EXAMPLE SHEET 1: NONLINEAR WAVE EQUATIONS

Throughout this example sheet, we take $\Box = -\partial_t^2 + \Delta$.

(1) (Solving Maxwell's equations as wave equations) Given vector fields $E_0, B_0 : \mathbb{R}^3 \to \mathbb{R}^3$ which are divergence free. Show that the solution (E, B) to

$$\Box E = 0, \quad \Box B = 0$$

with initial data

$$E \restriction_{t=0} = E_0, \quad B \restriction_{t=0} = B_0,$$
$$\partial_t E \restriction_{t=0} = \nabla \times B_0, \quad \partial_t B \restriction_{t=0} = -\nabla \times E_0$$

is indeed a solution to the Maxwell's equations.

(2) (Compressible irrotational Euler equations) Recall from lecture that the for irrotational fluid flow, the Euler equations reduce to

$$\eta^{-2}\partial_t^2\phi - 2\eta^{-2}\sum_{i=1}^{3}\partial_i\phi\partial_{ti}^2\phi + \eta^{-2}\sum_{i,j=1}^{3}\partial_i\phi\partial_j\phi\partial_{ij}^2\phi - \sum_{i=1}^{3}\partial_i^2\phi = 0.$$

Find the metric associated with the equation. When is it Lorentzian?

(3) Show that $\Box\phi = (\partial_t\phi)^2$ has a blow up solution. (Hint: consider intial data independent of x.) Now, can you construct a blow up solution with compactly supported initial data? (Hint: use finite speed of propagation.) Given $\epsilon > 0$, can you moreover construct a blow up solution with compactly supported initial data (ϕ_0, ϕ_1) such that its $H^1 \times L^2$ norm $\leq \epsilon$? (Hint: use scaling.)

(4) (The wave map equation) Recall that the wave map equation is given by

$$\Box\phi = \phi(\partial_t\phi^t\partial_t\phi - \sum_{i=1}^{n}\partial_i\phi^t\partial_i\phi),$$

where $\phi : I \times \mathbb{R}^n \to \mathbb{S}^m := \{x \in \mathbb{R}^{m+1} : |x| = 1\}$.

(a) Suppose that $|\phi_0|^2 = 1$ and $\phi_0^t\phi_1 = 0$. Then, if a solution ϕ exists in $I \times \mathbb{R}^n$, then $|\phi|^2 = 1$, i.e., ϕ is indeed a map to the sphere.

(b) Show that the inverse stereographic projection $\mathbb{R}^2 \to \mathbb{S}^2$ is a time-independent wave map.

(5) Given smooth and compactly supported initial data to the linear wave equation in any dimensions. Use the Fourier representation formula to show that the $\dot{H}^k$ norm is uniformly bounded in time for any $k \geq 1$.

(6) (Decay for Klein-Gordon equation) Suppose ϕ satisfies the Klein-Gordon equation, i.e.,

$$\Box\phi - \phi = 0$$

on $\mathbb{R} \times \mathbb{R}^n$ with initial data given by

$$(\phi, \partial_t\phi)_{t=0} = (\phi_0, \phi_1) \in C_c^\infty \times C_c^\infty.$$

(a) First, show using the Fourier transform that if the solution is sufficiently regular, it is given by the formula

$$\hat{\phi}(t,\xi) = \hat{\phi}_0(\xi)\cos(\sqrt{4\pi^2|\xi|^2 + 1}t) + \frac{\hat{\phi}_1(\xi)}{\sqrt{4\pi^2|\xi|^2+1}}\sin(\sqrt{4\pi^2|\xi|^2+1}t).$$

(b) For the $n = 1$ case, prove that $(\sup_x |\phi|)(t) \leq \frac{C}{(1+t)^{\frac{1}{2}}}$, where $C = C(\phi_0, \phi_1) > 0$ is independent of t. Note that it suffices (why?) to show that

$$|\int_{\mathbb{R}} e^{it(\frac{2\pi x\xi}{t} + \sqrt{4\pi^2\xi^2+1})}\hat{\phi}_0(\xi)d\xi| \leq \frac{C}{t^{\frac{1}{2}}}$$

for t large. Let

$$\varphi := \frac{2\pi x\xi}{t} + \sqrt{4\pi^2\xi^2 + 1}.$$

Show that

$$\frac{\partial}{\partial \xi}\varphi = 0 \implies \xi^2 = \frac{x^2}{4\pi^2(t^2 - x^2)}.$$

