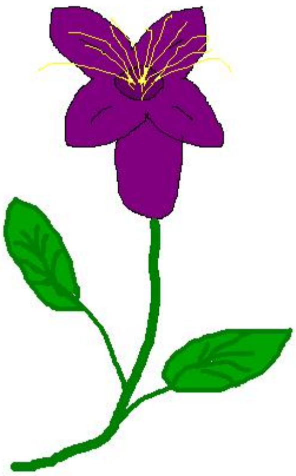
Category Theory

Julia Goedecke (jg352)



Contents

Preamble

The Notes

These notes are *not* verbatim what I will write on the blackboards. They will have more detail here and there, and more complete sentences. You can read ahead of lectures, you can use them for revision, you can use them to look up a little detail which you can't figure out from the more compressed notes from the actual lectures, and probably in many more ways. It is up to you to find out how they are most useful. If you don't like taking notes at all in lectures, use these. If you (like me) find that taking down notes in lectures is actually the best way to learn something, set these notes aside for a while and use them just to fill in gaps later. If you try to read these notes while I'm lecturing the same material, you may get confused and probably won't hear what I say.

There are probably still some errors and typos in the notes, please do let me know (jg352) if you find any, even if they look trivial. I would like to thank Zhen Lin Low, Tamar von Glehn and Achilleas Kryfties for helping me proofread the notes.

The Exam

Past papers will be a good guide to questions. The last two years were set by me; however 2012 was a bit too easy. The structure will be a choice of 5 questions out of 8 possibilities. There will be a mixture of some bookwork and some problem type questions. Examinable material covers not just the lectured material but also all material from the example sheets, and anything in the course which is left as an exercise.

Books

Here is a list of books which may be useful:

1. Mac Lane, S. *Categories for the Working Mathematician*, Springer 1971 (second edition 1998). Still the best one-volume book on the subject, written by one of its founders.
2. Awodey, S. *Category Theory*, Oxford U.P. 2006. A new treatment very much in the spirit of Mac Lane's classic, but rather more gently paced.
3. Borceux, F. *Handbook of Categorical Algebra*, Cambridge U.P. 1994. Three volumes which together provide the best modern account of everything an educated mathematician should know about categories: volume 1 covers most but not all of the Part III course.
4. McLarty, C. *Elementary Categories, Elementary Toposes* (chapters 1–12 only), Oxford U.P. 1992. A very gently-paced introduction to categorical ideas, written by a philosopher for those with little mathematical background.

To get into the subject, people have told me that the Awodey book is very good. Mac Lane is very dense but has a lot of material and examples in it (if you can find them), and Borceux suits my personal style the best, but there are some typos in it.

Example Sheets

There will be four example sheets. The questions vary in difficulty and length. You can find them on my website `https://www.dpmms.cam.ac.uk/~jg352/teaching`. Doing example sheet questions is the best way to understand the material. However, if you think the sheets are too long, just pick some of the questions. If you think the sheets are too short, find your own additional questions in books. You are responsible for your own learning, and these example sheets are just what I offer you to help your learning.

There will be examples classes, each with roughly 12 students in it. Arrangements will be advertised in lectures and on my website `https://www.dpmms.cam.ac.uk/~jg352/teaching.html`

The Course

What is Category Theory?

- ◇ It's one level more abstraction than other pure maths.

  One could call it "Mathematics about Mathematics". It is however still Mathematics! In pure maths, we for example abstract from symmetries of polyhedra to group theory and integers to ring theory, and in Category Theory we abstract from groups, rings, modules, ... to categories.
- ◇ It's a language for mathematicians.

  Notation is important! For example $\frac{\mathrm{d}}{\mathrm{d}x}$ suggests the right properties of differentiation. Category Theory is a subject-agnostic abstract notation system for pure mathematics.
- ◇ It's a way of thinking.

  We study structure, find common patterns, and try to understand how and why things work. We want to understand things so well that we can make them "look obvious". In this sense a lot of work goes into definitions!

Category Theory is not only interested in one particular mathematical object, but in how objects of a similar kind interact with each other, in global structures and connections. So for example we study morphisms of a similar kind such as sets or groups or modules, but with interaction between them, i.e. with *morphisms* of an appropriate kind as well.

To get a flavour of the "wider world" of Category Theory, you can go to the Category Theory Seminars, on Tuesdays, 2:15pm, in MR5. You may not understand everything or even anything, but you will still get an idea about what category theorists do. There is also the Junior Seminar (run by PhD students), which is on Thursdays 2pm. This should be more accessible to Part III students, and our PhD students are a very friendly and lively lot who will be happy to answer questions.

CHAPTER 1

Categories, Functors and Natural Transformations

A Categories

Definition: A **category** $\mathscr{C}$ consists of:

- a collection $\operatorname{ob}\mathscr{C}$ of **objects** (denoted $A, B, C, \ldots$)
- for each pair $A, B \in \operatorname{ob}\mathscr{C}$, a collection $\mathscr{C}(A,B) = \operatorname{Hom}_{\mathscr{C}}(A,B)$ of **morphisms** (denoted $f\colon A \longrightarrow B$, g, h,...)

equipped with

- for each $A \in \operatorname{ob}\mathscr{C}$, an identity morphism $\mathrm{id}_A = 1_A \in \mathscr{C}(A,A)$.
- for each $A, B, C \in \operatorname{ob}\mathscr{C}$, a composition law:

$$\begin{aligned} \mathscr{C}(A,B) \times \mathscr{C}(B,C) &\longrightarrow \mathscr{C}(A,C) \\ (f,g) &\longmapsto g\circ f = gf, \end{aligned}$$

satisfying

- identity axioms: if $f\colon A \longrightarrow B$, then $1_B\circ f = f = f\circ 1_A$.
- associativity: if $f\colon A \longrightarrow B$, $g\colon B \longrightarrow C$, $h\colon C \longrightarrow D$ then

$$h\circ(g\circ f) = (h\circ g)\circ f$$

Definition: A category $\mathscr{C}$ is said to be **small** if $\operatorname{ob}\mathscr{C}$ and all of the $\mathscr{C}(A,B)$ are sets, and **locally small** if each $\mathscr{C}(A,B)$ is a set (in which case we also call them "hom sets").

Remarks:

- If $f\colon A \longrightarrow B$, we call A the **domain** (or **source**) of f and B the **codomain** (or **target**) of f.
- Morphisms are also referred to as **maps** or **arrows**.
- Most of the time, we won't worry too much about the intricacies of set theory.
- We could define categories just considering morphisms (with the objects defined by the identities), but in most examples the objects "come first".
- We may write $\operatorname{mor}\mathscr{C}$ for the collection of all the morphisms in $\mathscr{C}$, and $\operatorname{dom}, \operatorname{cod}\colon \operatorname{mor}\mathscr{C} \longrightarrow \operatorname{ob}\mathscr{C}$ for the domain and codomain operations (see Example Sheet 1).

Definition: We say a square such as

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "g"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

is **commutative** (or **commutes**) when the composites $g\circ f$ and $k\circ h$ give the same morphism $A \longrightarrow D$.

This terminology also applies to other shapes of diagrams. To indicate that a diagram commutes, we often write a little square into it, or use ↻.

Examples:

a) Set of sets and functions.

b) Categories of algebraic structures such as:
  - Gp: groups and group homomorphisms,

- ◇ AbGp: abelian groups and group homomorphisms,
- ◇ Rng: rings and ring homomorphisms,
- ◇ R-Mod: R-modules and R-module homomorphisms for a given ring R.

c) Categories of topological structures such as:

- ◇ Top: topological spaces and continuous maps,
- ◇ Haus: Hausdorff spaces and continuous maps,
- ◇ Met: metric spaces and uniformly continuous maps (or Lipschitz maps, for a different category),
- ◇ Htpy: topological spaces and homotopy classes of continuous maps.

Note that the only maps we really *need* in a category (so as to have a category) are the identities.

Definition: A category with only identities is called **discrete**.

Examples: d) Mathematical structures viewed as categories:

- ◇ Sets: Any set can be viewed as a discrete category with the elements as objects.
- ◇ Posets: A poset $(P, \leqslant)$ can be regarded as a category with the elements of P as objects, and with $\mathrm{Hom}(a, b)$ being a singleton if $a \leqslant b$ and empty otherwise. Then reflexivity implies the existence of identity morphisms, and transitivity gives us composition.
  Any category in which there is at most one morphism between any two objects is a **preorder**. Note that a preorder doesn't need to satisfy antisymmetry.
- ◇ Monoids[1]: A locally small category with just one object is a monoid. The morphisms are the elements of the monoid, composition of morphisms is multiplication in the monoid and the identity morphism is the unit of multiplication.
- ◇ Groups: A group can be considered as a category with one object, just as for monoids. The difference is that every morphism now has a (two-sided) inverse.

Definition: A morphism $f\colon A \longrightarrow B$ in a category $\mathscr{C}$ is called an *isomorphism* if it has a two-sided inverse, i.e. a $g\colon B \longrightarrow A$ satisfying $gf = 1_A$ and $fg = 1_B$. A category in which every morphism is an isomorphism is called a **groupoid**.

This means that a group is a groupoid with only one object. Note that in a poset, only the identities are isomorphisms.

Examples:

- ◇ Iso$\mathscr{C}$: Any category gives rise to a groupoid: just take all objects and all isomorphisms.
- ◇ Fundamental groupoid: Given a space X, the fundamental groupoid $\pi(X)$ has objects the points of X, and morphisms $x \longrightarrow y$ are homotopy classes of continuous paths $u\colon [0,1] \longrightarrow X$ from x to y. Composition of $u\colon x \longrightarrow y$ with $v\colon y \longrightarrow z$ is defined as

$$vu(t) = \begin{cases} u(2t) & (0 \leqslant t \leqslant \frac{1}{2}) \\ v(2t-1) & (\frac{1}{2} \leqslant t \leqslant 1) \end{cases}$$

  The identity morphism is a constant path at x; inverses are paths traversed backwards.

1 Examples: ("New from old")

a) Given any category $\mathscr{C}$, the **opposite category** $\mathscr{C}^{\mathrm{op}}$ has the same objects and morphisms as $\mathscr{C}$, but the direction of the morphisms is reversed: $\mathscr{C}^{\mathrm{op}}(A, B) = \mathscr{C}(B, A)$. This gives us a **"duality principle"**: if some statement P holds in any category, so does the statement P^* obtained by "reversing all arrows in P".

---

[1] A monoid is like a group, but without inverses.

b) **Subcategories**: $\mathscr{D}$ is a subcategory of $\mathscr{C}$ if $\operatorname{ob}\mathscr{D} \subseteq \operatorname{ob}\mathscr{C}$ and for each $A, B \in \operatorname{ob}\mathscr{D}$, $\mathscr{D}(A,B) \subseteq \mathscr{C}(A,B)$. E.g. $\mathsf{AbGp} \hookrightarrow \mathsf{Gp}$.

c) **Product categories**: Given categories $\mathscr{C}$ and $\mathscr{D}$, the product $\mathscr{C} \times \mathscr{D}$ has objects (A,B) with $A \in \operatorname{ob}\mathscr{C}$ and $B \in \operatorname{ob}\mathscr{D}$, and morphisms $(f,g)\colon (A,B) \longrightarrow (C,D)$ with $f\colon A \longrightarrow C$ in $\mathscr{C}$ and $g\colon B \longrightarrow D$ in $\mathscr{D}$.

d) **Slice categories**: Given a category $\mathscr{C}$ and an object B of $\mathscr{C}$, the slice category $\mathscr{C}/B$ has as objects those morphisms in $\mathscr{C}$ with codomain B, and “morphisms are commutative triangles”:

$$h\colon \begin{pmatrix} A \\ \downarrow f \\ B \end{pmatrix} \longrightarrow \begin{pmatrix} C \\ \downarrow g \\ B \end{pmatrix} \qquad \text{satisfies}$$

$$\begin{tikzcd}
A \arrow[rr, "h"] \arrow[ddr, "f"'] & & C \arrow[ddl, "g"] \\
& \square & \\
& B &
\end{tikzcd}$$

Dually we have the **coslice category** $B\backslash\mathscr{C} = (\mathscr{C}^{\mathrm{op}}/B)^{\mathrm{op}}$ with

$$\begin{tikzcd}
& B \arrow[dl, "f"'] \arrow[dr, "g"] & \\
A \arrow[rr, "h"'] & & C.
\end{tikzcd}$$

For example:

- Set/B can be regarded as the category of “B-indexed families of sets”: An object $\begin{pmatrix} A \\ \downarrow f \\ B \end{pmatrix}$ may be identified with the family $(f^{-1}(b) \mid b \in B)$.
- $1\backslash\mathsf{Set}$ (with $1 = \{*\}$ a one-point set) is the category of **pointed sets**: objects are pairs (A, a) of sets with a distinguished element $a \in A$, and morphisms $f\colon (A,a) \longrightarrow (B,b)$ must preserve this: $f(a) = b$.

e) **Arrow categories**: Given a category $\mathscr{C}$, the arrow category $\operatorname{Arr}\mathscr{C}$ has as objects the morphisms of $\mathscr{C}$, and as morphisms commutative squares

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "v"] \\
C \arrow[r, "g"'] & D.
\end{tikzcd}$$

f) **Quotient categories**: Given an equivalence relation $\sim$ on each collection of morphisms $\mathscr{C}(A,B)$ of a category $\mathscr{C}$ satisfying

$$f \sim g \quad \Rightarrow \quad fh \sim gh \text{ and } kf \sim kg$$

whenever these composites are defined, then we can form the quotient category $\mathscr{C}/\sim$.

2 Examples: (“Unusual maps”)

Here are some categories where the morphisms are not just functions.

- *Matrices:* Given a field k, let Mat_k be the category with objects the natural numbers and $\mathsf{Mat}_k(n,m)$ being $m \times n$ matrices with entries in k. Then composition is matrix multiplication.
- *Relations:* Rel is the category which has sets as objects, and morphisms $A \longrightarrow B$ are triples (A,R,B) where $R \subseteq A \times B$ is an arbitrary subset (a relation on A and B). Composition of (A,R,B) and (B,S,C) is $(A, S \circ R, C)$ with

$$S \circ R = \{(a,c) \mid \exists b \in B \text{ s.t. } (a,b) \in R \text{ and } (b,c) \in S\}.$$

- *Partial functions:* Part has sets as objects and partial functions as morphisms. You can view a partial function as a relation $R \subseteq A \times B$ satisfying $((a,b) \in R \text{ and } (a,b') \in R) \Rightarrow b = b'$.

- *Formal proofs:* We can form a category Proofs with objects being logical statements (in some language) and morphisms being formal proofs of one statement from another (in a given logical system), modulo a suitable notion of equivalence.

3 Examples: (Finite categories)

a) A discrete category with 2 (or n) objects: $j \qquad j'$

b) A category with only one non-identity morphism: $j \longrightarrow j'$

c) A category with two non-identity morphisms: $j \rightrightarrows j'$

d)
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \\ \cdot & \end{tikzcd}$$

d')
$$\begin{tikzcd} & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

etc.

B Functors

Definition: Let C and $\mathscr{D}$ be categories. A **functor** $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ consists of:

- a mapping $A \longmapsto FA\colon \operatorname{ob}\mathscr{C} \longrightarrow \operatorname{ob}\mathscr{D}$ and
- mappings $f \longmapsto Ff\colon \mathscr{C}(A,B) \longrightarrow \mathscr{D}(FA,FB)$

such that

- $F1_A = 1_{FA}$ and
- $F(gf) = Fg \circ Ff$ (whenever gf is defined).

Examples: a) Any category $\mathscr{C}$ has an **identity functor**. We can also compose functors. This allows us to form the category Cat of small categories and functors between them.

b) If $\mathscr{D}$ is a subcategory of $\mathscr{C}$, there is an **inclusion functor** $\mathscr{D} \hookrightarrow \mathscr{C}$. If $\mathscr{A} \times \mathscr{B}$ is a product category, there are **projection functors** $\pi_1\colon \mathscr{A} \times \mathscr{B} \longrightarrow \mathscr{A}$ and $\pi_2\colon \mathscr{A} \times \mathscr{B} \longrightarrow \mathscr{B}$.

c) **Forgetful functors**: We can define a functor $U\colon \mathsf{Gp} \longrightarrow \mathsf{Set}$ which sends a group to its underlying set and a homomorphism to its underlying function: it "forgets" the group structure. Similarly, there are forgetful functors $\mathsf{Rng} \longrightarrow \mathsf{Set}$, R-$\mathsf{Mod} \longrightarrow \mathsf{Set}$, $\mathsf{Top} \longrightarrow \mathsf{Set}$, ... and $\mathsf{Rng} \longrightarrow \mathsf{Gp}$ forgetting the multiplication.

d) **Free functors**: For any set A, we can form the free group FA generated by A. Any function $f\colon A \longrightarrow B$ induces a unique group homomorphism $\overline{f}\colon FA \longrightarrow FB$ which sends any $a \in A$ to $f(a) \in B$. Given also $g\colon B \longrightarrow C$, we see that $\overline{gf} = \overline{g} \circ \overline{f}$, as they agree on the generators of FA. This gives a functor $F\colon \mathsf{Set} \longrightarrow \mathsf{Gp}$.

e) There is a functor $\mathsf{Set} \longrightarrow \mathsf{Top}$ sending a set X to the discrete space on X.

f) There is a functor $\operatorname{ab}\colon \mathsf{Gp} \longrightarrow \mathsf{AbGp}$ sending G to $G/[G,G]$, the **abelianisation functor**.

g) **Powerset functor**: Define $\mathscr{P}\colon \mathsf{Set} \longrightarrow \mathsf{Set}$ by setting $\mathscr{P}A$ to be the set of all subsets of A, and if $f\colon A \longrightarrow B$, then $(\mathscr{P}f)(A') = \{b \in B \mid \exists a \in A' \text{ s.t. } b = f(a)\} = f(A')$, the image of A' under f.

We can also make the powerset operation into a functor $\mathscr{P}^*\colon \mathsf{Set} \longrightarrow \mathsf{Set}^{\mathrm{op}}$ (or $\mathsf{Set}^{\mathrm{op}} \longrightarrow \mathsf{Set}$) by setting $(\mathscr{P}^* f)(B') = f^{-1}(B)$. Check that $\mathscr{P}^*(fg) = \mathscr{P}^*(g) \circ \mathscr{P}^*(f)$.

Definition: A **contravariant** functor from $\mathscr{C}$ to $\mathscr{D}$ is a functor $\mathscr{C}^{\mathrm{op}} \longrightarrow \mathscr{D}$ (or $\mathscr{C} \longrightarrow \mathscr{D}^{\mathrm{op}}$). A functor which does not reverse the direction of arrows is also called **covariant**.

Examples: h) **Duals**: Given a field k, we can form a functor $(-)^*\colon k$-$\mathsf{Mod} \longrightarrow k$-$\mathsf{Mod}^{\mathrm{op}}$ by sending a vectorspace V to its dual vectorspace V^* and a linear map $f\colon V \longrightarrow W$ to $f^*\colon W^* \longrightarrow V^*$, which sends a linear functional $\phi \in W^*$ to $\phi f \in V^*$.

