EXAMPLE SHEET 1: NONLINEAR WAVE EQUATIONS

Throughout this example sheet, we take $\Box = -\partial_t^2 + \Delta$.

(1) (Solving Maxwell’s equations as wave equations) Given vector fields $E_0, B_0 : \mathbb{R}^3 \to \mathbb{R}^3$ which are divergence free. Show that the solution $(E, B)$ to

$$\Box E = 0, \quad \Box B = 0$$

with initial data

$$E \mid_{t=0} = E_0, \quad B \mid_{t=0} = B_0,$$

$$\partial_t E \mid_{t=0} = \nabla \times B_0, \quad \partial_t B \mid_{t=0} = -\nabla \times E_0$$

is indeed a solution to the Maxwell’s equations.

(2) (Compressible irrotational Euler equations) Recall from lecture that the for irrotational fluid flow, the Euler equations reduce to

$$\eta^{-2} \partial_t^2 \phi - 2\eta^{-2} \sum_{i=1}^3 \partial_i \phi \partial_i^2 \phi + \eta^{-2} \sum_{i,j=1}^3 \partial_i \phi \partial_j \phi \partial_i^2 \phi - \sum_{i=1}^3 \partial_i^2 \phi = 0.$$

Find the metric associated with the equation. When is it Lorentzian?

(3) Show that $\Box \phi = (\partial_t \phi)^2$ has a blow up solution. (Hint: consider initial data independent of $x$.) Now, can you construct a blow up solution with compactly supported initial data? (Hint: use finite speed of propagation.) Given $\epsilon > 0$, can you moreover construct a blow up solution with compactly supported initial data $(\phi_0, \phi_1)$ such that its $H^1 \times L^2$ norm $\leq \epsilon$? (Hint: use scaling.)

(4) (The wave map equation) Recall that the wave map equation is given by

$$\Box \phi = \phi(\partial_t \phi' \partial_i \phi - \sum_{i=1}^n \partial_i \phi' \partial_i \phi),$$

where $\phi : I \times \mathbb{R}^n \to S^m := \{ x \in \mathbb{R}^{m+1} : |x| = 1 \}$.

(a) Suppose that $|\phi_0|^2 = 1$ and $\phi_0' \phi_1 = 0$. Then, if a solution $\phi$ exists in $I \times \mathbb{R}^n$, then $|\phi|^2 = 1$, i.e., $\phi$ is indeed a map to the sphere.

(b) Show that the inverse stereographic projection $\mathbb{R}^2 \to S^2$ is a time-independent wave map.

(5) Given smooth and compactly supported initial data to the linear wave equation in any dimensions. Use the Fourier representation formula to show that the $\dot{H}^k$ norm is uniformly bounded in time for any $k \geq 1$.

(6) (Decay for Klein-Gordon equation) Suppose $\phi$ satisfies the Klein-Gordon equation, i.e.,

$$\Box \phi - \phi = 0$$

on $\mathbb{R} \times \mathbb{R}^n$ with initial data given by

$$(\phi, \partial_t \phi)_{t=0} = (\phi_0, \phi_1) \in C_c^\infty \times C_c^\infty.$$

(a) First, show using the Fourier transform that if the solution is sufficiently regular, it is given by the formula

$$\hat{\phi}(t, \xi) = \hat{\phi}_0(\xi) \cos(\sqrt{4\pi^2|\xi|^2 + 1}t) + \frac{\hat{\phi}_1(\xi)}{\sqrt{4\pi^2|\xi|^2 + 1}} \sin(\sqrt{4\pi^2|\xi|^2 + 1}t).$$

(b) For the $n = 1$ case, prove that (sup $|\phi|$) $\leq \frac{C}{(1+t)^{\frac{1}{2}}}$, where $C = C(\phi_0, \phi_1) > 0$ is independent of $t$. Note that it suffices (why?) to show that

$$\left| \int_{\mathbb{R}} e^{it(\frac{2\pi x}{t} + \sqrt{4\pi^2\xi^2 + 1})} \hat{\phi}_0(\xi) d\xi \right| \leq \frac{C}{t^{\frac{1}{2}}}$$

for $t$ large. Let

$$\varphi := \frac{2\pi x \xi}{t} + \sqrt{4\pi^2 \xi^2 + 1}.$$
Show that
\[ \frac{\partial}{\partial \xi} \phi = 0 \implies \xi^2 = \frac{x^2}{4\pi^2(t^2 - x^2)}. \]
Show also that
\[ \left( \frac{\partial}{\partial \xi} \right)^2 \phi \geq \frac{4\pi^2}{(4\pi^2(\xi^2 + 1))^2}. \]

Now, split the integral into \( \int_{|\xi^2 - \frac{x^2}{4\pi^2(t^2 - x^2)}| \leq \delta} + \int_{|\xi^2 - \frac{x^2}{4\pi^2(t^2 - x^2)}| \geq \delta} \), estimate each part with the above observations and optimized in \( \delta \) to obtain the desired decay result.