Show also that

$$(\frac{\partial}{\partial \xi})^2\varphi \geq \frac{4\pi^2}{(4\pi^2|\xi|^2+1)^{\frac{3}{2}}}.$$

Now, split the integral into $\int_{|\xi^2 - \frac{x^2}{4\pi^2(t^2-x^2)}|\leq\delta} + \int_{|\xi^2 - \frac{x^2}{4\pi^2(t^2-x^2)}|\geq\delta}$, estimate each part with the above observations and optimized in δ to obtain the desired decay result.

(c) Finally, can you prove that

$$|\phi| \leq \frac{C}{(1+t)^{\frac{n}{2}}}$$

holds for some $C = C(\phi_0, \phi_1) > 0$ for arbitrary dimensions?

(7) (Precise constant for the forward fundamental solution) Recall from the lectures that the forward fundamental solution to the wave equation in $\mathbb{R}\times\mathbb{R}^n$ is given by

$$E_+ = -c_n 1_{\{t\geq 0\}}\chi_+^{-\frac{n-1}{2}}(t^2 - |x|^2).$$

We now show that the precise constant c_n is given by $c_n = \frac{\pi^{\frac{1-n}{2}}}{2}$. Let $r = |x|$, $u = t - r$, $v = t + r$. Let f and g be rotationally symmetric, smooth, compactly supported functions in $\mathbb{R}\times\mathbb{R}^n$, i.e., f, g are functions of t and r alone. Begin by proving the identity

$$< \Box f, g >= \omega_{n-1}\int\int(\partial_u f\partial_v g + \partial_u g\partial_v f)(\frac{v-u}{2})_+^{n-1}du\, dv,$$

where

$$\omega_{n-1} = \text{Area of } n-1 \text{ dimensional sphere}.$$

Apply this to $f = E_+$ and $g = 1_{t+r\leq 1}$ (Why is this allowed?). Show therefore that

$$c_n^{-1} = 2^{-n+1}(n-1)!\omega_{n-1}(\chi^{-\frac{n+1}{2}} * \chi_+^{n-1})(1).$$

Now conclude using the following facts:

$$\chi_+^a * \chi_+^b = \chi_+^{a+b+1}, \quad \chi_+^a(1) = \frac{1}{\Gamma(a+1)},$$

$$\omega_{n-1} = \text{Area of } n-1 \text{ dimensional sphere} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})},$$

$$\Gamma(\frac{n}{2})\Gamma(\frac{n+1}{2}) = 2^{-n+1}\sqrt{\pi}(n-1)!.$$

(I learnt this proof from P. Isett and S.-J. Oh.)

(8) We used in class the forward fundamental solution to solve the homogeneous wave equation

$$\Box\phi = 0.$$

In fact, this can also be used to solve the equation $\Box\phi = F$ for given $F \in C^\infty$. Show that if E_+ is the forward fundamental solutions, then

$$\phi = -E_+ * (F1_{\{t\geq 0\}})$$

is a solution to $\Box\phi = F$ with zero data, i.e., $(\phi, \partial_t\phi)\restriction_{\{t=0\}} = (0, 0)$.

(9) Show that the bound $\sup_{t,x}|\partial\phi| \leq C$ for the solutions to the wave equation in $\mathbb{R}\times\mathbb{R}$ with smooth and compactly supported data is sharp in the following sense: Suppose the initial data are given by $(\phi, \partial_t\phi)\restriction_{\{t=0\}} = (\phi_0, \phi_1) \in C_c^\infty \times C_c^\infty$. Then if there exists a non-increasing function $A(t)$ such that $\lim_{t\to\infty} A(t) = 0$ and

$$(\sup_x |\partial\phi|)(t) \leq A(t),$$

then $\phi_0 = \phi_1 = 0$.

(10) Consider the linear wave equation in $\mathbb{R} \times \mathbb{R}^3$. First, show that spherically symmetric (smooth, compactly supported) data lead to spherically symmetric solutions. Moreover, the wave equation becomes $\partial_u \partial_v (r\phi) = 0$. Then, use this to construct solutions such that

$$(\sup_x |\phi|)(t) \geq \frac{C}{1+t}$$

for some $C > 0$.

