) norm of the initial data \( (\phi_0, \phi_1) \).

(17) (Geometric formulation of \( L^2 \) type estimates for the wave equation) This is intended for students who know some differential geometry. Given a wave equation

\[ \Box_g \phi := \frac{1}{\sqrt{-g}} \partial_\mu ((g^{-1})^{\mu\nu} \sqrt{-g} \partial_\nu \phi) = 0. \]

Introduce the stress-energy-momentum tensor

\[ T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (g^{-1})^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi. \]

(a) Show that if \( \phi \) is a solution to the wave equation, then

\[ \nabla^\mu T_{\mu\nu} := (g^{-1})^\alpha\mu \nabla_\alpha T_{\mu\nu} := (g^{-1})^\alpha\mu (\partial_\alpha T_{\mu\nu} + \Gamma^\beta_{\alpha\mu} T_{\beta\nu} + \Gamma^\beta_{\alpha\nu} T_{\beta\mu}) = 0. \]

Here, repeated indices are automatically summed over. Also, \( \Gamma^\beta_{\alpha\mu} \) is the Christoffel symbol given by

\[ \Gamma^\beta_{\alpha\mu} = \frac{1}{2} (g^{-1})^{\beta\sigma} (\partial_\alpha g_{\sigma\mu} + \partial_\mu g_{\alpha\sigma} - \partial_\sigma g_{\alpha\mu}). \]

(b) Given a vector field \( X \) which is Killing, i.e.,

\[ (\mathcal{L}_X g)_{\mu\nu} := X^\alpha \partial_\alpha g_{\mu\nu} + g_{\alpha\mu} \partial_\alpha X^\nu + g_{\alpha\nu} \partial_\alpha X^\mu = 0 \]

Show that \( \text{div} (T(X, \cdot)) = 0 \) (i.e., \( (g^{-1})^{\alpha\beta} \nabla_\alpha (T_{\beta\gamma}) = 0 \)). Use the divergence theorem to get a conservation law.
(c) Assume that $g$ is a metric given by $g_{tt} = -1$, $g_{ti} = 0$ for $i = 1, \ldots, n$ and $g_{ij} = h_{ij}$ for $i, j = 1, \ldots, n$, where $h$ is a positive definite symmetric matrix such that $\sum_{i,j=1}^{n} |h_{ij} - \delta_{ij}| \leq \frac{1}{2}$. Moreover, assume that all components of $g$ is independent of $t$. Show that an appropriately defined “energy” is conserved.

(18) Finally, please look for mistakes in the above problems and let me know about them immediately!