(c) Finally, you can prove that
\[ |\phi| \leq \frac{C}{(1+t)^{\frac{n}{2}}} \]
holds for some \( C = C(\phi_0, \phi_1) > 0 \) for arbitrary dimensions?

(7) (Precise constant for the forward fundamental solution) Recall from the lectures that the forward fundamental solution to the wave equation in \( \mathbb{R} \times \mathbb{R}^n \) is given by
\[ E_+ = -c_n 1_{\{t \geq 0\}} \chi_+^{\frac{n-1}{2}}(t^2 - |x|^2). \]
We now show that the precise constant \( c_n \) is given by \( c_n = \frac{2^{1-n}}{2} \). Let \( r = |x|, u = t - r, v = t + r \). Let \( f \) and \( g \) be rotationally symmetric, smooth, compactly supported functions in \( \mathbb{R} \times \mathbb{R}^n \), i.e., \( f, g \) are functions of \( t \) and \( r \) alone. Begin by proving the identity
\[ \langle \Box f, g \rangle = \omega_{n-1} \int \int (\partial_u f \partial_v g + \partial_v g \partial_u f)(\frac{v-u}{2})_+^{n-1} du dv, \]
where
\[ \omega_{n-1} = \text{Area of } n - 1 \text{ dimensional sphere}. \]
Apply this to \( f = E_+ \) and \( g = 1_{t+r \leq 1} \) (Why is this allowed?). Show therefore that
\[ c_n^{-1} = 2^{-n+1}(n-1)! \omega_{n-1}(\chi_+^{\frac{n}{2}} \ast \chi_+^{n-1})(1). \]
Now conclude using the following facts:
\[ \chi_+^a \ast \chi_+^b = \chi_+^{a+b+1}, \quad \chi_+^a(1) = \frac{1}{\Gamma(a+1)}, \]
\[ \omega_{n-1} = \text{Area of } n - 1 \text{ dimensional sphere} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \]
\[ \Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2}\right) = 2^{-n+1}\sqrt{\pi}(n - 1)!. \]
(I learnt this proof from P. Isett and S.-J. Oh.)

(8) We used in class the forward fundamental solution to solve the homogeneous wave equation
\[ \Box \phi = 0. \]
In fact, this can also be used to solve the equation \( \Box \phi = F \) for given \( F \in C_\infty \). Show that if \( E_+ \) is the forward fundamental solutions, then
\[ \phi = -E_+ \ast (F 1_{\{t \geq 0\}}) \]
is a solution to \( \Box \phi = F \) with zero data, i.e., \( (\phi, \partial_t \phi) \big|_{\{t = 0\}} = (0, 0) \).

(9) Show that the bound \( \sup_{t,x} |\partial \phi| \leq C \) for the solutions to the wave equation in \( \mathbb{R} \times \mathbb{R} \) with smooth and compactly supported data is sharp in the following sense: Suppose the initial data are given by \( (\phi, \partial_t \phi) \big|_{\{t = 0\}} = (\phi_0, \phi_1) \in C_\infty \times C_\infty^\infty \). Then if there exists a non-increasing function \( A(t) \) such that
\[ \lim_{t \to \infty} A(t) = 0 \]
and
\[ (\sup_x | \partial \phi |)(t) \leq A(t), \]
then \( \phi_0 = \phi_1 = 0 \).
(10) Consider the linear wave equation in $\mathbb{R} \times \mathbb{R}^3$. First, show that spherically symmetric (smooth, compactly supported) data lead to spherically symmetric solutions. Moreover, the wave equation becomes $\partial_t \partial_r (r \phi) = 0$. Then, use this to construct solutions such that
\[
(\sup_x |\phi|)(t) \geq \frac{C}{1 + t}
\]
for some $C > 0$.