(11) Construct solutions to the linear wave equation in $\mathbb{R}\times\mathbb{R}^3$ with smooth (and necessarily non-compactly supported!) initial data satisfying $\lim_{|x|\to\infty} |\phi_0(x)| + |\phi_1(x)| = 0$ such that

$$(\sup_x |\phi|)(t) \to \infty$$

as $t \to \infty$.

(12) Show using the fundamental solution that that solutions to the linear wave equation in $(1 + 1)$-dimensions, i.e., $\mathbb{R} \times \mathbb{R}$ satisfy

$$\sup_{t,x} |\partial\phi| \leq C \sup_x |\partial\phi|(0, x)$$

for some $C > 0$ independent of ϕ. Show that the corresponding estimate fails in $\mathbb{R} \times \mathbb{R}^3$ (in fact all dimensions $n \geq 2$!), i.e.,

$$\sup_{\{(\phi_0,\phi_1):\sup_x |\partial\phi|(0,x)=1\}} (\sup_{t,x} |\partial\phi|) = \infty.$$

This shows that unlike the L^2 norm for the derivatives of ϕ, which is propagated by the equation in all dimensions, the L^∞ norm for the derivative of ϕ is not propagated.

(13) Show that the solutions to the linear wave equation in any dimensions with initial data $(\phi_0, \phi_1) \in H^1 \times L^2$ satisfy the estimate

$$\sum_{|\alpha|=2} \|\partial_x^\alpha \int_I \phi(t, x) dt\|_{L^2_x(\mathbb{R}^n)} \leq C\|(\phi_0, \phi_1)\|_{H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}$$

for any interval I and for some $C > 0$. (Hint: integrate by parts.) (I learnt this from Exercise 2.25 in *Nonlinear dispersive equations* by T. Tao.)

(14) (Finite speed of propagation for non-constant coefficient wave equations) Consider the equation

$$-\partial_t^2\phi + \partial_i((h^{-1})^{ij}\sqrt{h}\partial_j\phi) = 0,$$

where h_{ij} is a positive definite matrix such that

$$\sum_{i,j=1}^n |h_{ij} - \delta_{ij}| \leq \frac{1}{2}.$$

Fix x_0. Suppose there is a smooth solution to

$$\sum_{i,j=1}^n (h^{-1})^{ij}\partial_i q \partial_j q = 1, \quad q(x_0) = 0$$

in $B(x_0, R)$ such that $q > 0$ in $B(x_0, R) \setminus \{x_0\}$. For every $r < R$, define the set

$$S := \{(t, x) : q(x) < r - t,\ 0 \leq t \leq r\}.$$

Show that if the initial data $(\phi_0, \phi_1) = (0, 0)$ in $\{x : q(x) \leq r\}$, then the solution $\phi = 0$ in S. Show that the same conclusion holds for solutions to

$$-\partial_t^2\phi + \partial_i((h^{-1})^{ij}\sqrt{h}\partial_j\phi) = (\partial_t\phi)^2$$

assuming that the solutions are smooth. (Notice that geometrically, S is the past of the point $(t = r, x_0)$.)

(15) In this problem, we consider the equation

$$\sum_{\mu,\nu=0}^{n} \frac{1}{\sqrt{-g}}\partial_\mu((g^{-1})^{\mu\nu}\sqrt{-g}\partial_\nu\phi) + \sum_{\mu=0}^{n} b^\mu\partial_\mu\phi + d\phi = F,$$

(i.e., it has an extra zeroth order $d\phi$ term compared to what is done in lecture) in $(3+1)$-dimensions. In the formula above, we have also used g to denote the determinant of g. (This convention will also be used throughout the example sheet.)

(a) In $\mathbb{R}^3$, prove the Hardy inequality

$$\|\frac{\phi}{|x|}\|_{L^2(\mathbb{R}^3)} \leq C\|\partial_r\phi\|_{L^2(\mathbb{R}^3)}$$

for ϕ smooth and compactly supported where $C > 0$ is independent of ϕ (or its support). (Hint: Use the fact that $0 \leq \int_0^\infty(\partial_r\phi + \beta\phi)^2 r^2 dr$ for all functions β. Choose $\beta(r)$ and integrate by parts.)