Similarly, there is a functor $(-)^*\colon \mathsf{Rel} \longrightarrow \mathsf{Rel}^{\mathrm{op}}$ defined on objects by $A^* = A$ and on morphisms by $R^* = \{(b,a) \mid (a,b) \in R\}$.

i) We can regard the operation $\mathscr{C} \longmapsto \mathscr{C}^{\mathrm{op}}$ as a functor $\mathsf{Cat} \longrightarrow \mathsf{Cat}$. If F is a functor $F\colon \mathscr{C} \longrightarrow \mathscr{D}$, then F^{op} denotes the same data regarded as a functor $\mathscr{C}^{\mathrm{op}} \longrightarrow \mathscr{D}^{\mathrm{op}}$. Note that this is a covariant functor!

j) A functor between monoids is a monoid homomorphism.

k) A functor between partially orderd sets is an order-preserving map.

l) **Hom-functor**: Given a locally small category $\mathscr{C}$, there is, for every object A of C, a **hom-functor** $\mathscr{C}(A,-)\colon \mathscr{C} \longrightarrow \mathsf{Set}$:

$\mathscr{C}(A,-)$ applied to an object B gives the set $\mathscr{C}(A,B)$. $\mathscr{C}(A,-)$ applied to $g\colon C \longrightarrow D$ gives "post-composition with g":

$$\mathscr{C}(A,g)\colon \mathscr{C}(A,C) \longrightarrow \mathscr{C}(A,D)$$

$$A\xrightarrow{f}C \longmapsto A\xrightarrow{f}C\xrightarrow{g}D$$

Simiarly, we have a contravariant hom-functor $\mathscr{C}(-,A)\colon \mathscr{C}^{\mathrm{op}} \longrightarrow \mathsf{Set}$.

m) Let $\mathscr{G}$ be a group, considered as a category with one object $*$. What is a functor $\mathscr{G} \longrightarrow \mathsf{Set}$? We have a set $A = F(*)$ and for each $g \in \mathscr{G}$, a function $\overline{g} = Fg\colon A \longrightarrow A$ satisfying $\overline{1} = 1_A$ and $\overline{g}\overline{h} = \overline{gh}$. This forces $\overline{g^{-1}} = (\overline{g})^{-1}$, so all $\overline{g}$ are bijections. So F is a **permutation representation** (or **action**) of $\mathscr{G}$ on the set A. Similarly, for a given field k, functors $\mathscr{G} \longrightarrow k\text{-}\mathsf{Mod}$ are the same thing as k-linear representations of $\mathscr{G}$.

n) The fundamental group of a space defines a functor

$$\pi_1\colon (1\backslash\mathsf{Top}) \longrightarrow \mathsf{Gp}$$

(in fact $(1\backslash\mathsf{Top})/\sim \longrightarrow \mathsf{Gp}$ where $\sim$ is base-point preserving homotopy).

The homology groups define functors

$$H_n\colon \mathsf{Top}/\sim \longrightarrow \mathsf{Gp}$$

(in fact $H_n\colon \mathsf{Top}/\sim \longrightarrow \mathsf{AbGp}$).

Remark: Functors preserve commutative diagrams, so also properties defined by commutative diagrams, such as isomorphisms.

C Natural Transformations

Natural transformations give a way of "moving between the images of two functors".

Definition: Let $\mathscr{C},\mathscr{D}$ be categories and $F,G\colon \mathscr{C} \longrightarrow \mathscr{D}$ two functors. A **natural transformation** α from F to G is a collection of morphisms in $\mathscr{D}$ $\{\alpha_A\colon FA \longrightarrow GA \mid A \in \mathrm{ob}\,\mathscr{C}\}$ satisfying $(Gf)\circ\alpha_A = \alpha_B\circ(Ff)$ for all $f\colon A \longrightarrow B$ in $\mathscr{C}$.

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\square"] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\alpha_B"'] & GB \end{tikzcd}$$

(Naturality condition)

If $\beta\colon G \longrightarrow H$ is another natural transformation, then the composite $\beta\alpha$ (given by $(\beta\alpha)_A = \beta_A\alpha_A$) is also natural[2]. For every functor F, there is an identity natural transformation $1_F\colon F \longrightarrow F$. So, given two categories $\mathscr{C}$ and $\mathscr{D}$, we have a **functor category** $[\mathscr{C},\mathscr{D}]$: objects are functors $F\colon \mathscr{C} \longrightarrow \mathscr{D}$, morphisms are natural transformations between them. Note that $[\mathscr{C},\mathscr{D}](F,G) = \mathrm{Nat}(F,G)$ is the class of natural transformations from F to G.

If each α_A is an isomorphism in $\mathscr{D}$, then we have another natural transformation $G \longrightarrow F$ given by $\{\alpha_A^{-1}\colon GA \longrightarrow FA\}$, since

$$(Ff)\alpha_A^{-1} = (\alpha_B^{-1})\alpha_B(Ff)\alpha_1^{-1} = \alpha_B^{-1}(Gf)\alpha_A(\alpha_A^{-1}) = \alpha_B^{-1}(Gf).$$

This makes α an isomorphism in $[\mathscr{C},\mathscr{D}]$, and we call it a **natural isomorphism**.

---

[2] This is called "vertical composition". For another way of composing natural transformations, see Example Sheet 1.

Examples:

a) For any vectorspace V we have a "natural" mapping $\alpha_V\colon V \longrightarrow V^{**}$ sending $v \in V$ to $(\phi \longmapsto \phi(v))$. This is the V-component of a natural transformation $1_{k\text{-}\mathsf{Mod}} \longrightarrow (-)^{**}$, i.e. for any linear map $f\colon V \longrightarrow W$, the diagram

$$\begin{tikzcd}
V \arrow[r, "\alpha_V"] \arrow[d, "f"'] & V^{**} \arrow[d, "f^{**}"] \\
W \arrow[r, "\alpha_W"'] & W^{**}
\end{tikzcd}$$

commutes.

b) Recall the covariant powerset-functor $\mathscr{P}\colon \mathsf{Set} \longrightarrow \mathsf{Set}$. For each set A, let $\{\ \}_A\colon A \longrightarrow \mathscr{P}A$ be the function $a \longmapsto \{a\}$. Then $\{\ \}$ is a natural transformation $1_{\mathsf{Set}} \longrightarrow \mathscr{P}$.

c) Let G, H be groups and $f, g\colon G \longrightarrow H$ group homomorphisms. A natural transformation $\alpha\colon f \longrightarrow g$ consists of an element $c = \alpha_* \in H$ such that, for any $x \in G$, we have

$$\begin{tikzcd}
* \arrow[r, "c"] \arrow[d, "f(x)"'] \arrow[dr, phantom, "\square"] & * \arrow[d, "g(x)"] \\
* \arrow[r, "c"'] & *
\end{tikzcd}$$

i.e. $g(x) = cf(x)c^{-1}$, so α is a conjugacy between f and g.

d) The Hurewicz homomorphism

$$h\colon \pi_n(X, x) \longrightarrow H_n(X)$$

is a natural transformation $\pi_n \longrightarrow IH_nU$, where $U\colon (1\backslash\mathsf{Top})/\sim \longrightarrow \mathsf{Top}/\sim$ forgets the basepoint and $I\colon \mathsf{AbGp} \longrightarrow \mathsf{Gp}$ is the inclusion.

D Equivalences

Definition: Let $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ be a functor.

a) We say F is **faithful** if, for each $A \underset{g}{\overset{f}{\rightrightarrows}} B$ in $\mathscr{C}$, the equation $Ff = Fg$ implies $f = g$. (i.e. "F": $\mathscr{C}(A, B) \longrightarrow \mathscr{D}(FA, FB)$ is injective.)

b) We say F is **full** if, for all objects A, B of $\mathscr{C}$ and morphisms $h\colon FA \longrightarrow FB$ in $\mathscr{D}$, there exists $f\colon A \longrightarrow B$ with $Ff = h$. ($\mathscr{C}(A, B) \longrightarrow \mathscr{D}(FA, FB)$ is surjective.)

c) We say F is **essentially surjective on objects** if for every $B \in \operatorname{ob}\mathscr{D}$, there exists $A \in \operatorname{ob}\mathscr{C}$ with $FA \cong B$.

d) We say a subcategory $\mathscr{C}'$ of $\mathscr{C}$ is **full** if the inclusion functor $\mathscr{C}' \longrightarrow \mathscr{C}$ is a full functor (i.e. $\mathscr{C}'(A, B) = \mathscr{C}(A, B)$ for all $A, B \in \mathscr{C}'$).

For example, Gp is a full subcategory of the category Mon of monoids, but Mon is not a full subcategory of semigroups[3].

Definition: Let $\mathscr{C}$ and $\mathscr{D}$ be categories. An **equivalence** between $\mathscr{C}$ and $\mathscr{D}$ is a pair of functors $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ and $G\colon \mathscr{D} \longrightarrow \mathscr{C}$ together with a pair of natural isomorphisms $\alpha\colon 1_{\mathscr{C}} \longrightarrow GF$ and $\beta\colon 1_{\mathscr{D}} \longrightarrow FG$. We say $\mathscr{C}$ and $\mathscr{D}$ are **equivalent**, write $\mathscr{C} \simeq \mathscr{D}$, if there is an equivalence between them.

4 Lemma: ("equivalence ⇔ f.f.+e.s.")

Let $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ be a functor.

i) If F is part of an equivalence (F, G, α, β), then F is full, faithful and essentially surjective on objects.

ii) The converse holds if we assume a 'sufficiently big' axiom of choice.

---

[3] Semigroups are monoids but not necessarily with a unit. Semigroup homomorphisms need not preserve the 1 in a monoid.

PROOF. i) F **faithful:** For any $f\colon A \longrightarrow B$ in $\mathscr{C}$, we can recover f from Ff:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha_A", "\cong"'] & B \arrow[d, "\alpha_B", "\cong"'] \\
GFA \arrow[r, "GFf"'] & GFB
\end{tikzcd}$$

So $f = \alpha_B^{-1} \circ GFf \circ \alpha_A$. So $Ff = Fg$ implies $f = g$. (Of course, this also shows that G is faithful.)

F **full:** Given $h\colon FA \longrightarrow FB$, define $f = \alpha_B^{-1} Gh \alpha_A$:

$$\begin{tikzcd}
A \arrow[r, dotted, "f"] \arrow[d, "\alpha_A", "\cong"'] & B \arrow[d, "\alpha_B", "\cong"'] \\
GFA \arrow[r, "Gh"'] & GFB
\end{tikzcd}$$

Then f also equals $\alpha_B^{-1} \circ (GFf) \circ \alpha_A$ as above, so $GFf = Gh$. But G is faithful by the above, so $h = Ff$ as required.

F **essentially surjective:** Given $B \in \operatorname{ob} \mathscr{D}$, we have an iso $\beta_B\colon B \longrightarrow FGB$.

ii) Suppose that F is full, faithful an essentially surjective. We construct a functor G and a natural iso $\beta\colon 1_{\mathscr{D}} \longrightarrow FG$: For each $C \in \operatorname{ob} \mathscr{D}$, choose a pair (GC, β_C) such that β_C is an iso $C \longrightarrow FGC$ in $\mathscr{D}$. (We can do this because F is essentially surjective.) Given $h\colon C \longrightarrow D$, the composite

$$\begin{tikzcd}
C \arrow[r, "h"] & D \arrow[d, "\beta_D", "\cong"'] \\
FGC \arrow[u, "\beta_C^{-1}", "\cong"'] \arrow[r, dotted, "F(Gh)"'] & FGD
\end{tikzcd}$$

can be written as $F(Gh)$ for a unique $Gh\colon GC \longrightarrow GD$ in $\mathscr{C}$, as F is full and faithful. We check whether G really is a functor: given $h'\colon D \longrightarrow E$, both $G(h'h)$ and $Gh' \circ Gh$ are the unique f that make

$$\begin{tikzcd}
C \arrow[r, "h'h"] \arrow[d, "\beta_C", "\cong"'] & E \arrow[d, "\beta_E", "\cong"'] \\
FGC \arrow[r, "Ff"'] & FGE
\end{tikzcd}$$

commute, so they must be equal.

By construction, β is a natural transformation $1_D \longrightarrow FG$. We obtain α_A from the component $\beta_{FA}\colon FA \longrightarrow FGFA$: as F is full and faithful, $\beta_{FA} = F(\alpha_A)$ for a unique $\alpha_A\colon A \longrightarrow GFA$. The facts that α_A is an isomorphism and that α is natural follow from F being full and faithful (**Exercise**).

□

Examples: a) The category Set/B is equivalent to Set^B (B-indexed families of sets). In one direction, the equivalence sends $f\colon A \longrightarrow B$ to $(f^{-1}(b) \mid b \in B)$ (c.f. Examples 1 "New from Old") and a morphism

$$\begin{tikzcd}
A \arrow[rr, "h"] \arrow[dr, "f"'] & & A' \arrow[dl, "f'"] \\
& B &
\end{tikzcd}$$

to the family $(h|_{f^{-1}(b)} \mid b \in B)$. In the other direction, we send $(A_b \mid b \in B)$ to the disjoint union $\coprod_{b \in B} A_b = \bigcup_{b \in B} \{A_b \times \{b\}\}$ equipped with its projection to B.

b) For a field k, the categories $k\text{-}\mathsf{Mod}_{\text{f.d.}}$ and $k\text{-}\mathsf{Mod}_{\text{f.d.}}^{\text{op}}$ (of finite dimensional vectorspaces and its opposite) are equivalent. The functors in both directions are $V \longmapsto V^*$, and the isomorphism $V \longrightarrow V^{**}$ is that of Example a) in Natural Transformations (1C).

c) The category Mat_k from the "unusual maps" Example 2 is equivalent to $k\text{-}\mathsf{Mod}_{\text{f.d.}}$: The functor $F\colon \mathsf{Mat}_k \longrightarrow k\text{-}\mathsf{Mod}_{\text{f.d.}}$ sends n to k^n and a matrix M to the linear map it presents with respect to the standard bases. To define a functor G in the other direction, we need to choose a basis for each finite dimensional vectorspace: $GV = \dim V$, and $G(f\colon V \longrightarrow W)$ is the matrix representing f wrt. our chosen bases. GF is the identity functor (if we choose the standard basis), and the chosen bases give us a natural isomorphism $1 \longrightarrow FG$.

E Representable Functors

Recall the hom-functors $\mathscr{C}(A,-)\colon \mathscr{C} \longrightarrow \mathsf{Set}$. We can put all these together into a functor:

Definition: Let $\mathscr{C}$ be a locally small category. We define a functor $Y\colon \mathscr{C}^{\text{op}} \longrightarrow [\mathscr{C}, \mathsf{Set}]$, called the **Yoneda embedding**, by setting $YA = \mathscr{C}(A,-)$, and $Y(f\colon A \longrightarrow B)$ is the natural transformation with components $(Yf)_C\colon \mathscr{C}(B,C) \xrightarrow{-\circ f} \mathscr{C}(A,C)$.[4]

Remark: We could also define a similar functor $\mathscr{C} \longrightarrow [\mathscr{C}^{\text{op}}, \mathsf{Set}]$.

We should check that Yf is really a natural transformation and Y is really a functor. Given $f\colon A \longrightarrow B$ and $g\colon C \longrightarrow D$, we need

$$\begin{tikzcd}
\mathscr{C}(B,C) \arrow[r, "-\circ f", "(Yf)_C"'] \arrow[d, "\mathscr{C}(B,g)"', "g\circ -"] & \mathscr{C}(A,C) \arrow[d, "\mathscr{C}(A,g)"', "g\circ -"] \\
\mathscr{C}(B,D) \arrow[r, "-\circ f", "(Yf)_D"'] & \mathscr{C}(A,D)
\end{tikzcd}$$

to commute. A morphism $h\colon B \longrightarrow C$ is sent to $g(hf)$ and $(gh)f$ respectively, so by associativity of composition, Yf really is a natural transformation. Similarly associativity of composition also implies that Y is a functor. (Check it!)

What is so special about the hom-functors $\mathscr{C}(A,-)$?

Given a natural transformation $\alpha\colon \mathscr{C}(A,-) \longrightarrow F$, let us look at the naturality square

$$\begin{tikzcd}
\mathscr{C}(A,A) \arrow[r, "\alpha_A"] \arrow[d, "f\circ -"'] & FA \arrow[d, "Ff"] \\
\mathscr{C}(A,B) \arrow[r, "\alpha_B"'] & FB
\end{tikzcd}$$

for some $f\colon A \longrightarrow B$. We see that

$$\alpha_B(f\circ 1_A) = Ff(\alpha_A(1_A)),$$

i.e. $\alpha_B(f)$ is completely determined by $\alpha_A(1_A)$, so α itself is completely determined by the element $\alpha_A(1_A) \in FA$.[5]

5 Theorem: (Yoneda Lemma)
Let $\mathscr{C}$ be a locally small category, $A \in \operatorname{ob}\mathscr{C}$ and $F\colon \mathscr{C} \longrightarrow \mathsf{Set}$ a functor. Then there is a bijection

$$\theta\colon \mathrm{Nat}(\mathscr{C}(A,-), F) \longrightarrow FA$$

between natural transformations $\mathscr{C}(A,-) \longrightarrow F$ and elements of FA. Moreover, this bijection is natural in A and F.

---

[4] Y is contravariant!

[5] Think of a group homomorphism $\mathbb{Z} \longrightarrow G$ being determined by where 1 goes.

Proof. Given a natural transformation $\alpha\colon \mathscr{C}(A,-) \longrightarrow F$, we set $\theta(\alpha) = \alpha_A(1_A)$.

Given an element $x \in FA$, we define a natural transformation $\psi(x)\colon \mathscr{C}(A,-) \longrightarrow F$ by $\psi(x)_B(f) = Ff(x)$, i.e.

$$\begin{aligned} \psi(x)_B\colon \mathscr{C}(A,B) &\longrightarrow FB \\ f &\longmapsto Ff(x) \end{aligned}$$

We check that $\psi(x)$ really is a natural transformation:

Given $g\colon B \longrightarrow C$ in $\mathscr{C}$, consider

$$\begin{tikzcd} \mathscr{C}(A,B) \arrow[r, "\psi(x)_B"] \arrow[d, "g\circ-"', "\mathscr{C}(A{,}g)"] & FB \arrow[d, "Fg"] \\ \mathscr{C}(A,C) \arrow[r, "\psi(x)_C"'] & FC \end{tikzcd}$$

Chasing $f \in \mathscr{C}(A,B)$ around the diagram, we see that we need $Fg(Ff(x)) = F(gf)(x)$, which is true as F is a functor.

We now show that θ and ψ are inverse to each other:

◇ $\psi(\theta(\alpha))_B = \psi(\alpha_A(1_A))_B\colon \mathscr{C}(A,B) \longrightarrow FB$ sends $f\colon A \longrightarrow B$ to $Ff(\alpha_A(1_A)) = \alpha_B(f\circ 1_A) = \alpha_B(f)$. So $\psi(\theta(\alpha)) = \alpha$.

◇ $\theta(\psi(x)) = \psi(x)_A(1_A) = F1_A(x) = 1_{FA}(x) = x$.

We now fix F and show that θ is natural in A:

Given $f\colon A \longrightarrow B$, we have a square

$$\begin{tikzcd} \mathrm{Nat}(\mathscr{C}(A,-),F) \arrow[r, "\theta_A"] \arrow[d, "-\circ Y(f)"'] & FA \arrow[d, "Ff"] \\ \mathrm{Nat}(\mathscr{C}(B,-),F) \arrow[r, "\theta_B"'] & FB \end{tikzcd}$$

A natural transformation $\alpha \in \mathrm{Nat}(\mathscr{C}(A,-),F)$ is mapped to $\theta_B(\alpha\circ Y(f)) = \alpha_B\circ Y(f)_B(1_B)$, going down and then across. Now $Y(f)_B = -\circ f$, so we get $\alpha_B Y(f)_B(1_B) = \alpha_B(f)$.

$$\begin{tikzcd} \mathscr{C}(B,B) \arrow[r, "-\circ f"] & \mathscr{C}(A,B) \arrow[r, "\alpha_B"] & FB \\ 1_B \arrow[r, mapsto] & f \arrow[r, mapsto] & \alpha_B(f) \end{tikzcd}$$

On the other hand, $Ff\circ\theta_A(\alpha) = Ff\circ\alpha_A(1_A) = \alpha_B(f)^6$ So the square above commutes, and θ is natural in A.