(11) Construct solutions to the linear wave equation in $\mathbb{R} \times \mathbb{R}^3$ with smooth (and necessarily non-compactly supported) initial data satisfying $\lim_{|x| \to \infty} |\phi_0(x)| + |\phi_1(x)| = 0$ such that
\[
(\sup_x |\phi|)(t) \to \infty
\]
as $t \to \infty$.

(12) Show using the fundamental solution that that solutions to the linear wave equation in $(1 + 1)$-dimensions, i.e., $\mathbb{R} \times \mathbb{R}$ satisfy
\[
\sup_{t,x} |\partial_\phi| \leq C \sup_{x} |\partial_\phi|(0, x)
\]
for some $C > 0$ independent of $\phi$. Show that the corresponding estimate fails in $\mathbb{R} \times \mathbb{R}^3$ (in fact all dimensions $n \geq 2$!), i.e.,
\[
\sup_{\{(\phi_0, \phi_1) : \sup_x |\partial_\phi|(0, x) = 1\}} (\sup_{t,x} |\partial_\phi|) = \infty.
\]
This shows that unlike the $L^2$ norm for the derivatives of $\phi$, which is propagated by the equation in all dimensions, the $L^\infty$ norm for the derivative of $\phi$ is not propagated.

(13) Show that the solutions to the linear wave equation in any dimensions with initial data $(\phi_0, \phi_1) \in H^1 \times L^2$ satisfy the estimate
\[
\sum_{|\alpha| = 2} \|\partial^\alpha_\phi\|_{L^2(I \times \mathbb{R}^n)} \leq C \|\phi_1\|_{H^1(I \times \mathbb{R}^n)} \times L^2(I \times \mathbb{R}^n)
\]
for any interval $I$ and for some $C > 0$. (Hint: integrate by parts.) (I learnt this from Exercise 2.25 in Nonlinear dispersive equations by T. Tao.)

(14) (Finite speed of propagation for non-constant coefficient wave equations) Consider the equation
\[
-\partial_t^2 \phi + \partial_i ((h^{-1})^{ij} \sqrt{h} \partial_j \phi) = 0,
\]
where $h_{ij}$ is a positive definite matrix such that
\[
\sum_{i,j=1}^n |h_{ij} - \delta_{ij}| \leq \frac{1}{2}.
\]
Fix $x_0$. Suppose there is a smooth solution to
\[
\sum_{i,j=1}^n (h^{-1})^{ij} \partial_i q \partial_j q = 1, \quad q(x_0) = 0
\]
in $B(x_0, R)$ such that $q > 0$ in $B(x_0, R) \setminus \{x_0\}$. For every $r < R$, define the set
\[
S := \{(t,x) : q(x) < r - t, \; 0 \leq t \leq r\}.
\]
Show that if the initial data $(\phi_0, \phi_1) = (0, 0)$ in $\{x : q(x) \leq r\}$, then the solution $\phi = 0$ in $S$. Show that the same conclusion holds for solutions to
\[
-\partial_t^2 \phi + \partial_i ((h^{-1})^{ij} \sqrt{h} \partial_j \phi) = (\partial_t \phi)^2
\]
assuming that the solutions are smooth. (Notice that geometrically, $S$ is the past of the point $(t = r, x_0)$.)
(15) In this problem, we consider the equation
\[ \sum_{\mu, \nu=0}^{n} \frac{1}{\sqrt{-g}} \partial_{\mu}((g^{-1})^{\mu\nu} \sqrt{-g} \partial_{\nu} \phi) + \sum_{\mu=0}^{n} b^{\mu} \partial_{\mu} \phi + d\phi = F, \]

(i.e., it has an extra zeroth order \( d\phi \) term compared to what is done in lecture) in \((3+1)\)-dimensions. In the formula above, we have also used \( g \) to denote the determinant of \( g \). (This convention will also be used throughout the example sheet.)

(a) In \( \mathbb{R}^3 \), prove the Hardy inequality
\[ \| \phi \|_{L^2(\mathbb{R}^3)} \leq C \| \partial_r \phi \|_{L^2(\mathbb{R}^3)} \]

for \( \phi \) smooth and compactly supported where \( C > 0 \) is independent of \( \phi \) (or its support). (Hint: Use the fact that \( 0 \leq \int_{0}^{\infty} (\partial_{\phi} + \beta \phi)^2 r^2 dr \) for all functions \( \beta \). Choose \( \beta(r) \) and integrate by parts.)

