Predicative toposes revisited

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How constructive is a topos?

In this talk *topos* will mean: an elementary topos with a natural numbers object (nno).

Is a topos constructive?

- *Yes*, because the internal logic is, in general, only intuitionistic. Indeed, toposes are models of **HAH**.
- *No*, because the internal logic of toposes is impredicative. Therefore it is not constructive in the sense of Martin-Löf Type Theory (MLTT) or Aczel's constructive set theory **CZF**.

What makes toposes impredicative is the presence of subobject classifiers (and power objects, generally).

Could there be a predicative analogue of the notion of a topos?

A predicative topos?

What should a predicative topos be?

Desiderata:

- At least a locally cartesian closed pretopos with nno.
- Every topos should be a predicative topos.
- The setoids in MLTT should be an example of a predicative topos.
- The sets in **CZF** should be an example of a predicative topos.
- One should be able to do formal topology (predicative locale theory) internally in a predicative topos.
- Predicative toposes should be stable under sheaves and realizability.
- There should be interesting examples which are not toposes.

Not so easy to satisfy all of these!

Two dilemmas . . .

The main issues are:

- How much *induction* (inductively generated sets) do we add?
- How much *choice* do we add?

. . . and one real problem

- If you want to do locale theory predicatively, the general notion of a formal space is too wild: in fact, only those that are *set-presented* (in Peter Aczel's terminology) behave well. For example, building inside a model of **CZF** a category of sheaves over a formal space that is not set-presented, does not necessarily give rise to another model of **CZF** (Grayson).
- But many formal spaces cannot be shown to be set-presented inside **CZF**. For example, formal Baire space cannot be shown to be set-presented in **CZF**; and the same applies to inductively generated formal spaces generally.

To prove the statement "Inductively generated formal spaces are set-generated" we need some induction and choice.

Note: Peter Aczel solves this problem by introducing a new axiom (the Regular Extension Axiom **REA**). This axiom is highly set-theoretic and uncategorical and I will not follow him in this.

W-types

How to add some induction?

The obvious choice is: add the W-types of MLTT. As observed by Moerdijk and Palmgren, they are initial algebras for polynomial functors. I think this is a good idea and then we end up with: locally cartesian closed pretoposes with W-types, aka ΠW*-pretoposes*.

But we still seem to need some choice. Without some form of choice, we do not seem to be able to prove that

- inductively generated formal spaces are set-presented.
- taking sheaves over a set-presented formal space gives you a category which again has W-types.

Choice

How to add some choice?

If we follow the lead of MLTT again, we would add the axiom that predicative toposes have enough projectives (because that is what happens in setoids). But that would not be such a good idea, because this axiom is not stable under sheaves.

For solving precisely this issue, Moerdijk and Palmgren formulated their Axiom of Multiple Choice **AMC**. However, that axiom is quite complicated.

Fortunately, *a weaker and much simpler axiom does the job just as well*.

The axiom of multiple choice

It is this weaker and simpler version that I will call **AMC**. In set-theoretic terms, it says:

Axiom of Multiple Choice

For every set X there is a set of surjections onto X such that any surjection onto X is refined by one in this set (where we say that g is refined by f if f factors through g).

(On the nLab this axiom is called WISC.)

Formulating this axiom categorically is a bit of a hassle (in particular because we need to ensure its stability under slicing).

AMC categorically

Call

$$\begin{array}{ccc} D & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & A \end{array}$$

a *collection square*, if

- the square is a quasi-pullback (meaning that the inscribed map $D \to B \times_A C$ is epic).
- (in the internal logic it holds that) for every epi $p : E \to B_a$ there is a $c \in C_a$ such that $D_c \to B_a$ factors through p.

AMC categorically

For every map $f : Y \to X$ there is an epi $q : A \to X$ such that $q^* f$ fits into the right-hand side of a collection square.

Predicative topos

So this is my proposed definition of a predicative topos:

A *predicative topos* is ΠW-pretopos satisfying **AMC**.

Then:

- Internally to a predicative topos one can prove that inductively generated formal spaces are set-presented. And this allows one to develop a fair amount of formal topology.
- The category of setoids in MLTT is an example of a predicative topos.
- Predicative toposes are closed under internal sheaves and realizability.

Exact completion

We also have:

Theorem

Predicative toposes are closed under exact (ex/lex) completion. In fact, an exact completion of a ΠW-pretopos is a predicative topos.

This leads to many examples of predicative toposes that are not toposes, viz. exact completions of toposes.

Other examples

We also have:

Theorem

If $\mathcal{E}$ is a locally cartesian closed regular category with W-types satisfying **AMC**, then $\mathcal{E}_{ex/reg}$ is a predicative topos.

Therefore we have as other examples of predicative toposes that are not toposes:

- the subcountable objects in the effective topos.
- exact (ex/lex) completion of the category of topological spaces.

Conjecture

More examples can be built using typed pcas. Not every typed pca will be suitable, but categories of domains in which you have weakly initial algebras for functors of the form $FX = A \times (B \to X)$ will do the job.

An objection

But is every topos automatically also a predicative topos in my sense?

Probably not. Indeed, I conjecture:

Conjecture

ZF $\nvdash$ **AMC**.

This would follow from:

Conjecture

Internally to a predicative topos every algebraic theory has free algebras.

Theorem (Blass)

ZF cannot show that every algebraic theory has free algebras.

My response

- Note first of all that **AMC** follows from the axiom of choice, so that **ZFC** proves **AMC**. Since the validity of **AMC** is preserved by realizability and sheaves, every realizability or sheaf topos defined over $\mathcal{S}ets$ will satisfy **AMC**.
- It is actually quite desirable to have free algebras for every algebraic theory. So perhaps the problem is rather with the notion of a topos.

Summary

We have defined a predicative topos as a locally cartesian closed pretopos with W-types satisfying **AMC**. This was our list of desiderata:

- At least a locally cartesian closed pretopos with nno.
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