Exercise: Check that θ is natural in F for fixed A. □

Remark: This means that θ is a natural transformation $\mathrm{Nat}(\mathscr{C}(\,\cdot\,,-),F) \longrightarrow F$ for fixed F and also $\mathrm{Nat}(\mathscr{C}(A,-),\,\cdot\,) \longrightarrow \mathrm{ev}_A$ for fixed A, where ev_A means **evaluation at** A. These can also be combined into a more complicated natural transformation.

Definition: A functor $F\colon \mathscr{C} \longrightarrow \mathsf{Set}$ is called **representable** if it is isomorphic to $\mathscr{C}(A,-)$ for some $A \in \mathrm{ob}\,\mathscr{C}$. A **representation** of F is a pair (A,x), where $A \in \mathrm{ob}\,\mathscr{C}$, $x \in FA$ and $\psi(x)$ is a natural isomorphism $\mathscr{C}(A,-) \longrightarrow F$). We also call x a universal element of F.

Corollary: *The Yoneda embedding is full and faithful.*

Proof. Putting $F = \mathscr{C}(B,-)$ in the Yoneda Lemma gives us a bijection between morphisms $\mathscr{C}(A,-) \longrightarrow \mathscr{C}(B,-)$ in $[\mathscr{C},\mathsf{Set}]$ and elements in $\mathscr{C}(B,A)$, i.e. morphisms $B \longrightarrow A$ in $\mathscr{C}$. The inverse is exactly the action of the Yoneda embedding on morphisms. (Check this!) This shows that the Yoneda embedding is full and faithful. □

---

[6] Use the square before the statement of the Yoneda Lemma.

6 Corollary: ("Representations are unique up to unique isomorphism.")
If (A, x) *and* (B, y) *are both representations of* $F\colon \mathscr{C} \longrightarrow \mathsf{Set}$*, then there is a unique isomorphism* $f\colon A \longrightarrow B$ *in* $\mathscr{C}$ *with* $Ff(x) = y$*.*

Proof. We have a composite isomorphism

$$\mathscr{C}(B,-) \xrightarrow{\psi(y)} F \xrightarrow{\psi(x)^{-1}} \mathscr{C}(A,-).$$

As the Yoneda embedding is full and faithful, this is of the form $Y(f)$ for a unique isomorphism $f\colon A \longrightarrow B$ in C (c.f. Example Sheet 1 Question 1(e)). So $Y(f) = \psi(x)^{-1}\psi(y)$, or $\psi(x)Y(f) = \psi(y)$. Via the bijection in the Yoneda Lemma this is equivalent to $Ff(x) = y$. ☐

Examples:

a) The forgetful functor $\mathsf{Gp} \longrightarrow \mathsf{Set}$ is representable by $(\mathbb{Z}, 1)$, since homomorphisms $f\colon \mathbb{Z} \longrightarrow G$ correspond bijectively to elements $f(1)$ of the underlying set of G. Similarly, $\mathsf{Rng} \longrightarrow \mathsf{Set}$ is representable by $(\mathbb{Z}[x], x)$, etc.

b) The covariant powerset functor $\mathscr{P}\colon \mathsf{Set} \longrightarrow \mathsf{Set}$ isn't representable. (**Exercise:** prove this!) But $\mathscr{P}^*\colon \mathsf{Set}^{\mathrm{op}} \longrightarrow \mathsf{Set}$ is represented by $(2, \{1\})$, where $2 = \{0, 1\}$, since subsets $A' \subseteq A$ correspond bijectively to (indicator) functions $\chi'_A\colon A \longrightarrow 2$.

c) The dual-space functor $(\ \ )^*\colon k\text{-}\mathsf{Mod}^{\mathrm{op}} \longrightarrow k\text{-}\mathsf{Mod}$, when composed with the forgetful functor $k\text{-}\mathsf{Mod} \longrightarrow \mathsf{Set}$, is representable by $(k, 1_k)$.

CHAPTER 2

Limits and Colimits

A Terminal objects and Products

Definition: A **terminal object** in a category $\mathscr{C}$ is an object 1 such that for every object $A \in \operatorname{ob}\mathscr{C}$, there is a unique morphism $A \longrightarrow 1$.[1]

Proposition: *Any terminal object is unique up to unique isomorphism.*

Proof. Suppose 1 and $1'$ are two terminal objects in the category $\mathscr{C}$. Then there is a unique morphism $f\colon 1 \longrightarrow 1'$ and a unique morphism $g\colon 1' \longrightarrow 1$. This gives a morphism $gf\colon 1 \longrightarrow 1$, but as there is a *unique* morphism $1 \longrightarrow 1$, we must have $gf = \mathrm{id}_1$. Similarly $fg = \mathrm{id}_{1'}$, so 1 and $1'$ are isomorphic. □

The dual notion is an **initial object**: 0 is initial if there is a unique morphism $0 \longrightarrow A$ for each object A.

Examples: In Set, any one-element set is terminal, and of course they are all isomorphic. The empty set is initial.

In Top, the one-element topological space is terminal and the empty topological space is initial.

In Gp, the one-element group is both initial add terminal. We write it as 0 $(= \{*\})$ and call it a **zero object**. Similarly in R-Mod.

In Rng, the one-element ring is terminal, and $\mathbb{Z}$ is initial.

Definition: A **product** of two objects $A, B \in \operatorname{ob}\mathscr{C}$ is a triple (P, π_A, π_B) of an object P in $\mathscr{C}$ and two morphisms $\pi_A\colon P \longrightarrow A$ and $\pi_B\colon P \longrightarrow B$, such that, if there is any other triple $(C, f\colon C \longrightarrow A, g\colon C \longrightarrow B)$, then there is a unique morphism $c\colon C \longrightarrow P$ such that $\pi_A c = f$ and $\pi_B c = g$.[2]

Proposition: *A product of A and B is unique up to unique isomorphism.*

Proof. Similar to terminal object, or note:

A product of A and B is a representation of the functor $C \longmapsto \mathscr{C}(C, A) \times \mathscr{C}(C, B)\colon \mathscr{C}^{\mathrm{op}} \longrightarrow \mathsf{Set}$. We already saw that representations are unique up to unique isomorphism. □

We write $A \times B$ for "the" product of A and B.

Examples: In Set, the product of two sets A, B is their cartesian product

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

In Gp, R-Mod, Rng, Top, ... we can equip the cartesian product with the appropriate structure.

In Proofs, "and" is the product.

This generalises to products of any family of objects.

---

[1] We call this a **universal property**.

[2] Another universal property.

The dual notion is a **coproduct**: $(A+B, \iota_A, \iota_B)$ with $\iota_A\colon A \longrightarrow A+B$, $\iota_B\colon B \longrightarrow A+B$ such that for any C with $f\colon A \longrightarrow C$ and $g\colon B \longrightarrow C$ there is a unique morphism $h\colon A+B \longrightarrow C$ such that $h\iota_A = f$ and $h\iota_B = g$.

Examples: In Set, the coproduct $A+B$ is the disjoint union $A \sqcup B$. The same will work in Top, but not in Gp: There the coproduct $A+B$ is the free product $A * B$.

In R-Mod (and AbGp), the coproduct is the same as the product. We also call it **biproduct** or **direct sum** and write $A \oplus B$.

In Proofs, "or" is the coproduct.

B Cones and Limits

Terminal objects and products are examples of limits, which we shall now define.

Definition: Let $\mathscr{J}$ be a particular category (usually small, often finite). A **diagram of shape $\mathscr{J}$ in $\mathscr{C}$** is a functor $\mathscr{J} \longrightarrow \mathscr{C}$.

Remember the examples of finite categories from Section 1A (Example 3). If $\mathscr{J} = (\,\cdot \rightrightarrows \cdot\,)$, a diagram of shape $\mathscr{J}$ is a pair of parallel arrows $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in $\mathscr{C}$. If $\mathscr{J} = \left(\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}\right)$, then a diagram of shape $\mathscr{J}$ is a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

in $\mathscr{C}$.

We sometimes call the objects **vertices** and the morphisms **edges** of the diagram.

Definition: Let $D\colon \mathscr{J} \longrightarrow \mathscr{C}$ be a diagram. A **cone** over D is an object $A \in \mathscr{C}$ together with morphisms (called **legs**) $\mu_j\colon A \longrightarrow D(j)$ for all $j \in \operatorname{ob}\mathscr{J}$, such that for any morphism $\alpha\colon j \longrightarrow j'$ in $\mathscr{J}$, the triangle

$$\begin{tikzcd} & A \arrow[dl, "\mu_j"'] \arrow[dr, "\mu_{j'}"] & \\ D(j) \arrow[rr, "D(\alpha)"'] & & D(j') \end{tikzcd}$$

commutes (i.e. $D(\alpha)\mu_j = \mu_{j'}$).

Remark: A cone is really a special sort of natural transformation. Consider the **constant functor** $\Delta_A\colon \mathscr{J} \longrightarrow \mathscr{C}$ which sends each $j \in \operatorname{ob}\mathscr{J}$ to $A \in \operatorname{ob}\mathscr{C}$ and each morphism α to 1_A in $\mathscr{C}$. Then a cone is a natural transformation $\mu\colon \Delta_A \longrightarrow D$.[3]

Definition: Given two cones (A,μ) and (B,ν) over a diagram D, a **morphism of cones** is a morphism $f\colon A \longrightarrow B$ such that

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\mu_j"'] & & B \arrow[dl, "\nu_j"] \\ & D(j) & \end{tikzcd}$$

commutes for all $j \in \operatorname{ob}\mathscr{J}$.

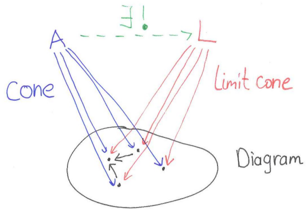
---

[3] One side of the naturality square collapses to give a triangle.

The cones over a particular diagram form a category.

Definition: A **limit** of D is a terminal cone, i.e. a terminal object in this category of cones (often written $(\lambda_j : L \longrightarrow D(j))_{j \in \mathrm{ob}\,\mathscr{J}}$).

In pictures:



Dually, we have cocones under a diagram D (some people just say cone under D), and a **colimit** is an initial cocone.

Proposition: *Limits (and colimits) are unique up to unique isomorphism.*

Proof. Exercise. ☐

So we can speak of "the" limit of D (if it exists). We say $\mathscr{C}$ **has limits of shape** $\mathscr{J}$ if any diagram $D\colon \mathscr{J} \longrightarrow \mathscr{C}$ has a limit.

Examples:

a) A terminal object is the limit of the empty diagram. A product is the limit of a discrete diagram with two objects. More generally, we say product for the limit of any discrete diagram. We write $\prod_{j \in \mathrm{ob}\,\mathscr{J}} D(j)$ (or e.g. $\prod_{i \leqslant n} A_i$). The legs are called **product projections**.

b) The limit of a diagram of shape $\bullet \rightrightarrows \bullet$ is called an **equaliser**: Given a pair of arrows $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in $\mathscr{C}$, a cone over this diagram is

$$\begin{tikzcd} & C \arrow[dl, "\mu_1"'] \arrow[dr, "\mu_2"] & \\ A \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & B \end{tikzcd}$$

such that $\mu_2 = f\mu_1 = g\mu_1$, or (simpler) just

$$\begin{tikzcd} C \arrow[r, "c"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

with $fc = gc$. A limit cone is a pair (E, e) with $e\colon E \longrightarrow A$, $fe = ge$, such that any other cone (C, c) factors through (E, e): there is a unique morphism $l\colon C \longrightarrow E$ satisfying $el = c$.

$$\begin{tikzcd} E \arrow[rr, "e"] & & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ & C \arrow[ul, dotted, "\exists! l"] \arrow[ur, "c"'] & & \end{tikzcd}$$

A colimit of this diagram is called a **coequaliser**.

In Set, the equaliser of f and g is the set $E = \{a \in A \mid f(a) = g(a)\}$ equipped with the inclusion map into A.

c) The limit of a diagram of shape
$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$
is called a **pullback**. A cone over such a diagram is just a commutative square:

$$\begin{tikzcd} K \arrow[r, "\mu_1"] \arrow[d, "\mu_2"'] \arrow[dr, "\mu_3"] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd} \qquad \text{with} \qquad \mu_3 = f\mu_1 = g\mu_2$$

i.e. the square commutes.

We write a pullback square as follows:

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd} \qquad \text{or} \qquad \begin{tikzcd} A \times_C B \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

Pullbacks are also called **fibred products**.

We say $(A \times_C B, \pi_1, \pi_2)$ is the **pullback of** f **and** g and π_2 is the **pullback of** f **along** g. A pullback of f with itself is also called the **kernel pair of** f.

In Set, we can construct pullbacks by first forming the product $A \times B$ and then the equaliser $P \longrightarrow A \times B$ of $A \times B \overset{f\pi_1}{\underset{g\pi_2}{\rightrightarrows}} C$, i.e. the set $\{(a, b) \in A \times B \mid f(a) = g(b)\}$.

Notice that the colimit under this diagram is trivial (**Exercise:** find it!).

The appropriate dual is a **pushout**: the colimit of a diagram of shape
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$
.

7 Theorem: ("constructing limits")

i) If $\mathscr{C}$ has equalisers and all small products, then $\mathscr{C}$ has all small limits.
ii) If $\mathscr{C}$ has equalisers and all finite products, then $\mathscr{C}$ has all finite limits.
iii) If $\mathscr{C}$ has pullbacks and a terminal object, then $\mathscr{C}$ has all finite limits.

PROOF.

(i) and (ii) Let $D\colon \mathscr{J} \longrightarrow \mathscr{C}$ be a diagram with $\mathscr{J}$ small (resp. finite). Form the products

$$P = \prod_{j \in \operatorname{ob} \mathscr{J}} D(j) \qquad \text{and} \qquad Q = \prod_{\alpha \in \operatorname{mor} \mathscr{J}} D(\operatorname{cod} \alpha),$$

and the morphisms $P \overset{f}{\underset{g}{\rightrightarrows}} Q$ defined by

$$\pi_\alpha f = \pi_{\operatorname{cod} \alpha} \qquad \text{and} \qquad \pi_\alpha g = D(\alpha)\pi_{\operatorname{dom} \alpha}.$$

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[dr, "\pi_{j'}"'] & Q \arrow[d, "\pi_\alpha"] \\ & D(j') \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "g"] \arrow[d, "\pi_j"'] & Q \arrow[d, "\pi_\alpha"] \\ D(j) \arrow[r, "D(\alpha)"'] & D(j') \end{tikzcd}$$

Let $e\colon L \longrightarrow P$ be an equaliser of (f, g). We claim that the family $(\lambda_j = \pi_j e\colon L \longrightarrow D(j))$ forms a limit cone over D. It is indeed a cone, because, for any $\alpha\colon j \longrightarrow j'$ in $\mathscr{J}$, we have

$$D(\alpha)\lambda_j = D(\alpha)\pi_j e = \pi_\alpha g e = \pi_\alpha f e = \pi_{j'} e = \lambda_{j'}.$$

Given a cone $(\mu_j\colon M \longrightarrow D(j) \mid j \in \operatorname{ob} \mathscr{J})$, there is a unique morphism $m\colon M \longrightarrow P$ satisfying $\pi_j m = \mu_j$ for all j. Then $fm = gm$, as $\pi_\alpha fm = \pi_\alpha gm$ for all α. (**Exercise:** Check this carefully!) So there is a unique $n\colon M \longrightarrow L$ with $\lambda_j n = \mu_j$ for all j.

(iii) It is enough to construct finite products and equalisers. Any finite product $\prod_{i=1}^n A_i$ can be constructed from products of pairs: $((A_1 \times A_2) \times A_3) \times A_4 \dots$

The product of the empty family (which is also finite) is the terminal object 1.

Given two objects A and B, their product can be constructed as a pullback of

$$\begin{tikzcd} & A \arrow[d] \\ B \arrow[r] & 1. \end{tikzcd}$$

Given a pair of parallel morphisms $A \underset{g}{\overset{f}{\rightrightarrows}} B$, their equaliser can be constructed as the pullback of

$$\begin{tikzcd} & A \arrow[d, "{(1_A, f)}"] \\ A \arrow[r, "{(1_A, g)}"'] & A \times B. \end{tikzcd}$$

A cone on this is $\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "k"'] & A \\ A & \end{tikzcd}$ satisfying $h = k$ and $fh = gk$, so it is equivalent to a cone over $A \underset{g}{\overset{f}{\rightrightarrows}} B$.

□

The categories Set, Gp, Rng, R-Mod, Top, … all have small products and equalisers, so they have all small limits. We call a category with all small limits **complete**, and a category with all finite limits **finitely complete**.

Similarly, the categories have small coproducts and coequalisers, so they are **cocomplete**.

C Special morphisms

Definition: A morphism $f\colon A \longrightarrow B$ in a category $\mathscr{C}$ is a **monomorphism** if, given any $C \underset{h}{\overset{g}{\rightrightarrows}} A$ with $fg = fh$, we necessarily have $g = h$. (f is left-cancellable.)

Dually, f is called an **epimorphism** if, given $B \underset{l}{\overset{k}{\rightrightarrows}} D$ with $kf = lf$, we necessarily have $k = l$.

Examples: In Set, monos are injective functions and epis are surjective functions. In Gp, monos are injective group homomorphisms and epis are surjective group homomorphisms. Similarly in Top monos are injective and epis are surjective.

HOWEVER it is not always this simple: for example $\mathbb{Z} \longrightarrow \mathbb{Q}$ is an epimorphism in CRng, and in Mon, the inclusion $(\mathbb{N}, +) \longrightarrow (\mathbb{Z}, +)$ is epic (epimorphic).

Proposition: *If $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow A$ satisfy $gf = 1_A$, then f is monic and g is epic.*

Proof. If we have $C \underset{k}{\overset{h}{\rightrightarrows}} A$ with $fh = fk$, then also $gfh = gfk$, i.e. $h = k$. So f is monic. The statement that g is epic is dual, i.e.

$$g \text{ epic in } \mathscr{C} \quad \Leftrightarrow \quad g \text{ monic in } \mathscr{C}^{\text{op}}.$$

□

Definition: a) If $gf = 1_A$ as above, we call f a **split monomorphism** and g a **split epimorphism**.

b) We say $f\colon A \longrightarrow B$ is a **regular monomorphism** if it is an equaliser of some pair $B \underset{h}{\overset{g}{\rightrightarrows}} C$. Dually, a **regular epimorphism** is the coequaliser of some pair $D \underset{l}{\overset{k}{\rightrightarrows}} A$.

Exercise: Prove that any regular mono is indeed monic. Prove that every split mono is a regular mono (consider fg and 1_B).

In Set, every mono is a regular mono, but not in Top. (In Top, regular monos are injections $f\colon Y \longrightarrow X$ for which Y has the subspace topology of X.)

In Set, any mono with non-empty domain is split, and the fact that every epi is split in Set is equivalent to the axiom of choice. In k-$\mathsf{Mod}_{\mathrm{f.d.}}$, all monos and epis are split.

8 Proposition: ("epi + regular mono ⇒ iso")
If f is both an epi and regular monic, then it is an iso.

Proof. If $f\colon A \longrightarrow B$ is the equaliser of $B \underset{h}{\overset{g}{\rightrightarrows}} C$, then $g = h$ as f is epic. But 1_B is an equaliser of (g, g), so by uniqueness of limits, f is an iso. □

Definition: A category is called **balanced** if every morphism which is monic and epic is an isomorphism.