(b) Consider now the equation

$$\sum_{\mu,\nu=0}^{n} \frac{1}{\sqrt{-g}}\partial_\mu((g^{-1})^{\mu\nu}\sqrt{-g}\partial_\nu\phi) + \sum_{\mu=0}^{n} b^\mu\partial_\mu\phi + d\phi = F$$

in $(3+1)$-dimensions with finite energy data $(\phi_0, \phi_1) \in H^1 \times L^2$. Use the Hardy inequality to show that as long as

$$\sum_{\mu,\nu} |(g^{-1})^{\mu\nu}\sqrt{-g} - m^{\mu\nu}| + |\sqrt{-g} - 1| \leq \frac{1}{2},$$

and

$$\|(1+|x|)d\|_{L^1([0,T];L^\infty(\mathbb{R}^3))} + \|\partial(g^{-1}\sqrt{-g})\|_{L^1([0,T];L^\infty(\mathbb{R}^3))} + \|b\|_{L^1([0,T];L^\infty(\mathbb{R}^3))} = C_1 < \infty$$

and

$$\|F\|_{L^1([0,T];L^2(\mathbb{R}^3))} = C_2 < \infty,$$

the solution $(\phi, \partial_t\phi) \in L^\infty([0,T]; H^1(\mathbb{R}^3)) \times L^\infty([0,T]; L^2(\mathbb{R}^3))$. Moreover, prove that the $L^\infty([0,T]; H^1(\mathbb{R}^3)) \times L^\infty([0,T]; L^2(\mathbb{R}^3))$ norm is bounded by a constant depending only on C_1, C_2 and the $H^1 \times L^2$ norm of the initial data (ϕ_0, ϕ_1).

(16) (Geometric formulation of L^2 type estimates for the wave equation) This is intended for students who know some differential geometry. Given a wave equation

$$\Box_g\phi := \frac{1}{\sqrt{-g}}\partial_\mu((g^{-1})^{\mu\nu}\sqrt{-g}\partial_\nu\phi) = 0.$$

Introduce the stress-energy-momentum tensor

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(g^{-1})^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi.$$

(a) Show that if ϕ is a solution to the wave equation, then

$$\nabla^\mu T_{\mu\nu} := (g^{-1})^{\alpha\mu}\nabla_\alpha T_{\mu\nu} := (g^{-1})^{\alpha\mu}(\partial_\alpha T_{\mu\nu} + \Gamma^\beta_{\alpha\mu}T_{\beta\nu} + \Gamma^\beta_{\alpha\nu}T_{\beta\mu}) = 0.$$

Here, repeated indices are automatically summed over. Also, $\Gamma^\beta_{\alpha\mu}$ is the Christoffel symbol given by

$$\Gamma^\beta_{\alpha\mu} = \frac{1}{2}(g^{-1})^{\beta\sigma}(\partial_\alpha g_{\sigma\mu} + \partial_\mu g_{\alpha\sigma} - \partial_\sigma g_{\alpha\mu}).$$

(b) Given a vector field X which is Killing, i.e.,

$$(\mathcal{L}_X g)_{\mu\nu} := X^\alpha\partial_\alpha g_{\mu\nu} + g_{\alpha\mu}\partial_\nu X^\alpha + g_{\alpha\nu}\partial_\mu X^\alpha = 0$$

Show that div $(T(X,\cdot)) = 0$ (i.e., $(g^{-1})^{\alpha\beta}\nabla_\alpha(T_{\beta\gamma}X^\gamma) = 0$). Use the divergence theorem to get a conservation law.

(c) Assume that g is a metric given by $g_{tt} = -1$, $g_{ti} = 0$ for $i = 1, ..., n$ and $g_{ij} = h_{ij}$ for $i, j = 1, ..., n$, where h is a positive definite symmetric matrix such that $\sum_{i,j=1}^{n} |h_{ij} - \delta_{ij}| \leq \frac{1}{2}$. Moreover, assume that all components of g is independent of t. Show that an appropriately defined "energy" is conserved.

(17) Finally, please look for mistakes in the above problems and let me know about them immediately!