(b) Consider now the equation
\[ \sum_{\mu, \nu=0}^{n} \frac{1}{\sqrt{-g}} \partial_{\mu}((g^{-1})^{\mu\nu} \sqrt{-g} \partial_{\nu} \phi) + \sum_{\mu=0}^{n} b^{\mu} \partial_{\mu} \phi + d\phi = F \]
in \((3 + 1)\)-dimensions with finite energy data \((\phi_0, \phi_1) \in H^1 \times L^2\). Use the Hardy inequality to show that as long as
\[ \sum_{\mu, \nu}|(g^{-1})^{\mu\nu} \sqrt{-g} - m^{\mu\nu}| + |\sqrt{-g} - 1| \leq \frac{1}{2}, \]

and
\[ \| (1 + |x|)d\|_{L^1([0,T];L^\infty(\mathbb{R}^3))} + \| \partial(g^{-1} \sqrt{-g}) \|_{L^1([0,T];L^\infty(\mathbb{R}^3))} \]
\[ + \| b \|_{L^1([0,T];L^\infty(\mathbb{R}^3))} = C_1 < \infty \]

and
\[ \| F \|_{L^1([0,T];L^2(\mathbb{R}^3))} = C_2 < \infty, \]

the solution \((\phi, \partial_t \phi) \in L^\infty([0,T];H^1(\mathbb{R}^3)) \times L^\infty([0,T];L^2(\mathbb{R}^3))\). Moreover, prove that the \( L^\infty([0,T];H^1(\mathbb{R}^3)) \times L^\infty([0,T];L^2(\mathbb{R}^3)) \) norm is bounded by a constant depending only on \( C_1, C_2 \) and the \( H^1 \times L^2 \) norm of the initial data \((\phi_0, \phi_1)\).

(16) (Geometric formulation of \( L^2 \) type estimates for the wave equation) This is intended for students who know some differential geometry. Given a wave equation
\[ \Box g \phi := \frac{1}{\sqrt{-g}} \partial_{\mu}((g^{-1})^{\mu\nu} \sqrt{-g} \partial_{\nu} \phi) = 0. \]

Introduce the stress-energy-momentum tensor
\[ T_{\mu\nu} = \partial_{\mu} \phi \partial_{\nu} \phi - \frac{1}{2} g_{\mu\nu}(g^{-1})^{\alpha\beta} \partial_{\alpha} \phi \partial_{\beta} \phi. \]

(a) Show that if \( \phi \) is a solution to the wave equation, then
\[ \nabla^\mu T_{\mu\nu} := (g^{-1})^{\alpha\mu} \nabla_{\alpha} T_{\mu\nu} := (g^{-1})^{\alpha\mu}(\partial_{\alpha} T_{\mu\nu} + \Gamma_{\alpha\nu}^{\beta} T_{\beta\mu} + \Gamma_{\alpha\mu}^{\beta} T_{\beta\nu}) = 0. \]

Here, repeated indices are automatically summed over. Also, \( \Gamma_{\alpha\mu}^{\beta} \) is the Christoffel symbol given by
\[ \Gamma_{\alpha\mu}^{\beta} = \frac{1}{2}(g^{-1})^{\beta\sigma}(\partial_{\alpha} g_{\sigma\mu} + \partial_{\mu} g_{\sigma\alpha} - \partial_{\sigma} g_{\alpha\mu}). \]

(b) Given a vector field \( X \) which is Killing, i.e.,
\[ (\mathcal{L}_X g)_{\mu\nu} := X^{\alpha} \partial_{\alpha} g_{\mu\nu} + g_{\alpha\nu} \partial_{\mu} X^{\alpha} + g_{\alpha\mu} \partial_{\nu} X^{\alpha} = 0 \]

Show that \( \text{div} (T(X, \cdot)) = 0 \) (i.e., \( (g^{-1})^{\alpha\beta} \nabla_{\alpha}(T_{\beta\gamma} X^{\gamma}) = 0 \)). Use the divergence theorem to get a conservation law.
(c) Assume that $g$ is a metric given by $g_{tt} = -1$, $g_{ti} = 0$ for $i = 1, ..., n$ and $g_{ij} = h_{ij}$ for $i, j = 1, ..., n$, where $h$ is a positive definite symmetric matrix such that $\sum_{i,j=1}^{n}|h_{ij} - \delta_{ij}| \leq \frac{1}{2}$. Moreover, assume that all components of $g$ is independent of $t$. Show that an appropriately defined “energy” is conserved.

(17) Finally, please look for mistakes in the above problems and let me know about them immediately!