Set and Gp are balanced categories, but Mon and Top are not. (Top: continuous bijections need not be homeomorphisms.)

In diagrams, we write $A \overset{f}{\rightarrowtail} B$ for monos and $A \overset{f}{\twoheadrightarrow} B$ for epis.

9 Lemma: ("Pullbacks preserve monos.")
Given a pullback square

$$\begin{tikzcd}
P \arrow[r, "h"] \arrow[d, "k"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, tail, "f"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}$$

if f is monic, then k is monic.

Proof. Suppose $D \underset{m}{\overset{l}{\rightrightarrows}} P$ satisfy $kl = km$. Then $fhl = gkl = gkm = fhm$, so $hl = hm$. So l and m correspond to the same cone over

$$\begin{tikzcd}
 & A \arrow[d] \\
B \arrow[r] & C
\end{tikzcd}$$

and hence $l = m$. So k is monic. □

Definition: A **subobject** of an object A in a category $\mathscr{C}$ is either a monomorphism $A' \rightarrowtail A$ in $\mathscr{C}$, or an isomorphism class (in $\mathscr{C}/A$) of such monomorphisms[4]. We write $\mathrm{Sub}_{\mathscr{C}}(A)$ for the full subcategory of $\mathscr{C}/A$ whose objects are the monomorphisms $A' \rightarrowtail A$. (Note that this category is a preorder.)

A category $\mathscr{C}$ is **well-powered** if each $\mathrm{Sub}_{\mathscr{C}}(A)$ is equivalent to a partially ordered set, i.e. there exists a set $\{A_i \rightarrowtail A \mid i \in I\}$ of monomorphisms meeting every isomorphism class in $\mathrm{Sub}(A)$.

Examples: Set is well-powered since $\mathrm{Sub}_{\mathsf{Set}}(A) \simeq PA$. Similarly, Gp, Rng, Top, ... are all well-powered.

---

[4] It should be clear from the context which of these is meant.

D Preserving Limits

Definition: Let $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ be a functor.

a) We say F **preserves limits** of shape $\mathscr{J}$ if, given any diagram $D\colon \mathscr{J} \longrightarrow \mathscr{C}$ and a limit cone $(\lambda_j\colon L \longrightarrow D(j) \mid j \in \operatorname{ob}\mathscr{J})$ for D, the cone $(F\lambda_j\colon FL \longrightarrow FD(j))_j$ is a limit for FD.

b) We say F **reflects limits** of shape $\mathscr{J}$ if, given $D\colon \mathscr{J} \longrightarrow \mathscr{C}$ and a cone $(\lambda_j\colon L \longrightarrow D(j))_j$ such that $(F\lambda_j\colon FL \longrightarrow FD(j))_j$ is a limit for FD, then $(L, \lambda_j)_j$ forms a limit for D.

c) We say F **creates limits** of shape $\mathscr{J}$ if, given $D\colon \mathscr{J} \longrightarrow \mathscr{C}$ and a limit $(\mu_j\colon M \longrightarrow FD(j))_j$ for FD, there exists a cone $(\lambda_j\colon L \longrightarrow D(j))_j$ over D in $\mathscr{C}$ whose image is isomorphic to $(M, \mu_j)_j$; and any such cone is a limit in $\mathscr{C}$.[5]

Corollary: *In any of the version of the "constructing limits" Theorem 7, we can replace "$\mathscr{C}$ has" with either "$\mathscr{C}$ has and $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ preserves" or "$\mathscr{D}$ has and $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ creates".*

PROOF. Exercise. □

10 Examples: ("Creating limits")

a) The forgetful functor $\mathsf{Gp} \longrightarrow \mathsf{Set}$ creates all small limits; for example, if $\{G_j \mid j \in \mathscr{J}\}$ is a family of groups, then the product set $\prod_{j\in\mathscr{J}} G_j$ has a unique group structure making the projections into homomorphisms, and this structure makes it into a product in Gp. But $\mathsf{Gp} \longrightarrow \mathsf{Set}$ doesn't preserve coproducts (or other colimits)[6].

b) The forgetful functor $\mathsf{Top} \longrightarrow \mathsf{Set}$ preserves all small limits and colimits, but doesn't reflect them: given spaces X and Y, there are (in general) other topologies on the set $X \times Y$ making the projections
$$\begin{tikzcd} & X \times Y \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$
continuous, but not making it into a product in Top. This functor also does not create products: while the choice of topology on $X \times Y$ does not change its image under the forgetful functor, not any such choice turns $X \times Y$ into a limit in Top, so the last part of the definition is not satisfied.

c) The inclusion functor $\mathsf{AbGp} \longrightarrow \mathsf{Gp}$ reflects coproducts, but doesn't preserve them. A coproduct $\sum_{i\in I} A_i$ in Gp is non-abelian, unless all but one of the A_i are trivial, and then it coincides with the coproduct in AbGp.

d) Let $\mathscr{C}$ be a category and $B \in \operatorname{ob}\mathscr{C}$. The forgetful functor $U\colon \mathscr{C}/B \longrightarrow \mathscr{C}$ sending $\begin{pmatrix} A \\ \downarrow f \\ B \end{pmatrix}$ to A creates all colimits which exist in $\mathscr{C}$. A diagram $D\colon \mathscr{J} \longrightarrow \mathscr{C}/B$ is essentially a diagram UD of shape $\mathscr{J}$ in $\mathscr{C}$, together with a cocone $(UD(j) \longrightarrow B)_{j\in\operatorname{ob}\mathscr{J}}$ under it. Given a colimit cocone $(UD(j) \longrightarrow L)$ for UD, we get a unique $L \longrightarrow B$ making all the $UD(j) \longrightarrow L$ into morphisms of $\mathscr{C}/B$, which "lifts" the colimit cocone to a colimit cocone in $\mathscr{C}/B$. However, $\mathscr{C}/B \longrightarrow \mathscr{C}$ doesn't preserve all limits; e.g. if $\begin{pmatrix} A \\ \downarrow f \\ B \end{pmatrix}$ and $\begin{pmatrix} C \\ \downarrow g \\ B \end{pmatrix}$ are objects of $\mathscr{C}/B$, their product in $\mathscr{C}/B$ is the diagonal of the pullback square

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr] & A \arrow[d, "f"] \\ C \arrow[r, "g"'] & B \end{tikzcd}$$

if this exists in $\mathscr{C}$, and $P \not\cong A \times C$ in general.

---

[5] This last part of the sentence is very important, see e.g. the example on topological spaces.

[6] It does create filtered colimits (of which directed limits are a special case). If you don't know what that is, either look it up or ignore this comment.

e) "Limits in functor categories are constructed object by object."

Let $\mathscr{C}$ and $\mathscr{D}$ be two categories, and write $\mathscr{C}^{\operatorname{ob}\mathscr{D}}$ for the category of functors from the discrete category on the objects of $\mathscr{D}$ to $\mathscr{C}$, or "the product of $\operatorname{ob}\mathscr{D}$ copies of $\mathscr{C}$". Then the forgetful functor $U\colon [\mathscr{D},\mathscr{C}] \longrightarrow \mathscr{C}^{\operatorname{ob}\mathscr{D}}$ creates all limits (and colimits) that exist in $\mathscr{C}$.

To see this, let $D\colon \mathscr{J} \longrightarrow [\mathscr{D},\mathscr{C}]$ be a diagram in the functor category, and suppose that for every object A of $\mathscr{D}$, the diagram UD_A (i.e. UD evaluated at A)[7] has a limit (LA, λ_j^A) in $\mathscr{C}$. Then clearly $L\colon \operatorname{ob}\mathscr{D} \longrightarrow \mathscr{C}$ is a limit of UD.[8] We want to show that L is actually a functor $L\colon \mathscr{D} \longrightarrow \mathscr{C}$ and is the limit of D in $[\mathscr{D},\mathscr{C}]$. Given a morphism $f\colon A \longrightarrow B$ in $\mathscr{D}$, we have, for any morphism $\alpha\colon j \longrightarrow j'$ in $\mathscr{J}$, a commutative square

$$\begin{tikzcd}
D(j)A \arrow[r, "D(\alpha)_A"] \arrow[d, "D(j)f"'] & D(j')A \arrow[d, "D(j')f"] \\
D(j)B \arrow[r, "D(\alpha)_B"'] & D(j')B
\end{tikzcd}$$

Here in the "usual" view, $D(\alpha)$ is a natural transformation from $D(j)$ to $D(j')$, which are functors $\mathscr{D} \longrightarrow \mathscr{C}$. But we can also view it as saying that $D(-)f$ is a natural transformation from "evaluation at A" to "evaluation at B". So $(LA, D(j)f \circ \lambda_j^A)$ forms a cone on UD_B, which gives a unique morphism $Lf\colon LA \longrightarrow LB$ making

$$\begin{tikzcd}
LA \arrow[r, "Lf"] \arrow[d, "\lambda_j^A"'] & LB \arrow[d, "\lambda_j^B"] \\
D(j)A \arrow[r, "D(j)f"'] & D(j)B
\end{tikzcd}$$

commute for each $j \in \operatorname{ob}\mathscr{J}$. This makes L into a functor $\mathscr{D} \longrightarrow \mathscr{C}$, the λ_j into natural transformations $L \longrightarrow D(j)$, and L into the limit of D in $[\mathscr{D},\mathscr{C}]$.

(**Exercise:** Check all this.)

Note that this also shows that the functor "evaluation at A" $\operatorname{ev}_A\colon [\mathscr{D},\mathscr{C}] \longrightarrow \mathscr{C}$ preserves all limits which exist in $\mathscr{C}$.

11 Remark: ("Monos in functor categories")

In any category, a morphism $f\colon A \longrightarrow B$ is monic if and only if

$$\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "1_A"'] & A \arrow[d, "f"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

is a pullback (i.e. iff its kernel pair is $(A, 1_A, 1_A)$.) Hence a functor which preserves pullbacks must preserve monos. Therefore, supposing $\mathscr{C}$ has pullbacks[9], a morphism $\alpha\colon F \longrightarrow G$ in a functor category $[\mathscr{D},\mathscr{C}]$ is monic if and only if each component $\alpha_C\colon FC \longrightarrow GC$ is a mono in $\mathscr{C}$. (c.f. Example Sheet 1 Question 7.)[10]

There is a connection between initial objects and limits:

---

[7] Note that $UD_A = D_A$, because evaluation at A doesn't involve any morphisms of $\mathscr{D}$.

[8] I.e. this is just defined on objects.

[9] Or at least it must have kernel pairs, i.e. specific pullbacks.

[10] $\Leftarrow$ is obvious, and $\Rightarrow$ follows from ev_A preserving pullbacks (or kernel pairs).

12 Lemma: ("Initial object as limit")
Let $\mathscr{C}$ be an arbitrary category. Then $\mathscr{C}$ has an initial object if and only if the diagram $1_{\mathscr{C}}\colon \mathscr{C} \longrightarrow \mathscr{C}$ has a limit.[11]

Proof. "$\Rightarrow$" Let I be an initial object of $\mathscr{C}$, and write $\lambda_A\colon I \longrightarrow A$ for the unique morphism from I to each object A. Then we claim that (I, λ_A) forms a terminal cone on $1_{\mathscr{C}}$. Indeed, it is a cone as

$$\begin{tikzcd}
& I \arrow[dl, "\lambda_A"'] \arrow[dr, "\lambda_B"] & \\
A \arrow[rr, "f"'] & & B
\end{tikzcd}$$

commutes for each morphism f in $\mathscr{C}$, by uniqueness of λ_B.

Given another cone (B, μ_A) over $1_{\mathscr{C}}$, the morphism $\mu_I\colon B \longrightarrow I$ satisfies

$$\begin{tikzcd}
B \arrow[rr, "\mu_I"] \arrow[dr, "\mu_A"'] & & I \arrow[dl, "\lambda_A"] \\
& A &
\end{tikzcd}$$

(i.e. $\lambda_A \mu_I = \mu_A$) for all A (as the μ are a cone), so μ_I is a morphism of cones. But any morphism of cones ν satisfies

$$\begin{tikzcd}
B \arrow[rr, "\nu"] \arrow[dr, "\mu_I"'] & & I \arrow[dl, "\lambda_I = 1_I"] \\
& I &
\end{tikzcd}$$

, so $\nu = \mu_I$. So μ_I is the *unique* morphism of cones, so (I, λ_A) is the limit as claimed.

"$\Leftarrow$" If we have a limit (I, λ_A) for $1_{\mathscr{C}}$, we want to show I is initial. As we already have a morphism $\lambda_A\colon I \longrightarrow A$ for each object A, we must show that it is unique, i.e. given $f\colon I \longrightarrow A$, we have $f = \lambda_A$.

We certainly have $f\lambda_I = \lambda_A$,

$$\begin{tikzcd}
& I \arrow[dl, "\lambda_I"'] \arrow[dr, "\lambda_A"] & \\
I \arrow[rr, "f"'] & & A
\end{tikzcd}$$

so we just have to show that $\lambda_I = 1_I$. Putting $f = \lambda_A$, we get $\lambda_A \lambda_I = \lambda_A$ for all objects A, so λ_I is a morphism of cones from the limit cone to itself.

$$\begin{tikzcd}
I \arrow[rr, "\lambda_I"] \arrow[dr, "\lambda_A"'] & & I \arrow[dl, "\lambda_A"] \\
& A &
\end{tikzcd}$$

So as there is a *unique* one, $\lambda_I = 1_I$. $\square$

E Projectives

Definition: An object P of a category $\mathscr{C}$ is **projective** if given any diagram (of solid arrows)

$$\begin{tikzcd}
& P \arrow[dl, dotted, "\exists h"'] \arrow[d, "g"] \\
A \arrow[r, two heads, "f"'] & B
\end{tikzcd}$$

with f epic, there exists $h\colon P \longrightarrow A$ with $fh = g$.

Dually, I is **injective** in $\mathscr{C}$ if it is projective in $\mathscr{C}^{\text{op}}$.

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d] & B \arrow[dl, dotted, "\exists"] \\
I &
\end{tikzcd}$$

---

[11] Notice that if $\mathscr{C}$ is not small, this is not a small diagram.

Remark: Note that h need not be unique![12]

If $\mathscr{C}$ is locally small, P is projective iff $\mathscr{C}(P, -)$ preserves epimorphisms.

Lemma: *For any locally small $\mathscr{C}$, all representable functors are projective in $[\mathscr{C}, \mathsf{Set}]$.*

PROOF. The dual of "monos in functor categories"(Remark 11) says that $\alpha\colon F \longrightarrow G$ is epic in $[\mathscr{C}, \mathsf{Set}]$ iff $\alpha_A\colon FA \longrightarrow GA$ is surjective for all A. Now, given

$$\begin{tikzcd} & \mathscr{C}(A,-) \arrow[d, "\beta"] \\ F \arrow[r, two heads, "\alpha"'] & G, \end{tikzcd}$$

by the Yoneda Lemma β corresponds to an element $y \in GA$. As α is epic, there is an $x \in FA$ with $\alpha_A(x) = y$. Then x corresponds to $\gamma\colon \mathscr{C}(A, -) \longrightarrow F$ with $\alpha\gamma = \beta$. ☐

Lemma: *A coproduct of projectives is projective.*

PROOF. Exercise. ☐

Examples: In Set, every object is projective (as any epi is split, which uses the Axiom of Choice).

In Gp, any free group is projective. In fact these are the only projective objects in Gp.

In R-Mod, a module M is projective if and only if it is a direct summand of a free module.

---

[12] This is called a *weak* universal property.

CHAPTER 3

Adjunctions

A Definitions and examples

Definition: (D.M. Kan) Let $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ and $G\colon \mathscr{D} \longrightarrow \mathscr{C}$ be two functors. An **adjunction** between F and G is a specification, for each pair $(A \in \operatorname{ob}\mathscr{C}, B \in \operatorname{ob}\mathscr{D})$, of a bijection betwen morphisms $FA \longrightarrow B$ in $\mathscr{D}$ and morphisms $A \longrightarrow GB$ in $\mathscr{C}$, which is natural in A and B.

(If $\mathscr{C}$ and $\mathscr{D}$ are locally small, this means that the functors $\mathscr{C}^{\mathrm{op}} \times \mathscr{D} \longrightarrow \mathsf{Set}$ sending (A, B) to $\mathscr{D}(FA, B)$ and to $\mathscr{C}(A, GB)$ are naturally isomorphic.)

We say that F is **left adjoint** to G, or that G is **right adjoint** to F, and write $(F \dashv G)$ to indicate that there is such an adjunction.

Notation: Given $\begin{tikzcd}\mathscr{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{D} \arrow[l, shift left, "G"]\end{tikzcd}$, we sometimes write $\dfrac{FA \longrightarrow B}{A \longrightarrow GB}$ for the bijection, and we write $\overline{f}\colon A \longrightarrow GB$ for the morphism corresponding to $f\colon FA \longrightarrow B$, and $\overline{g}\colon FA \longrightarrow B$ corresponds to $g\colon A \longrightarrow GB$. Notice that $\overline{\overline{f}} = f$ and $\overline{\overline{g}} = g$.[1]

13 Examples: (Adjunctions)

a) The free functor $F\colon \mathsf{Set} \longrightarrow \mathsf{Gp}$ is left adjoint to the forgetful functor $G\colon \mathsf{Gp} \longrightarrow \mathsf{Set}$, as homomorphisms $FA \longrightarrow B$ are uniquely determined by mappings $A \longrightarrow GB$. Similarly for free rings, free R-modules, etc. (We will look at the meaning of the naturality in Section B.)

b) The forgetful functor $\mathsf{Top} \longrightarrow \mathsf{Set}$ has both left and right adjoints: The left adjoint D equips a set A with its discrete topology, since all functions $DA \longrightarrow X$ (for X an arbitrary space) are continuous. The right adjoint I equips A with the indiscrete topology.

c) The functor $\operatorname{ob}\colon \mathsf{Cat} \longrightarrow \mathsf{Set}$ has a left adjoint D sending a set A to the discrete category DA (with objects the elements of A and only identity morphisms), since a functor $DA \longrightarrow \mathscr{C}$ is determined by its effect on objects. The functor ob also has a right adjoint I, which sends A to the category with objects given by the elements of A, and exactly one morphism $a \longrightarrow b$ for each pair $(a, b) \in A \times A$. (This makes all morphisms into isomorphisms!)[2]

The functor D itself also has a left adjoint π_0. $\pi_0(\mathscr{C})$ is the set of connected components of $\mathscr{C}$, i.e. the quotient of $\operatorname{ob}\mathscr{C}$ by the smallest equivalence relation which identifies c and d whenever there exists a morphism $c \longrightarrow d$ in $\mathscr{C}$. (Given a functor $F\colon \mathscr{C} \longrightarrow DA$, F is necessarily constant on each connected component of $\mathscr{C}$, as each morphism must go to an identity morphism. So F induces a function $\pi_0\mathscr{C} \longrightarrow A$.)

d) Let 1 denote the category with one object $*$ and one morphism. A functor $F\colon 1 \longrightarrow \mathscr{C}$ picks out an object $F*$ of $\mathscr{C}$. This F is left adjoint to the unique functor $\mathscr{C} \longrightarrow 1 \Leftrightarrow F*$ is an initial object of $\mathscr{C}$.

F is right adjoint to $\mathscr{C} \longrightarrow 1 \Leftrightarrow F*$ is a terminal object of $\mathscr{C}$.

---

[1] We sometimes call this adjunction operation $\overline{(\ \ )}$ "transpose".

[2] So you could think of DA as lots of completely separated objects and IA as "one big connected blob" of isomorphic objects.

e) Let Idem be the category with objects being pairs (A, e), where A is a set and $e\colon A \longrightarrow A$ satisfies $e \circ e = e$ (is **idempotent**). (Morphisms $(A, e) \longrightarrow (A', e')$ are functions $f\colon A \longrightarrow A'$ satisfying

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "e"'] & A' \arrow[d, "e'"] \\ A \arrow[r, "f"'] & A' \end{tikzcd}$$

.)

We have a functor $F\colon \mathsf{Set} \longrightarrow \mathsf{Idem}$ sending A to $(A, 1_A)$, and a functor $G\colon \mathsf{Idem} \longrightarrow \mathsf{Set}$ sending (A, e) to $\{e(a) \mid a \in A\} = \{a \in A \mid e(a) = a\}$ (the image of e, or the fixed points of e). G is both left and right adjoint to F:

- ◇ morphisms $f\colon (A, 1_A) \longrightarrow (B, e)$ must satisfy

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal, "1_A"'] & B \arrow[d, "e"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

, i.e. f must land in the image of e. This gives a bijection $\dfrac{(A, 1_A) \longrightarrow (B, e)}{A \longrightarrow \{e(b) \mid b \in B\}}$.

- ◇ morphisms $f\colon (B, e) \longrightarrow (A, 1_A)$ must satisfy

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "e"'] & A \arrow[d, equal, "1_A"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

so f is completely determined by what it does on the image of e, which gives a bijection $\dfrac{(B, e) \longrightarrow (A, 1_A)}{\{e(b) \mid b \in B\} \longrightarrow A}$.

f) Let X be a topological space, $\mathscr{C}X$ the ordered set of closed subsets of X and $\mathscr{P}X$ the set of all subsets of X.[3] The inclusion $\mathscr{C}X \longrightarrow \mathscr{P}X$ has a left adjoint $A \longmapsto \overline{A}$, since for any closed set C we have $A \leqslant C \Leftrightarrow \overline{A} \leqslant C$.

(An adjunction between posets

$$\begin{tikzcd} P \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & Q \arrow[l, shift left, "G"] \end{tikzcd}$$

always looks like $Fa \leqslant b \Leftrightarrow a \leqslant Gb$.)

g) (Adjunctions of contravariant functors)

Consider two sets A and B and a relation $R \subseteq A \times B$. We have mappings $r\colon \mathscr{P}A \longrightarrow \mathscr{P}B$ sending

$$A' \longmapsto r(A') = \{b \in B \mid (\forall a \in A')((a, b) \in R)\}^4$$

and $l\colon \mathscr{P}B \longrightarrow \mathscr{P}A$ sending

$$B' \longmapsto l(B') = \{a \in A \mid (\forall b \in B')((a, b) \in R)\}.$$

r and l are contravariant functors between posets, and we have

$$A' \subseteq l(B') \Leftrightarrow A' \times B' \subseteq R \Leftrightarrow B' \subseteq r(A')^5$$

We can regard $l\colon \mathscr{P}B \longrightarrow \mathscr{P}A^{\mathrm{op}}$ as left adjoint to $r\colon \mathscr{P}A^{\mathrm{op}} \longrightarrow \mathscr{P}B$. (We sometimes say that l and r are contravariant functors adjoint on the right.)

h) The contravariant powerset functor $\mathscr{P}^*\colon \mathsf{Set}^{\mathrm{op}} \longrightarrow \mathsf{Set}$ is right adjoint to $\mathscr{P}^*\colon \mathsf{Set} \longrightarrow \mathsf{Set}^{\mathrm{op}}$, since functions $A \longrightarrow \mathscr{P}B$ correspond to relations $R \subseteq A \times B$, and hence to functions $B \longrightarrow \mathscr{P}A$.

B Properties

What does the naturality in A and B of the bijection $\dfrac{FA \longrightarrow B}{A \longrightarrow GB}$ mean?

---

[3] Remember how posets can be regarded as categories.

[4] Those b which are related to everything in A'.

[5] All $a \in A'$ are related to all $b \in B'$.

Naturality in A says that for $a\colon A' \longrightarrow A$ in $\mathscr{C}$,

$$\begin{tikzcd}
\mathscr{C}(A,GB) \arrow[r, "\overline{(\ )}"] \arrow[d, "-\circ a"'] & \mathscr{D}(FA,B) \arrow[d, "-\circ Fa"] \\
\mathscr{C}(A',GB) \arrow[r, "\overline{(\ )}"'] & \mathscr{D}(FA',B)
\end{tikzcd}$$

commutes, and naturality in B says that for $b\colon B \longrightarrow B'$ in $\mathscr{D}$,

$$\begin{tikzcd}
\mathscr{D}(FA,B) \arrow[r, "\overline{(\ )}"] \arrow[d, "b\circ-"'] & \mathscr{C}(AG,B) \arrow[d, "Gb\circ-"] \\
\mathscr{D}(FA,B') \arrow[r, "\overline{(\ )}"'] & \mathscr{C}(A,GB')
\end{tikzcd}$$

commutes, i.e.

$$\overline{g\circ a} = \overline{g}\circ Fa \qquad \text{and} \qquad \overline{b\circ f} = Gb\circ\overline{f}.$$

So in fact we have natural transformations like the ones appearing in the Yoneda Lemma:

$$\mathscr{D}(FA,-) \longrightarrow \mathscr{C}(A,G-)$$

$$\text{and } \mathscr{C}(-,GB) \longrightarrow \mathscr{D}(F-,B).$$

So these isomorphisms are completely determined by where the identity goes:

$$FA \xrightarrow{1_{FA}} FA \qquad \text{corresponds to} \qquad A \xrightarrow{\eta_A} GFA.$$

$$\text{Any } FA \xrightarrow{f} B \qquad \text{corresponds to} \qquad A \xrightarrow{\eta_A} GFA \xrightarrow{Gf} GB.$$

(I.e. $\overline{f} = \overline{f1_{FA}} = Gf\overline{1_{FA}} = Gf\eta_A$.)

$$GB \xrightarrow{1_{GB}} GB \qquad \text{corresponds to} \qquad FGB \xrightarrow{\epsilon_B} B.$$

$$\text{Any } A \xrightarrow{g} GB \qquad \text{corresponds to} \qquad FA \xrightarrow{Fg} FGB \xrightarrow{\epsilon_B} B.$$

Lemma: *The $\eta_A\colon A \longrightarrow GFA$ form a natural transformation $\eta\colon 1_{\mathscr{C}} \longrightarrow GF$. (Dually, the ϵ_B form a natural transformation $\epsilon\colon FG \longrightarrow 1_{\mathscr{D}}$.)*

PROOF. Given $a\colon A \longrightarrow A'$, we have:

$$A \xrightarrow{\eta_A} GFA \xrightarrow{GFa} GFA' \qquad \text{corresponds to} \qquad FA \xrightarrow{Fa} FA'$$

$$A \xrightarrow{a} A' \xrightarrow{\eta_{A'}} GFA' \qquad \text{corresponds to} \qquad FA \xrightarrow{Fa} FA' \xrightarrow{1_{FA'}} FA'$$

So the following square commutes

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "\eta_A"'] & A' \arrow[d, "\eta_{A'}"] \\
GFA \arrow[r, "GFa"'] & GFA'
\end{tikzcd}$$

and η is natural. □

Notation: Given a functor $G\colon \mathscr{D} \longrightarrow \mathscr{C}$ and an object A of $\mathscr{C}$, we write $(A \downarrow G)$ for the category whose objects are pairs (B,f), where $B \in \operatorname{ob}\mathscr{D}$ and $f\colon A \longrightarrow GB$ in $\mathscr{C}$, and whose morphisms $(B,f) \longrightarrow (B',f')$ are morphisms $g\colon B \longrightarrow B'$ in $\mathscr{D}$ such that

$$\begin{tikzcd}
& A \arrow[dl, "f"'] \arrow[dr, "f'"] & \\
GB \arrow[rr, "Gg"'] & & GB'
\end{tikzcd}$$

commutes.

(Similarly, there is a category $(G \downarrow A)$.)

14 Theorem: ("Adjunctions via initial objects")
Let $G\colon \mathscr{D} \longrightarrow \mathscr{C}$ be a functor. Then specifying a left adjoint for G is equivalent to specifying, for each object $A \in \operatorname{ob}\mathscr{C}$, an initial object of $(A \downarrow G)$.

Proof. "$\Rightarrow$" Let $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ be a left adjoint for G. We show that (FA, η_A) is an initial object of $(A \downarrow G)$.

Given an object (B, f) of $(A \downarrow G)$, the triangle

$$\begin{tikzcd} A \arrow[d, "\eta_A"'] \arrow[dr, "f"] & \\ GFA \arrow[r, "Gh"'] & GB \end{tikzcd} \quad \text{commutes iff} \quad \begin{tikzcd} FA \arrow[d, "1_A"'] \arrow[dr, "\overline{f}"] & \\ FA \arrow[r, "h"'] & B \end{tikzcd} \quad \text{commutes.}$$

So there is a unique morphism $h\colon (FA, \eta_A) \longrightarrow (B, f)$ in $(A \downarrow G)$, namely $\overline{f}$.

"$\Leftarrow$" Given an initial object (FA, η_A) of each category $(A \downarrow G)$, we already have the action of F on objects. We want to see what F does on morphisms, that it is a functor and that it is adjoint to G.

Given $f\colon A \longrightarrow A'$, we get an object $(A \xrightarrow{f} A' \xrightarrow{\eta_{A'}} GFA')$ of $(A \downarrow G)$. So there is a unique morphism $g\colon FA \longrightarrow FA'$ making

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"'] & A' \arrow[d, "\eta_{A'}"] \\ GFA \arrow[r, "Gg"'] & GFA' \end{tikzcd}$$

commute. So we define $Ff = g$. The uniqueness of g makes F functorial (**check this!**). To see that F is adjoint to G, take any $h\colon FA \longrightarrow B$. Then the composite $A \xrightarrow{\eta_A} GFA \xrightarrow{Gh} GB$ is a morphism $A \longrightarrow GB$. Given $k\colon A \longrightarrow GB$, there is a unique morphism $h\colon FA \longrightarrow B$ making

$$\begin{tikzcd} A \arrow[d, "\eta_A"'] \arrow[dr, "k"] & \\ GFA \arrow[r, "Gh"'] & GB \end{tikzcd}$$

commute. So we get a bijection.

Naturality in B is built in:

$$\text{Given} \quad \begin{tikzcd} FA \arrow[r, "h"] \arrow[dr, "h'"'] & B \arrow[d, "b"] \\ & B' \end{tikzcd}, \text{ we get} \quad \begin{tikzcd} A \arrow[r, "\eta_A"] & GFA \arrow[r, "Gh"] \arrow[dr, "Gh'"'] & GB \arrow[d, "Gb"] \\ & & GB' \end{tikzcd}.$$

Naturality in A needs η to be a natural transformation, which was built in to the definition of F:

$$\text{Given} \quad \begin{tikzcd} A \arrow[r, "k"] \arrow[d, "a"'] & GB \\ A' \arrow[ur, "k'"'] & \end{tikzcd}, \text{ we get} \quad \begin{tikzcd} FA \arrow[r, "h"] \arrow[d, dotted, "Fa"'] & B \\ FA' \arrow[ur, "h'"'] & \end{tikzcd} \quad \text{satisfying}$$

$$\begin{tikzcd} & A' \arrow[d, "\eta_{A'}"] \arrow[dr, "k'"] & \\ A \arrow[ur, dotted, "a"] \arrow[d, "\eta_A"'] \arrow[dr, "k"] & GFA' \arrow[r, "Gh'"] & GB \arrow[dl, equal] \\ GFA \arrow[ur, dotted, "GFa"] \arrow[r, "Gh"'] & GB & \end{tikzcd}$$

i.e. both h and $h'Fa$ are morphisms $(FA, \eta_A) \longrightarrow (B, k) = (B, k'a)$ in $(A \downarrow G)$, so they are the same. So $F \dashv G$. $\square$

Example: $\mathscr{C}$ has limits (resp. colimits) of shape $\mathscr{J}$ if and only if the functor $\Delta\colon \mathscr{C} \longrightarrow [\mathscr{J}, \mathscr{C}]$ sending an object A to the constant diagram Δ_A has a right (resp. left) adjoint.

15 Corollary: ("Uniqueness of Adjoints")
Any two left adjoints of a given functor $G\colon \mathscr{D} \longrightarrow \mathscr{C}$ *are canonically naturally isomorphic.*

Proof. Suppose F and F' are both left adjoints of G. Then (FA, η_A) and $(F'A, \eta'_A)$ are both initial objects of $(A \downarrow G)$, so there is a unique isomorphism $\alpha_A\colon (FA, \eta_A) \longrightarrow (F'A, \eta'_A)$ in $(A \downarrow G)$. The fact that α is a natural transformation follows from uniqueness. ☐

16 Lemma: ("Adjoints compose")
Given $\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D} \underset{K}{\overset{H}{\rightleftarrows}} \mathscr{E}$ *with* $F \dashv G$ *and* $H \dashv K$*, then we have* $HF \dashv GK$*.*

Proof. We have bijections

$$\begin{array}{c} HFA \longrightarrow C \\ \hline FA \longrightarrow KC \\ \hline A \longrightarrow GKC \end{array}$$

natural in A and C. ☐

17 Corollary: ("Adjoints in squares")
Let

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[d, "G"'] & \mathscr{D} \arrow[d, "H"] \\ \mathscr{E} \arrow[r, "K"'] & \mathscr{F} \end{tikzcd}$$

be a commutative diagram where all of F*,* G*,* H*,* K *have left adjoints. Then the diagram*

$$\begin{tikzcd} \mathscr{C} & \mathscr{D} \arrow[l] \\ \mathscr{E} \arrow[u] & \mathscr{F} \arrow[l] \arrow[u] \end{tikzcd}$$

of left adjoints commutes up to natural isomorphism.

Proof. Both composites of the square are left adjoint to $HF = KG$, so they are isomorphic by uniqueness of adjoints (Corollary 15). ☐

C Units and Counits

Definition: Given an adjunction $(F \dashv G)$, the natural transformation $\eta\colon 1_{\mathscr{C}} \longrightarrow GF$ is called the **unit** of the adjunction. Dually, $\epsilon\colon FG \longrightarrow 1_D$ is the **counit** of the adjunction.

Recall that, given $F \dashv G$, we have the following correspondances:

$$FA \xrightarrow{f} B \qquad \longleftrightarrow \qquad A \xrightarrow{\eta_A} GFA \xrightarrow{Gf} GB$$

$$A \xrightarrow{g} GB \qquad \longleftrightarrow \qquad FA \xrightarrow{Fg} FGB \xrightarrow{\epsilon_B} B$$

Recall also that naturality in A and B means

$$\overline{ga} = \overline{g}Fa \qquad \text{and} \qquad \overline{bf} = Gb\overline{f}.$$

18 Theorem: ("Adjunctions via units and counits")
Given $\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}$*, specifying an adjunction* $F \dashv G$ *is equivalent to specifying natural transformations* $\eta\colon 1_{\mathscr{C}} \longrightarrow GF$ *and* $\epsilon\colon FG \longrightarrow 1_{\mathscr{D}}$ *satisfying the* ***triangular identities:*** η *and* ϵ *must make*

the diagrams

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\epsilon_F"] \\ & F \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} G \arrow[r, "\eta_G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\epsilon"] \\ & G \end{tikzcd} \quad \textit{commute.}$$

Proof. Given an adjunction $F \dashv G$, the unit $A \xrightarrow{\eta_A} GFA$ corresponds to $FA \xrightarrow{1_{FA}} FA$ and to $FA \xrightarrow{F\eta_A} FGFA \xrightarrow{\epsilon_{FA}} FA$, so the first triangular idetity follows. Dually, the second one follows using ϵ_B.

Conversely, given η and ϵ satisfying the triangular identities, we must show that the mappings $f \longmapsto Gf \circ \eta_A$ and $g \longmapsto \epsilon_B \circ Fg$ are inverse to each other, and natural in A and B. We have commutative diagrams

$$\begin{tikzcd} FA \arrow[r, "F\eta_A"] \arrow[dr, dotted, "1_{FA}"'] & FGFA \arrow[r, "FGf"] \arrow[d, dotted, "\epsilon_{FA}"] & FGB \arrow[d, "\epsilon_B"] \\ & FA \arrow[r, dotted, "f"'] & B \end{tikzcd}$$

and

$$\begin{tikzcd} A \arrow[r, dotted, "g"] \arrow[d, "\eta_A"'] & GB \arrow[d, dotted, "\eta_{GB}"] \arrow[dr, dotted, "1_{GB}"] & \\ GFA \arrow[r, "GFg"'] & GFGB \arrow[r, "G\epsilon_B"'] & GB \end{tikzcd}$$

which prove that the mappings are mutually inverse. Naturality in A and B follows easily from functoriality of F and G. □

Examples: a) Consider $\begin{tikzcd} \mathsf{Set} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{Gp} \arrow[l, shift left, "G"] \end{tikzcd}$, the "forgetful/free" adjunction. For a set A, the unit $\eta_A \colon A \longrightarrow GFA$ is the inclusion of the generators, and for a group B, $\epsilon_B \colon FGB \longrightarrow B$ is evaluation.

b) The abelianisation functor ab: $\mathsf{Gp} \longrightarrow \mathsf{AbGp}$ is left adjoint to the inclusion $I \colon \mathsf{AbGp} \longrightarrow \mathsf{Gp}$. For a group G, $\eta_G \colon G \longrightarrow I\mathrm{ab}G = G/[G,G]$ is the quotient map. For an abelian group A, $\epsilon_A \colon \mathrm{ab}IA \longrightarrow A$ is the canonical iso $A/[A,A] \longrightarrow A$ (note that $[A,A]$ is trivial).

c) Consider a space X and the adjunction $\begin{tikzcd} X \arrow[r, shift left, "\overline{(\ )}"] \arrow[r, phantom, "\perp"] & \mathscr{C}X \arrow[l, shift left, "I"] \end{tikzcd}$ given in the Adjunctions Example 13f). Then the unit is $A \leqslant \overline{A}$, i.e. any set is inside its closure, and the counit is $\overline{F} \leqslant F$, i.e. any closed set contains its closure.

d) Write down the unit and counit for any example of adjunction that you know.

19 Lemma: ("reflections")

Given an adjunction $F \dashv G$ with counit $\epsilon \colon FG \longrightarrow 1_{\mathscr{D}}$,

i) G is faithful $\Leftrightarrow$ ϵ_B is an epimorphism for all B.

ii) G is full and faithful $\Leftrightarrow$ ϵ_B is an isomorphism for all B.

Proof. i) Given $g \colon B \longrightarrow B'$, its image $Gg \colon GB \longrightarrow GB'$ corresponds under the adjunction to $FGB \xrightarrow{\epsilon_B} B \xrightarrow{g} B'$ (by naturality of ϵ). So if $g' \colon B \longrightarrow B'$ satisfies $Gg = Gg'$ and ϵ_B is an epi, then $g = g'$ and so G is faithful.

Conversely, if G is faithful and $g\epsilon_B = g'\epsilon_B$, then $Gg = Gg'$, so $g = g'$ and so ϵ_B is epic.

ii) Suppose ϵ is an isomorphism. Then by i) G is faithful. Given $f\colon GB \longrightarrow GB'$, we can form the composite

$$g = \begin{tikzcd} FGB \arrow[r, "Ff"] & FGB' \arrow[d, "\epsilon_{B'}"] \\ B \arrow[u, "\epsilon_B^{-1}"] & B' \end{tikzcd}$$

Then g satisfies $FGg = Ff$ (as ϵ_B and $\epsilon_{B'}$ are isos), and so Gg corresponds under the adjunction to $\epsilon_{B'}FGg = \epsilon_{B'}Ff$, which is also what f corresponds to, so $Gg = f$, so G is full.

Conversely suppose that G is full and faithful. We have a morphism $\eta_{GB}\colon GB \longrightarrow GFGB$, which is Gg for a unique $g\colon B \longrightarrow FGB$ (existence as G is full, uniqueness as G is faithful). We show that g is the inverse of ϵ_B: We have the triangular identity

$$\begin{tikzcd} GB \arrow[r, "\eta_{GB}", "=Gg"'] \arrow[dr, "1_{GB}"'] & GFGB \arrow[d, "G\epsilon_B"] \\ & GB \end{tikzcd}$$

which gives $\epsilon_B g = 1_B$ as G is faithful.

We can also use the other triangular identity and naturality of ϵ to show that $g\epsilon_B = 1_{FGB}$.

$$\begin{tikzcd} FGB \arrow[r, "\epsilon_B"] \arrow[d, dotted, "F\eta_{GB}=FGg"'] \arrow[dr, dotted, "1_{FGB}"] & B \arrow[d, "g"] \\ FGFGB \arrow[r, dotted, "\epsilon_{FGB}"'] & FGB \end{tikzcd}$$

So ϵ_B is an isomorphism.

□

Definition: a) An adjunction where G is full and faithful is called a **reflection**.

b) A **reflective subcategory** is a full subcategory $\mathscr{D}$ of $\mathscr{C}$ for which the inclusion functor $\mathscr{D} \longrightarrow \mathscr{C}$ has a left adjoint.

Examples: a) We have already seen that AbGp is reflective in Gp. Given a group G, the commutator subgroup $[G, G]$ has the property that $G/[G, G]$ is abelian and any homomorphism $G \longrightarrow A$ with A abelian factors uniquely through $G \longrightarrow G/[G, G]$.

b) Let $\mathscr{C}$ denote the full subcategory of AbGp whose objects are torsion groups (those in which every element has finite order). Then $\mathscr{C}$ is coreflective in AbGp: Given A, the subgroup A_t of torsion elements in A is the required coreflection, since any homomorphism $B \longrightarrow A$ with B a torsion group factors through the inclusion $A_t \longrightarrow A$.

c) Let $\mathscr{C} = \mathsf{Top}$ and let $\mathscr{D}$ be the full subcategory of compact Hausdorff spaces. Then the Stone-Čech compactification βX of an arbitrary space X is its reflection in $\mathscr{D}$.

D Adjoint Equivalence

An adjunction whose unit and counit are both isomorphisms is in particular an equivalence of categories; we call it an **adjoint equivalence**.

20 Lemma: ("Any equivalence can be made into an adjoint one.")

Consider an equivalence $\begin{tikzcd} \mathscr{C} \arrow[r, shift left, "F"] & \mathscr{D} \arrow[l, shift left, "G"] \end{tikzcd}$, $\alpha\colon 1_{\mathscr{C}} \xrightarrow{\sim} GF$, $\beta\colon 1_{\mathscr{D}} \xrightarrow{\sim} FG$. *Then there exists an adjoint equivalence* $(F \dashv G)$ *with unit* α.

Proof. We define ϵ as the composite

$$\epsilon\colon FG \xrightarrow{\beta_{FG}=FG\beta} FGFG \xrightarrow{(F\alpha_G)^{-1}} FG \xrightarrow{\beta^{-1}} 1_{\mathscr{D}}.$$

Note here that $\beta_{FG} = FG\beta$, since
$$\begin{tikzcd} 1_{\mathscr{D}} \arrow[r, "\beta"] \arrow[d, "\beta"'] & FG \arrow[d, "\beta_{FG}"] \\ FG \arrow[r, "FG\beta"'] & FGFG \end{tikzcd}$$
commutes and β is pointwise epic. (Similarly, $\alpha_{GF} = GF\alpha$.)

We have to verify the triangular identities. We have

$$\begin{tikzcd} F \arrow[r, dotted, "\beta_F"] \arrow[d, "F\alpha"'] & FGF \arrow[d, dotted, "FGF\alpha=F\alpha_{GF}"'] \arrow[dr, equal] & & \\ FGF \arrow[r, "\beta_{FGF}"'] & FGFGF \arrow[r, "(F\alpha_{GF})^{-1}"'] & FGF \arrow[r, "\beta_F^{-1}"'] & F \end{tikzcd}$$

which reduces $F \xrightarrow{F\alpha} FGF \xrightarrow{\epsilon_F} F$ to 1_F, and similarly $G \xrightarrow{\alpha_G} GFG \xrightarrow{G\epsilon} G$ is reduced to 1_G. □

E Adjunctions and Limits

21 Theorem: ("Right adjoints preserve limits")
Suppose $G\colon \mathscr{D} \longrightarrow \mathscr{C}$ has a left adjoint F. Then G preserves all limits which exist in $\mathscr{D}$.

Proof 1. "Apply adjunction to each leg." Consider a diagram $D\colon \mathscr{J} \longrightarrow \mathscr{D}$. Then cones over GD with summit A correspond to cones over D with summit FA. Hence, if D has a limit $(\lambda_j\colon L \longrightarrow D(j))_{j\in\mathrm{ob}\,\mathscr{J}}$, each such cone corresponds to a morphism $FA \longrightarrow L$, which in turn corresponds to a morphism $A \longrightarrow GL$. So $(G\lambda_j\colon GL \longrightarrow GD(j))$ is a limit cone in $\mathscr{C}$. □

Proof 2. [6] Recall that $\mathscr{D}$ has limits of shape $\mathscr{J}$ iff the "constant diagram" functor $\Delta\colon \mathscr{D} \longrightarrow [\mathscr{J}, \mathscr{D}]$ has a right adjoint. So suppose that $\mathscr{C}$ and $\mathscr{D}$ have limits of shape $\mathscr{J}$, for some $\mathscr{J}$. Form the commutative square

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[d, "\Delta"'] & \mathscr{D} \arrow[d, "\Delta"] \\ {[\mathscr{J}, \mathscr{C}]} \arrow[r, "{[\mathscr{J}, F]}"'] & {[\mathscr{J}, \mathscr{D}]} \end{tikzcd}$$

where all the functors have right adjoints. So by the "adjoints in squares" Corollary 17, the diagram of right adjoints

$$\begin{tikzcd} {[\mathscr{J}, \mathscr{D}]} \arrow[r, "{[\mathscr{J}, G]}"] \arrow[d, "\lim_j"'] & {[\mathscr{J}, \mathscr{C}]} \arrow[d, "\lim_j"] \\ \mathscr{D} \arrow[r, "G"'] & \mathscr{C} \end{tikzcd}$$

commutes up to isomorphism, i.e. G preserves limits of shape $\mathscr{J}$. □

For a converse to this theorem, we need to construct initial objects in the categories $(A \downarrow G)$, under the assumption that $\mathscr{D}$ has and G preserves suitable limits.

22 Lemma: ("limits in $(A \downarrow G)$")
Consider $G\colon \mathscr{D} \longrightarrow \mathscr{C}$ and $A \in \mathrm{ob}\,\mathscr{C}$. If $\mathscr{D}$ has and G preserves limits of shape $\mathscr{J}$, then $(A \downarrow G)$ has limits of shape $\mathscr{J}$, and the forgetful functor $U\colon (A \downarrow G) \longrightarrow \mathscr{D}$ creates them.

---

[6] This proof uses more assumptions: we need *all* limits of shape $\mathscr{J}$ to exist in $\mathscr{D}$ and in $\mathscr{C}$. But it gives the "moral reason" for this result to be true.

PROOF. Consider a diagram $D\colon \mathscr{J} \longrightarrow (A \downarrow G)$. Write the object $D(j)$ as $(UD(j), f_j)$ where $f_j\colon A \longrightarrow GUD(j)$.

Suppose $(\lambda_j\colon L \longrightarrow UD(j))_{j \in \mathrm{ob}\,\mathscr{J}}$ is a limit for UD. Then $(G\lambda_j\colon GL \longrightarrow GUD(j))$ is a limit for GUD as G preserves limits. But $(f_j)_{j\in \mathrm{ob}\,\mathscr{J}}$ is a cone over GUD, since the edges of UD lie in $(A \downarrow G)$. So we get a unique $f\colon A \longrightarrow GL$ such that $G\lambda_j \circ f = f_j$ for all j, i.e. such that the λ_j become morphisms $(L, f) \longrightarrow D(j)$ in $(A \downarrow G)$.

$$\begin{tikzcd}
A \arrow[rr, dashed, "f"] \arrow[d, "f_j"'] \arrow[drr, "f_{j'}"' near start] & & GL \arrow[dll, "G\lambda_j"' near start] \arrow[d, "G\lambda_{j'}"] \\
GUD(j) \arrow[rr] & & GUD(j')
\end{tikzcd}$$

They form a cone over D, since U is faithful (which implies that commutativity of diagrams carries over to cones over D), and it is straight forward to verify that this is a limit cone in $(A \downarrow G)$. [**Verify it!**] □

23 Theorem: (Primeval Adjoint Functor Theorem)
Suppose $\mathscr{D}$ has all limits. Then a functor $G\colon \mathscr{D} \longrightarrow \mathscr{C}$ has a left adjoint if and only if it preserves all limits.

PROOF. $\Rightarrow$ Any right adjoint preserves limits.

$\Leftarrow$ For each $A \in \mathrm{ob}\,\mathscr{C}$, $(A \downarrow G)$ has all limits by the "limits in $(A \downarrow G)$" Lemma 22, so it has an initial object by the "initial object as limit" Lemma 12 (Section 2D). Then by the "Adjunctions via Initial objects" Theorem 14, G has a left adjoint. □

However, if a category $\mathscr{D}$ has limits of all diagrams over categories "as big as itself", then $\mathscr{D}$ is a preorder.

The Primeval Adjoint Functor Theorem is useful for posets (c.f. Example Sheet 3 Question 2), but to get a result applicable to general categories we need to impose "size restrictions" on $\mathscr{D}$ and/or $\mathscr{C}$ to ensure that the "large" limit in the "initial object as limit" Lemma can be reduced to a small one.

Definition: Let $\mathscr{C}$ be a category. A set of objects $\{A_i \mid i \in I\}$ in $\mathscr{C}$ is called **weakly initial** if for any $B \in \mathrm{ob}\,\mathscr{C}$ there is an $i \in I$ and a morphism $h_i\colon A_i \longrightarrow B$ in $\mathscr{C}$.

24 Theorem: (General Adjoint Functor Theorem)
Suppose $\mathscr{D}$ is locally small and complete (i.e. $\mathscr{D}$ has all small limits). Then a functor $G\colon \mathscr{D} \longrightarrow \mathscr{C}$ has a left adjoint if and only if G preserves all small limits and for each $A \in \mathrm{ob}\,\mathscr{C}$, $(A \downarrow G)$ has a weakly initial set.

PROOF. $\Rightarrow$ G preserves small limits as a right adjoint, and for each A, $(FA, \eta_A\colon A \longrightarrow GFA)$ is an initial object of $(A \downarrow G)$, i.e. a singleton weakly initial set.

$\Leftarrow$ By the "Limits in $(A \downarrow G)$" Lemma 22, each $(A \downarrow G)$ is complete; also $(A \downarrow G)$ inherits local smallness from $\mathscr{D}$. so we just have to prove

Claim: *If $\mathscr{A}$ is complete, locally small and has a weakly initial set, then $\mathscr{A}$ has an initial object.*

PROOF OF CLAIM. Let $\{A_j, j \in J\}$ be the weakly initial set in $\mathscr{A}$. Form the product $P = \prod_{j\in J} A_j$. Then for any $C \in \mathrm{ob}\,\mathscr{A}$ there is a morphism $P \longrightarrow C$ (i.e. P is a weakly initial object[7]). Form the diagram

$$\begin{tikzcd}
P \arrow[r, shift left=1.5] \arrow[r, shift right=1.5] \arrow[r, phantom, "\vdots"] & P
\end{tikzcd} \tag{†}$$

---

[7] Just choose the appropriate projection from the product and the morphism given from the weakly initial set.

with edges all morphisms $P \longrightarrow P$ that exist in $\mathscr{A}$. Let $I \longrightarrow P$ be a limit for (†) (industrial strength equaliser). Note that $I \rightarrowtail P$ is monic[8].

For every $C \in \mathscr{A}$, there exists a morphism $I \longrightarrow C$, namely $I \rightarrowtail P \longrightarrow C$. We want to show that this is unique. Suppose there are two morphisms $I \overset{f}{\underset{g}{\rightrightarrows}} C$. We can form their equaliser $E \rightarrowtail I$. E is an object of $\mathscr{A}$, so there is a map $P \longrightarrow E$. Then the composition $P \longrightarrow E \rightarrowtail I \rightarrowtail P$ occurs as an arrow in (†), so $I \rightarrowtail P \longrightarrow E \rightarrowtail I \rightarrowtail P = I \rightarrowtail P$.[9] But $I \rightarrowtail P$ is monic, so $I \rightarrowtail P \longrightarrow E \rightarrowtail I = \mathrm{id}_I$. So $E \longrightarrow I$ is split epic, so $E \longrightarrow I \xrightarrow{f} C = E \longrightarrow I \xrightarrow{g} C$ implies $f = g$. So I is an initial object of $\mathscr{A}$. □

This proves that G has a left adjoint, using the "Adjunctions via initial objects" Theorem 14. □

For another version of the Adjoint Functor Theorem, we need:

Definition: A **coseparating family** $\mathscr{G}$ for a category $\mathscr{C}$ is a family of objects $\mathscr{G} = (G_i \mid i \in I)$ such that for any pair $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in $\mathscr{C}$ with $f \neq g$, there is an $i \in I$ and an $h \colon B \longrightarrow G_i$ such that $hf \neq hg$.

25 Theorem: (Special Adjoint Functor Theorem)
Suppose both $\mathscr{C}$ and $\mathscr{D}$ are locally small, and that $\mathscr{D}$ is complete and well-powered and has a coseparating set. Then a functor $G \colon \mathscr{D} \longrightarrow \mathscr{C}$ has a left adjoint if and only if G preserves small limits.

Idea of proof. $(A \downarrow G)$ inherits completeness, local smallness and well-poweredness from $\mathscr{D}$ and the coseparating set for $\mathscr{D}$ gives a coseparating set for $(A \downarrow G)$.

So we just need to prove that if $\mathscr{A}$ is complete, locally small and well-powered and has a coseparating set, then $\mathscr{A}$ has an initial object.

Take the product P of the coseparating set and a limit of a representing set of subobjects of P. This gives a smallest subobject $I \rightarrowtail P$. It is easy to show that there is at most one morphism $I \longrightarrow C$ for any C, but constructing one is more complicated and uses the coseparating set (and local smallness). □

Proof. "$\Rightarrow$" G preserves all limits that exist in $\mathscr{D}$ as it is a right adjoint.

"$\Leftarrow$" The "limits in $(A \downarrow G)$" Lemma 22 implies that each $(A \downarrow G)$ is complete; it also inherits local smallness from $\mathscr{D}$. The Remark 11 "Monos in functor categories" implies that the forgetful functor $(A \downarrow G) \longrightarrow \mathscr{D}$ preserves monos (as it creates and so preserves limits by "limits in $(A \downarrow G)$"), so the subobjects of (B, f) in $(A \downarrow G)$ are those subobjects $B' \rightarrowtail B$ in $\mathscr{D}$ for which $f \colon A \longrightarrow GB$ factors through $GB' \rightarrowtail GB$. So $(A \downarrow G)$ inherits well-poweredness from $\mathscr{D}$.

Given a coseparating set $\mathscr{S}$ for $\mathscr{D}$, the set $\mathscr{S}' = \{(B, f) \mid B \in \mathscr{S},\ f \colon A \longrightarrow GB\}$ (i.e. taking all possible such f) is a coseparating set for $(A \downarrow G)$: if we have $(C, f_C) \overset{g}{\underset{h}{\rightrightarrows}} (D, f_D)$ with $g \neq h$ in $(A \downarrow G)$, there exists $B \in \mathscr{S}$ and $k \colon D \longrightarrow B$ such that $kg \neq kh$. Taking $f = (Gk)f_D$, we have $(B, f) \in \mathscr{S}'$ and $kg \neq kh$ in $(A \downarrow G)$.

$$\begin{tikzcd}
& A \arrow[dl, "f_C"'] \arrow[d, "f_D"] \arrow[dr, "f"] & \\
GC \arrow[r, shift left, "Gg"] \arrow[r, shift right, "Gh"'] & GD \arrow[r, "Gk"] & GB
\end{tikzcd}$$

Note that $\mathscr{S}'$ really is a set, as $\mathscr{A}$ is locally small.

---

[8] This follows from the property of a limit.

[9] Because the identity is also a morphism in (†).

So we have to show that if a category $\mathscr{A}$ is complete, locally small, well-powered and has a coseparating set, then $\mathscr{A}$ has an initial object I.

Let $\{B_j,\ j \in J\}$ be a coseparating set for $\mathscr{A}$. Form $P = \prod_{j\in J} B_j$ (possible as $\mathscr{A}$ is complete), and a set $\{\, P_k \rightarrowtail P \mid k \in K\}$ of representatives of subobjects of P (possible as $\mathscr{A}$ is well-powered). Form the limit of the diagram with edges all the $P_k \rightarrowtail P$ for $k \in K$ (possible as $\mathscr{A}$ is complete).

$$\begin{tikzcd}
& P_k \arrow[dr, tail] & \\
I \arrow[ur] \arrow[r] \arrow[dr] & \vdots \arrow[r, tail] & P \\
& P_{k'} \arrow[ur, tail] &
\end{tikzcd}$$

The legs $I \longrightarrow P_k$ are also monos (proof similar to "Pullbacks preserve monos" Lemma 9). We have

$$(I \rightarrowtail P) \leqslant (\, P_k \rightarrowtail P\,)$$

as subobjects, for all $k \in K$. So $I \rightarrowtail P$ is the smallest subobject of P. We want to show that I is initial in $\mathscr{A}$.

First we show that there can be at most one morphism $I \longrightarrow C$ for any $C \in \mathrm{ob}\,\mathscr{A}$. Suppose we have $I \underset{g}{\overset{f}{\rightrightarrows}} C$. We can form the equaliser $E \rightarrowtail I \underset{g}{\overset{f}{\rightrightarrows}} C$. Then $E \rightarrowtail I \rightarrowtail P$ is a subobject of P, but $I \rightarrowtail P$ is the smallest, so $E \longrightarrow I$ is an isomorphism, and so $f = g$.

Now we want to construct a morphism $I \longrightarrow C$.

For $C \in \mathrm{ob}\,\mathscr{A}$, form the set $T = \{(j, f) \mid\ j \in J,\ f \colon C \longrightarrow B_j\}$, and the product $Q = \prod_{(j,f)} B_j$. We have a canonical morphism $h \colon C \longrightarrow Q$, defined by composition with the projections

$$\begin{tikzcd}
C \arrow[r, "h"] \arrow[dr, "f"'] & Q \arrow[d, "\pi_{(j,f)}"] \\
& B_j
\end{tikzcd}$$

for all $(j, f) \in T$. This h is monic: for $D \underset{g_2}{\overset{g_1}{\rightrightarrows}} C \xrightarrow{h} Q$ with $hg_1 = hg_2$, we have $fg_1 = fg_2$ for all $(j, f) \in T$.

$$\begin{tikzcd}
D \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & C \arrow[r, "h"] \arrow[dr, "f"'] & Q \arrow[d, "\pi_{(j,f)}"] \\
& & B_j
\end{tikzcd}$$

So as the B_j form a coseparating set, $g_1 = g_2$.

We also have a morphism $l \colon P \longrightarrow Q$ defined by

$$\begin{tikzcd}
P \arrow[r, "l"] \arrow[dr, "\pi_j"'] & Q. \arrow[d, "\pi_{(j,f)}"] \\
& B_j
\end{tikzcd}$$

Form a pullback

$$\begin{tikzcd}
R \arrow[r, "o"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, tail, "h"] \\
P \arrow[r, "l"'] & Q
\end{tikzcd}$$

Here m is also monic, as pullbacks preserve monos (Lemma 9), so R is a subobject of P. But $I \rightarrowtail P$ is the smallest, so there is a morphism $I \rightarrowtail R$,

$$\begin{tikzcd}
& R \arrow[r, "o"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, tail, "h"] \\
I \arrow[ur, dotted] \arrow[r, tail] & P \arrow[r, "l"'] & Q
\end{tikzcd}$$

which gives a morphism $I \longrightarrow R \longrightarrow C$ as desired. □

Examples: a) Consider the forgetful functor $U\colon \mathsf{Gp} \longrightarrow \mathsf{Set}$. From the "creating limits" Example 10a) we know that Gp has all small limits and U preserves them; and Gp is locally small. To show U has a left adjoint, we need to find a weakly initial set of $(A \downarrow U)$ (so we can use the General Adjoint Functor Theorem): given a set A, any function $f\colon A \longrightarrow UG$ factors through $U(H \longrightarrow G)$ where H is the subgroup generated by the image of f. And UH has cardinality $\leqslant \max\{\aleph_0, \operatorname{card} A\}$. But, up to isomorphism, there is only a set of groups of a given cardinality, and there is only a set of functions from A to any such group. However, this argument uses most of the machinery required for the explicit construction of free groups.

In fact, in many cases, verifying that each $(A \downarrow G)$ has a weakly initial set is "equivalent in work" to actually constructing a free functor. There are some (more complicated) examples where some cardinality arguments will work but not give you an explicit construction, but we can't cover those with our knowledge.

b) Consider the inclusion $G\colon \mathsf{KHaus} \longrightarrow \mathsf{Top}$. By Tychonoff's Theorem, KHaus has and G preserves all small products; similarly for equalisers, since if $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ is a parallel pair in Top with Y Hausdorff, then the equaliser $E \rightarrowtail X$ is a closed subspace of X, and so compact if X is. KHaus and Top are both locally small, and KHaus is well-powered, since subobjects of X correspond to closed subspaces of X. Moreover, $[0,1]$ is a coseparator for KHaus, by Uryson's Lemma. So by the Special Adjoint Functor Theorem, G has a left adjoint β, the **Stone-Čech compactification** functor.

In fact, Čech's original proof of existence of β goes as follows: given X, form the product $P = \prod_{f\colon X \longrightarrow [0,1]} [0,1]$, and the canonical map $h\colon X \longrightarrow P$ defined by $\pi_f h = f$, and then take βX to be the closure of the image of h. This is exactly the construction given by the SAFT.

CHAPTER 4

Monads

A Monads and their Algebras

Suppose we have an adjunction $\mathscr{C} \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathscr{D}$. How much of this can we describe without mentioning the category $\mathscr{D}$?

We have the composite $T = GF \colon \mathscr{C} \longrightarrow \mathscr{C}$, and the unit $\eta \colon 1_{\mathscr{C}} \longrightarrow T$ and the natural transformation $G\epsilon_F \colon GFGF \longrightarrow GF$ which we denote $\mu \colon TT \longrightarrow T$. These satisfy the identities

$$\begin{tikzcd}
T \arrow[r, "T\eta"] \arrow[dr, "1_T"'] \arrow[dr, phantom, "(1)", shift left=3] & TT \arrow[d, "\mu"] \\
 & T
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
T \arrow[r, "\eta_T"] \arrow[dr, "1_T"'] \arrow[dr, phantom, "(2)", shift left=3] & TT \arrow[d, "\mu"] \\
 & T
\end{tikzcd}$$

by the triangular identities of the adjunction, and

$$\begin{tikzcd}
TTT \arrow[r, "T\mu"] \arrow[d, "\mu_T"'] \arrow[dr, phantom, "(3)"] & TT \arrow[d, "\mu"] \\
TT \arrow[r, "\mu"'] & T
\end{tikzcd}$$

by naturality of ϵ.

Definition: A monad $\mathbb{T} = (T, \eta, \mu)$ on a category $\mathscr{C}$ consists of a functor $T \colon \mathscr{C} \longrightarrow \mathscr{C}$ and natural transformations $\eta \colon 1_{\mathscr{C}} \longrightarrow T$ (the **unit**) and $\mu \colon TT \longrightarrow T$ (the **multiplication**) satisfying the unit laws (1) and (2) and associativity (3).

Example: Given a monoid M, we have a monad structure on the functor $M \times (-) \colon \mathsf{Set} \longrightarrow \mathsf{Set}$; the unit $\eta_A \colon A \longrightarrow M \times A$ sends a to $(1, a)$, and multiplication $\mu_A \colon M \times M \times A \longrightarrow M \times A$ sends (m, n, a) to (mn, a).

Is this induced by an adjunction? Yes!

Consider the category M-Set of M-sets[1]; this has a forgetful functor $G \colon M\text{-}\mathsf{Set} \longrightarrow \mathsf{Set}$, which has a left adjoint F given by $FA = M \times A$ with M-action by multiplication on the left factor. This gives rise to the monad just described.

Definition: Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on a category $\mathscr{C}$. A **$\mathbb{T}$-algebra** is a pair (A, α) where $A \in \mathrm{ob}\,\mathscr{C}$ and $\alpha \colon TA \longrightarrow A$ satisfies

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "1_A"'] \arrow[dr, phantom, "(4)", shift left=3] & TA \arrow[d, "\alpha"] \\
 & A
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
TTA \arrow[r, "T\alpha"] \arrow[d, "\mu_A"'] \arrow[dr, phantom, "(5)"] & TA \arrow[d, "\alpha"] \\
TA \arrow[r, "\alpha"'] & A.
\end{tikzcd}$$

---

[1] These are sets with an action of M on them.

A homomorphism $f\colon (A,\alpha) \longrightarrow (B,\beta)$ of $\mathbb{T}$-algebras is a morphism $f\colon A \longrightarrow B$ in $\mathscr{C}$ satisfying

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "(6)"] & TB \arrow[d, "\beta"] \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

We write $\mathscr{C}^{\mathbb{T}}$ for the category of algebra and their homomorphisms.

Examples: a) The identity functor is a monad on $\mathscr{C}$, its category of algebras is $\mathscr{C}$.

b) There is a **list monad** $(\mathscr{L},\eta,\mu)$ on Set as follows:

$$\begin{aligned} \mathscr{L}\colon \mathsf{Set} &\longrightarrow \mathsf{Set} \\ X &\longmapsto \{\text{lists } (x_1,\dots,x_k) \mid k \geqslant 0,\ \text{each } x_i \in X\} \end{aligned}$$

and appropriately on morphisms. The unit is defined by

$$\begin{aligned} \eta_X\colon X &\longrightarrow \mathscr{L}X \\ x &\longmapsto (x) \qquad\qquad \text{“singleton list”} \end{aligned}$$

and the multiplication

$$\begin{aligned} \mu_X\colon \mathscr{L}\mathscr{L}X &\longrightarrow \mathscr{L}X \\ ((x_{11},\dots,x_{1n}),\dots,(x_{k1},\dots,x_{km})) &\longmapsto (x_{11},\dots,x_{1n},\dots,x_{km}) \end{aligned}$$

is concatenation.

An algebra for $\mathscr{L}$ is a monoid. Indeed, it is a set X with a map

$$\begin{aligned} \theta\colon \mathscr{L}X &\longrightarrow X \\ () &\longmapsto e \\ (x_1,\dots,x_k) &\longmapsto x_1\cdot x_2\cdots x_k \end{aligned}$$

giving multiplication[2].

c) **Powerset monad**: Take the covariant powerset functor $\mathscr{P}\colon \mathsf{Set} \longrightarrow \mathsf{Set}$; the unit is

$$\begin{aligned} \eta_X\colon X &\longrightarrow \mathscr{P}X \\ x &\longmapsto \{x\} \qquad\qquad \text{“singleton set”} \end{aligned}$$

and multiplication

$$\begin{aligned} \mu_X\colon \mathscr{P}\mathscr{P}X &\longrightarrow \mathscr{P}X \\ \{A_i, i\in I\} &\longmapsto \bigcup_{i\in I} A_i \end{aligned}$$

is union.

An algebra for $\mathscr{P}$ is a complete lattice:

$$\begin{aligned} \mathscr{P}X &\longrightarrow X \\ A &\longmapsto \bigvee A \qquad \text{(join of } A) \\ X &\longmapsto \top \\ \varnothing &\longmapsto \bot \end{aligned}$$

Indeed, we get a partial order on X: $a \leqslant b$ if $\bigvee\{a,b\} = b$. You can check that indeed $a \leqslant \top\ \forall a \in X$ and $\bot \leqslant a\ \forall a \in X$ using Diagram (5). As soon as we have all joins and a $\bot$, we also get all meets (by the join of the set of lower bounds, which is non-empty as we have $\bot$).

Algebra homomorphisms are those which preserve arbitrary joins, so the category of algebras is that of sup-complete semilattices.

---

[2] Of all arities at once. Here () is the empty list.

B Eilenberg-Moore Category

Proposition: (Eilenberg-Moore) *There is an adjunction*
$$\begin{tikzcd}\mathscr{C} \arrow[r, shift left, "F^{\mathbb{T}}"] \arrow[r, phantom, "\perp"] & \mathscr{C}^{\mathbb{T}} \arrow[l, shift left, "G^{\mathbb{T}}"]\end{tikzcd}$$
inducing the monad $\mathbb{T}$.

Proof. We define $G^{\mathbb{T}}$ as the forgetful functor $(A, \alpha) \longmapsto A$, $f \longmapsto f$, and $F^{\mathbb{T}}A = (TA, \mu_A)$, which is an algebra by (2) and (3) (called a **free $\mathbb{T}$-algebra**). We let $F^{\mathbb{T}}(f\colon A \longrightarrow B) = Tf$, which is a homomorphism by naturality of μ.

Clearly $G^{\mathbb{T}}F^{\mathbb{T}} = T$, so we take η to be the unit of the adjunction. The counit $\epsilon\colon F^{\mathbb{T}}G^{\mathbb{T}} \longrightarrow 1_{\mathscr{C}^{\mathbb{T}}}$ is defined by $\epsilon_{(A,\alpha)} = \alpha\colon (TA, \mu_A) \longrightarrow (A, \alpha)$ (which is a homomorphism by (5) and natural by (6)). The triangular identities for η and ϵ are just diagrams (1) and (4). Also, $G^{\mathbb{T}}\epsilon_{F^{\mathbb{T}}A} = \mu_A$ by definition of $F^{\mathbb{T}}$, so the monad induced by $(F^{\mathbb{T}} \dashv G^{\mathbb{T}})$ is (T, η, μ). $\square$

There may be other adjunctions inducing the monad (T, η, μ).

Example: Consider
$$\begin{tikzcd}\mathsf{Set} \arrow[r, shift left, "D"] \arrow[r, phantom, "\perp"] & \mathsf{Top} \arrow[l, shift left, "U"]\end{tikzcd}$$
. The monad this induces on Set is the identity monad, which has
$$\begin{tikzcd}\mathsf{Set} \arrow[r, shift left, "1"] \arrow[r, phantom, "\perp"] & \mathsf{Set} \arrow[l, shift left, "1"]\end{tikzcd}$$
as its Eilenberg-Moore adjunction.

But the Eilenberg-Moore adjunction is a terminal object in the category of adjunctions inducing $\mathbb{T}$. We will make this more precise.

Definition: Given a monad $\mathbb{T} = (T, \eta, \mu)$ on $\mathscr{C}$, let $\mathsf{Adj}(\mathbb{T})$ be the category whose objects are adjunctions
$$\begin{tikzcd}\mathscr{D} \arrow[d, shift left, "G"] \\ \mathscr{C} \arrow[u, shift left, "F"] \arrow[u, phantom, "\dashv"]\end{tikzcd}$$
inducing the monad $\mathbb{T}$, and whose morphisms
$$\begin{tikzcd}\mathscr{D} \arrow[d, shift left, "G"] \\ \mathscr{C} \arrow[u, shift left, "F"] \arrow[u, phantom, "\dashv"]\end{tikzcd} \longrightarrow \begin{tikzcd}\mathscr{D}' \arrow[d, shift left, "G'"] \\ \mathscr{C} \arrow[u, shift left, "F'"] \arrow[u, phantom, "\dashv"]\end{tikzcd}$$
are functors $H\colon \mathscr{D} \longrightarrow \mathscr{D}'$ such that $HF = F'$ and $G'H = G$.

26 Proposition: ("Eilenberg-Moore is terminal")

Given a monad $\mathbb{T} = (T, \eta, \mu)$ *on* $\mathscr{C}$ *and an object*
$$\begin{tikzcd}\mathscr{D} \arrow[d, shift left, "G"] \\ \mathscr{C} \arrow[u, shift left, "F"] \arrow[u, phantom, "\dashv"]\end{tikzcd}$$
of $\mathsf{Adj}(\mathbb{T})$*, there is a unique morphism*

$$\begin{tikzcd}\mathscr{D} \arrow[rr, "K"] \arrow[dr, shift left, "G"] & & \mathscr{C}^{\mathbb{T}} \arrow[dl, shift left, "G^{\mathbb{T}}"] \\ & \mathscr{C} \arrow[ul, shift left, "F"] \arrow[ul, phantom, "\dashv"] \arrow[ur, shift left, "F^{\mathbb{T}}"] \arrow[ur, phantom, "\dashv"] & \end{tikzcd}$$

in $\mathsf{Adj}(\mathbb{T})$.

Proof. **Existence:** We define K by $KB = (GB, G\epsilon_B)$ (check it is a $\mathbb{T}$-algebra) and $K(g\colon B \longrightarrow C) = Gg\colon (GB, G\epsilon_B) \longrightarrow (GC, G\epsilon_C)$ (check it is a homomorphism).

Clearly, $G^{\mathbb{T}}K = G$; and

- $KFA = (GFA, G\epsilon_{FA}) = (TA, \mu_A) = F^{\mathbb{T}}A$,
- $KF(f\colon A \longrightarrow A') = GFf = Tf = F^{\mathbb{T}}f$.

Uniqueness: Suppose we have another functor $K'\colon \mathscr{D} \longrightarrow \mathscr{C}^{\mathbb{T}}$ satisfying $G^{\mathbb{T}}K' = G$ and $K'F = F^{\mathbb{T}}$. Then we can write $K'B = (GB, \beta_B)$ for some algebra structure $\beta_B\colon GFGB \longrightarrow GB$ (this is because of the first equation K' satisfies). As $K'(g\colon B \longrightarrow C) = Gg\colon (GB, \beta_B) \longrightarrow (GC, \beta_C)$, β must be a natural transformation $\beta\colon GFG \longrightarrow G$. We also know that $\beta_{FA} = \mu_A = G\epsilon_{FA}$, since $K'F = F^{\mathbb{T}}$.

Now, for any B, the diagram

$$\begin{tikzcd}
GFGFGB \arrow[r, "GFG\epsilon_B"] \arrow[d, "G\epsilon_{FGB}=\mu_{GB}=\beta_{FGB}"'] & GFGB \arrow[d, "\beta_B"] \\
GFGB \arrow[r, "G\epsilon_B"'] & GB
\end{tikzcd}$$

must commute by naturality of β. However, it would commute if we substitue $G\epsilon_B$ for β_B, and $GFG\epsilon_B$ is (split) epic by one of the triangular identities. So $\beta_B = G\epsilon_B$ for all B, and $K' = K$. □

There is also an initial object in $\mathsf{Adj}(\mathbb{T})$.

C Kleisli Category

Given an adjunction $\begin{tikzcd}\mathscr{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{D} \arrow[l, shift left, "G"]\end{tikzcd}$ inducing $\mathbb{T}$ on $\mathscr{C}$, we could consider the full subcategory $\mathscr{D}'$ on objects of the form FA. Then morphisms $FA \longrightarrow FB$ in $\mathscr{D}'$ must correspond to morphisms $A \longrightarrow TB$ in $\mathscr{C}$. We can use this idea to construct a "smallest" adjunction inducing $\mathbb{T}$.

Definition: Given a monad $\mathbb{T} = (T, \eta, \mu)$, the **Kleisli category** $\mathscr{C}_\mathbb{T}$ is defined by:
$\mathrm{ob}\,\mathscr{C}_\mathbb{T} = \mathrm{ob}\,\mathscr{C}$;

Morphisms $A \rightsquigarrow B$ in $\mathscr{C}_\mathbb{T}$ are morphisms $A \longrightarrow TB$ in $\mathscr{C}$. The identity morphism $A \overset{1_A}{\rightsquigarrow} A$ in $\mathscr{C}_\mathbb{T}$ is $\eta_A \colon A \longrightarrow TA$. The composite of two morphisms $A \overset{f}{\rightsquigarrow} B$ and $B \overset{g}{\rightsquigarrow} C$ in $\mathscr{C}_\mathbb{T}$ is

$$A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} TC.$$

We check that this really is a category:

$$\begin{tikzcd}
A \arrow[r, squiggly, "f"] & B \arrow[r, squiggly, "1_B"] & B \arrow[r, phantom, "="] & A \arrow[r, "f"] & TB \arrow[r, "T\eta_B"] \arrow[dr, dotted, "1_{TB}"'] & TTB \arrow[d, "\mu_B"] \\
 & & & & & TB
\end{tikzcd}$$

using (1).

$$\begin{tikzcd}
A \arrow[r, squiggly, "1_A"] & A \arrow[r, squiggly, "f"] & B \arrow[r, phantom, "="] & A \arrow[r, dotted, "f"] \arrow[d, "\eta_A"'] & TB \arrow[d, dotted, "\eta_{TB}"] \arrow[dr, dotted, "1_{TB}"] & \\
 & & & TA \arrow[r, "Tf"] & TTB \arrow[r, "\mu_B"'] & TB
\end{tikzcd}$$

using naturality of η and (2). Given $A \overset{f}{\rightsquigarrow} B \overset{g}{\rightsquigarrow} C \overset{h}{\rightsquigarrow} D$, we have

$$\begin{tikzcd}
(hg)f \arrow[r, phantom, "="] & A \arrow[r, "f"] \arrow[rrd, dashed, "(gf)"'] & TB \arrow[r, "Tg"] \arrow[rrrr, dashed, bend left, "T(hg)"] & TTC \arrow[r, "TTh"] \arrow[d, dotted, "\mu_C"'] & TTTD \arrow[r, "T\mu_D"] \arrow[d, dotted, "\mu_{TD}"] & TTD \arrow[d, "\mu_D"] \arrow[r, phantom, "="] & h(gf) \\
 & & & TC \arrow[r, dotted, "Th"'] & TTD \arrow[r, dotted, "\mu_D"'] & TD &
\end{tikzcd}$$

using naturality of μ and (3).

Proposition: (Kleisli) *There exists an adjunction* $\begin{tikzcd}\mathscr{C} \arrow[r, shift left, "F_\mathbb{T}"] \arrow[r, phantom, "\perp"] & \mathscr{C}_\mathbb{T} \arrow[l, shift left, "G_\mathbb{T}"]\end{tikzcd}$ *inducing* $\mathbb{T}$.

Proof. We define $F_{\mathbb{T}}$ by $F_{\mathbb{T}}A = A$ and $F_{\mathbb{T}}(A \xrightarrow{f} B) = A \xrightarrow{f} B \xrightarrow{\eta_B} TB$. This clearly preserves identities; we check it preserves composition. Given $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathscr{C}$,

$$\begin{tikzcd}
(F_{\mathbb{T}}g)(F_{\mathbb{T}}f) = & A \arrow[r, "f"] \arrow[dr, dotted, "gf"'] & B \arrow[r, "\eta_B"] \arrow[d, dotted, "g"] & TB \arrow[d, "Tg"] & = F_{\mathbb{T}}(gf) \\
& & C \arrow[r, dotted, "\eta_C"'] & TC \arrow[d, "T\eta_C"'] \arrow[dr, dotted, "1_{TC}"] & \\
& & & TTC \arrow[r, "\mu_C"'] & TC
\end{tikzcd}$$

using naturality of η and (1).

We set $G_{\mathbb{T}}A = TA$ and $G_{\mathbb{T}}(A \overset{f}{\rightsquigarrow} B) = TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB$. Then

$$\begin{tikzcd}
G_{\mathbb{T}}(A \overset{1_A}{\rightsquigarrow} A) = & TA \arrow[r, "T\eta_A"] \arrow[dr, dotted, "1_{TA}"'] & TTA \arrow[d, "\mu_A"] & = 1_{TA} \\
& & TA &
\end{tikzcd}$$

using (1), and

$$\begin{tikzcd}
G_{\mathbb{T}}(A \overset{f}{\rightsquigarrow} B \overset{g}{\rightsquigarrow} C) = & TA \arrow[r, "Tf"] \arrow[dr, dashed, "G_{\mathbb{T}}(f)"'] & TTB \arrow[r, "TTg"] \arrow[d, dotted, "\mu_B"] & TTTC \arrow[r, "T\mu_C"] \arrow[d, dotted, "\mu_{TC}"] & TTC \arrow[d, "\mu_C"] & = G_{\mathbb{T}}(g)G_{\mathbb{T}}(f) \\
& & TB \arrow[r, dotted, "Tg"'] \arrow[rr, dashed, bend right, "G_{\mathbb{T}}(g)"'] & TTC \arrow[r, dotted, "\mu_C"'] & TC &
\end{tikzcd}$$

using naturality of μ and (3).

We have $G_{\mathbb{T}}F_{\mathbb{T}}A = TA$ and

$$\begin{tikzcd}
G_{\mathbb{T}}F_{\mathbb{T}}(A \xrightarrow{f} B) = & TA \arrow[r, "Tf"] & TB \arrow[r, "T\eta_B"] \arrow[dr, dotted, "1_{TB}"'] & TTB \arrow[d, "\mu_B"] & = Tf. \\
& & & TB &
\end{tikzcd}$$

So $G_{\mathbb{T}}F_{\mathbb{T}} = T$. We take η as the unit of the adjunction $(F_{\mathbb{T}} \dashv G_{\mathbb{T}})$. The counit ϵ is defined by $TA \overset{\epsilon_A}{\rightsquigarrow} A = 1_{TA}$. Check that this is a natural transformation $F_{\mathbb{T}}G_{\mathbb{T}} \longrightarrow 1_{\mathscr{C}_{\mathbb{T}}}$.

For the triangular identities, we have

$$\begin{tikzcd}
G_{\mathbb{T}}A \arrow[r, "\eta_{G_{\mathbb{T}}A}"] & G_{\mathbb{T}}F_{\mathbb{T}}G_{\mathbb{T}}A \arrow[d, "G_{\mathbb{T}}\epsilon_A"] & = & TA \arrow[r, "\eta_{TA}"] \arrow[ddr, "1_{TA}"'] & TTA \arrow[d, "T1_{TA}"] \\
& G_{\mathbb{T}}A & & & TTA \arrow[d, "\mu_A"] \\
& & & & TA
\end{tikzcd}$$

using (2), and

$$\begin{tikzcd}
F_{\mathbb{T}}A \arrow[r, squiggly, "F_{\mathbb{T}}\eta_A"] & F_{\mathbb{T}}G_{\mathbb{T}}F_{\mathbb{T}}A \arrow[d, squiggly, "\epsilon_{F_{\mathbb{T}}A}"] & = & A \arrow[r, "\eta_A"] \arrow[rrdd, dotted, "\eta_A"'] & TA \arrow[r, "\eta_{TA}"] \arrow[rdd, dotted, "1_{TA}"'] & TTA \arrow[d, "T1_{TA}"] & = & A \arrow[r, squiggly, "1_A"] & A \\
& F_{\mathbb{T}}A & & & & TTA \arrow[d, "\mu_A"] & & & \\
& & & & & TA & & &
\end{tikzcd}$$

also using (2). Finally $G_{\mathbb{T}}\epsilon_{F_{\mathbb{T}}A} = G_{\mathbb{T}}(1_{TA}) = TTA \xrightarrow{T1_{TA}} TTA \xrightarrow{\mu_A} TA = \mu_A$, so the adjunction $(F_{\mathbb{T}} \dashv G_{\mathbb{T}})$ induces $\mathbb{T}$. ☐

27 Proposition: ("Kleisli is initial")
The Kleisli adjunction is initial in $\mathsf{Adj}(\mathbb{T})$.

Proof. Given $\begin{tikzcd} \mathscr{D} \arrow[d, shift left=2, "G"] \\ \mathscr{C} \arrow[u, shift left=2, "F"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$ inducing $\mathbb{T}$, we define $H\colon \mathscr{C}_{\mathbb{T}} \longrightarrow \mathscr{D}$ by $HA = FA$ and $H(A \overset{f}{\rightsquigarrow} B) = FA \xrightarrow{Ff} FGFB \xrightarrow{\epsilon_{FB}} FB$. It is easy to see that H preserves identities, and more generally that $HF_{\mathbb{T}}(f) = Ff$ for any $f \in \operatorname{mor}\mathscr{C}$. We check that H preserves composition: Consider $A \overset{f}{\rightsquigarrow} B \overset{g}{\rightsquigarrow} C$. Then

$$\begin{tikzcd}
H(gf) = & FA \arrow[r, "Ff"] & FGFB \arrow[r, "FGFg", "FTg"'] \arrow[d, dotted, "\epsilon_{FB}"'] & FGFGFC \arrow[r, "FG\epsilon_{FC}", "F\mu_C"'] \arrow[d, dotted, "\epsilon_{FGFC}"'] & FGFC \arrow[d, "\epsilon_{FC}"] & = H(g)H(f) \\
& & FB \arrow[r, dotted, "Fg"'] & FGFC \arrow[r, dotted, "\epsilon_{FC}"'] & FC &
\end{tikzcd}$$

using naturality of ϵ twice. Also $GHA = GFA = TA = G_{\mathbb{T}}A$, and

$$\begin{aligned}
GH(A \overset{f}{\rightsquigarrow} B) &= GFA \xrightarrow{GFf} GFGFB \xrightarrow{G\epsilon_{FB}} GFB \\
&= TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB \\
&= G_{\mathbb{T}}(f).
\end{aligned}$$

So H is a morphism in $\mathsf{Adj}(\mathbb{T})$.

For uniqueness, suppose $H'\colon \mathscr{C}_{\mathbb{T}} \longrightarrow \mathscr{D}$ is a morphism of $\mathsf{Adj}(\mathbb{T})$. Since $H'F_{\mathbb{T}} = F$, we have $H'A = FA$ for all A (i.e. $H'A = HA$). Any morphism $A \overset{f}{\rightsquigarrow} B$ in $\mathscr{C}_{\mathbb{T}}$ can be rewritten as $A \overset{F_{\mathbb{T}}f}{\rightsquigarrow} TB \overset{\epsilon_B}{\rightsquigarrow} B$, and H' maps the counit ϵ_B of the Kleisli adjunction to the counit ϵ_{FB} of $(F \dashv G)^3$, so $H'f$ must be the composite $FA \xrightarrow{Ff} FTB \xrightarrow{\epsilon_{FB}} FB$, i.e. $H' = H$. ☐

The Kleisli category $\mathscr{C}_{\mathbb{T}}$ is equivalent to the full subcategory of $\mathscr{C}^{\mathbb{T}}$ given by the free $\mathbb{T}$-algebras (**Exercise**).

Since $F_{\mathbb{T}}$ is surjective on objects and (as a left adjoint) preserves coproducts, it follows that $\mathscr{C}_{\mathbb{T}}$ has coproducts if $\mathscr{C}$ has them. But in general, it has few other limits and colimits. In constrast:

D Limits and Colimits of Algebras

28 Proposition: ("Limits and colimits of algebras")

i) $G^{\mathbb{T}}\colon \mathscr{C}^{\mathbb{T}} \longrightarrow \mathscr{C}$ *creates all limits which exist in* $\mathscr{C}$.

ii) If $\mathscr{C}$ *has colimits of shape* $\mathscr{J}$, *then* $G^{\mathbb{T}}$ *creates colimits of shape* $\mathscr{J}$ *iff* T *preserves them.*

---

[3] Recall how the correspondance works: both correspond to $1_{TB}\colon TB \longrightarrow TB$.

Proof. i) Consider a diagram $D\colon \mathscr{J} \longrightarrow \mathscr{C}^{\mathbb{T}}$. (We write G for $G^{\mathbb{T}}$). Write $D(j) = (GD(j), \delta_j)$ with $\delta_j\colon TGD(j) \longrightarrow GD(j)$. Suppose $(\lambda_j\colon L \longrightarrow GD(j))_{j \in \text{ob}\,\mathscr{J}}$ is a limit for GD in $\mathscr{C}$. Then $(T\lambda_j\colon TL \longrightarrow TGD(j))$ is a cone over TGD, and the composites $TL \xrightarrow{T\lambda_j} TGD(j) \xrightarrow{\delta_j} GD(j)$ form a cone over GD.

$$\begin{tikzcd}
& TL \arrow[dl, "T\lambda_j"'] \arrow[dr, "T\lambda_{j'}"] & \\
TGD(j) \arrow[rr, "TGD\alpha"'] \arrow[d, "\delta_j"'] & & TGD(j') \arrow[d, "\delta_{j'}"] \\
GD(j) \arrow[rr, "GD\alpha"'] & & GD(j')
\end{tikzcd}$$

So there is a unique $\beta\colon TL \longrightarrow L$ such that

$$\begin{tikzcd}
TL \arrow[r, "\beta"] \arrow[d, "T\lambda_j"'] & L \arrow[d, "\lambda_j"] \\
TGD(j) \arrow[r, "\delta_j"'] & GD(j)
\end{tikzcd} \qquad (\dagger)$$

commutes for all j. We want to show that β gives L a $\mathbb{T}$-algebra structure, i.e. we have to show $\beta\eta_L = 1_L$ and $\beta T\beta = \beta\mu_L$. Both of these conditions mean showing that two morphisms with codomain L are equal, so by the limit property of L, it is enough to show that their composites with λ_j are equal for each j.

We have

$$\begin{tikzcd}
L \arrow[r, dotted, "\lambda_j"] \arrow[d, "\eta_L"'] & GD(j) \arrow[d, dotted, "\eta_{GD(j)}"'] \arrow[dd, dotted, bend left=60, "1"] \\
TL \arrow[r, dotted, "T\lambda_j"'] \arrow[d, "\beta"'] & TGD(j) \arrow[d, dotted, "\delta_j"'] \\
L \arrow[r, "\lambda_j"'] & GD(j)
\end{tikzcd}$$

so $\lambda_j\beta\eta_L = \lambda_j$ for all j, and

$$\begin{aligned}
\lambda_j\beta T\beta &= \delta_j T\lambda_j T\beta && \text{by } (\dagger)\\
&= \delta_j T\delta_j TT\lambda_j && \text{by T}(\dagger)\\
&= \delta_j \mu_{GD(j)} TT\lambda_j && \text{by } \delta_j \text{ being } \mathbb{T}\text{-algebra structure}\\
&= \delta_j T\lambda_j \mu_L && \text{by naturality of } \mu\\
&= \lambda_j \beta\mu_L && \text{by } (\dagger).
\end{aligned}$$

So (L, β) is a $\mathbb{T}$-algebra, and the λ_j are $\mathbb{T}$-algebra homomorphisms, by $(\dagger)$. To show that $(\lambda_j\colon (L,\beta) \longrightarrow D(j))$ is a limit for D in $\mathscr{C}^{\mathbb{T}}$, consider any cone $(\nu_j\colon (N,\gamma) \longrightarrow D(j))$ over

D in $\mathscr{C}^{\mathbb{T}}$.

$$\begin{tikzcd}
& TN \arrow[dl, "T\nu_j"'] \arrow[dr, "T\nu_{j'}"] \arrow[dd, dotted, "\gamma"] & \\
TGD(j) \arrow[rr, "TGD\alpha"'] \arrow[dd, "\delta_j"'] & & TGD(j') \arrow[dd, "\delta_{j'}"] \\
& N \arrow[dl, "\nu_j"'] \arrow[dr, "\nu_{j'}"] & \\
GD(j) \arrow[rr, "GD\alpha"'] & & GD(j')
\end{tikzcd}$$

Then $(\nu_j \colon N \longrightarrow GD(j))$ is a cone in $\mathscr{C}$, so there is a unique factorisation $n \colon N \longrightarrow L$ over $(\lambda_j \colon L \longrightarrow GD(j))$ in $\mathscr{C}$, and again composing with the λ_j shows that n is in fact a morphism in $\mathscr{C}^{\mathbb{T}}$.

The same argument shows that any cone over D whose image in $\mathscr{C}$ is a limit of GD is indeed a limit cone in $\mathscr{C}^{\mathbb{T}}$.

ii) The proof of $\Leftarrow$ is exactly like i), except that we need to know that T (and TT) preserve the colimit of GD. For $\Rightarrow$, we note that T is the composite $G^{\mathbb{T}}F^{\mathbb{T}}$, and $F^{\mathbb{T}}$ preserves colimits because it is a left adjoint.

□

Because of this proposition, it would be useful to know when the comparison functor K is part of an equivalence of categories.

E Monadicity

Definition: An adjunction $(F \dashv G)$ is **monadic** if K is part of an equivalence. We also say $G \colon \mathscr{D} \longrightarrow \mathscr{C}$ is a **monadic functor** if it has a left adjoint and the adjunction is monadic.

Lemma: *Monadic functors reflect isomorphisms.*

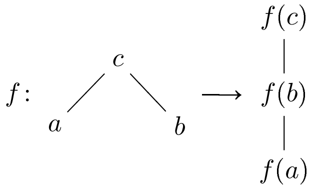
Proof. If $G \colon \mathscr{D} \longrightarrow \mathscr{C}$ is monadic, then there is $F \dashv G$ such that $G = G^{\mathbb{T}}K$. As K is part of an equivalence, it is enough to show that $G^{\mathbb{T}} \colon \mathscr{C}^{\mathbb{T}} \longrightarrow \mathscr{C}$ reflects isos (for any monad $\mathbb{T}$). Given $f \colon (A, \alpha) \longrightarrow (B, \beta)$ in $\mathscr{C}^{\mathbb{T}}$ with $f \colon A \longrightarrow B$ an iso in $\mathscr{C}$, then f^{-1} is also a morphism of $\mathbb{T}$-algebras:

$$\alpha T f^{-1} = f^{-1} f \alpha T f^{-1} = f^{-1} \beta T f T f^{-1} = f^{-1} \beta.$$

□

So this already tells us that some functors are *not* monadic.

Example: The forgetful functor Poset $\longrightarrow$ Set doesn't reflect isos.



is an iso in Set but not in Poset.

But to properly characterise monadic functors, we need more. The main idea is that algebras are coequalisers of morphisms between free algebras. We will make this more precise.

Definition: a) A **reflexive pair** in $\mathscr{C}$ is a parallel pair $A \underset{g}{\overset{f}{\rightrightarrows}} B$ for which there exists $r\colon B \longrightarrow A$ with $fr = gr = 1_B$ (such an r is called a common splitting). A **reflexive coequaliser** is a coequaliser of a reflexive pair.

b) A **split coequaliser diagram** is a diagram of the form

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] \arrow[l, bend left=50, "t"] & C \arrow[l, bend left=50, "s"]
\end{tikzcd}$$

satisfying $hf = hg$, $hs = 1_C$, $gt = 1_B$ and $ft = sh$. Recall from Example Sheet 2 that this makes h into the coequaliser of f and g.

c) Given a functor $G\colon \mathscr{D} \longrightarrow \mathscr{C}$, a parallel pair $A \underset{g}{\overset{f}{\rightrightarrows}} B$ in $\mathscr{D}$ is **G-split** if there exists a split coequaliser diagram in $\mathscr{C}$:

$$\begin{tikzcd}
GA \arrow[r, shift left, "Gf"] \arrow[r, shift right, "Gg"'] & GB \arrow[r, "h"] \arrow[l, bend left=50, "t"] & C \arrow[l, bend left=50, "s"]
\end{tikzcd}$$

29 Examples: ("Split coequalisers")

Given an adjunction $\mathscr{C} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathscr{D}$ inducing (T, η, μ) and a $\mathbb{T}$-algebra (A, α),

$$\begin{tikzcd}
TTA \arrow[r, shift left, "T\alpha"] \arrow[r, shift right, "\mu_A"'] & TA \arrow[r, "\alpha"] \arrow[l, bend left=50, "\eta_{TA}"] & A \arrow[l, bend left=50, "\eta_A"]
\end{tikzcd}$$

is a split coequaliser diagram. So $FGFA \underset{\epsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} FA$ is G-split.

Similarly

$$\begin{tikzcd}
GFGFGB \arrow[r, shift left, "GFG\epsilon_B"] \arrow[r, shift right, "G\epsilon_{FGB}"'] & GFGB \arrow[r, "G\epsilon_B"] \arrow[l, bend left=30, "\eta_{GFGB}"] & GB \arrow[l, bend left=50, "\eta_{GB}"]
\end{tikzcd}$$

is a split coequaliser diagram.

30 Lemma: ("$\mathbb{T}$-algebras are coequalisers")

Given a monad $\mathbb{T}$ on $\mathscr{C}$ and an algebra (A, α), the structure map $\alpha\colon (TA, \mu_A) \longrightarrow (A, \alpha)$ is a coequaliser in $\mathscr{C}^{\mathbb{T}}$.

Proof. Consider the diagram

$$\begin{tikzcd}
TTTA \arrow[r, shift left, "TT\alpha"] \arrow[r, shift right, "T\mu_A"'] \arrow[d, "\mu_{TA}"'] & TTA \arrow[r, "T\alpha"] \arrow[d, "\mu_A"] & TA \arrow[d, "\alpha"] \\
TTA \arrow[r, shift left, "T\alpha"] \arrow[r, shift right, "\mu_A"'] & TA \arrow[r, "\alpha"'] & A
\end{tikzcd}$$

in $\mathscr{C}^{\mathbb{T}}$. Here the bottom row is a split coequaliser in $\mathscr{C}$ and $T\alpha$ is (split) epic. Given any $f\colon (TA, \mu_A) \longrightarrow (B, \beta)$ in $\mathscr{C}^{\mathbb{T}}$ with $fT\alpha = f\mu_A$, we get a unique $g\colon A \longrightarrow B$ in $\mathscr{C}$ satisfying $g\alpha = f$. Then as $T\alpha$ is epic, g is an algebra homomorphism, so (A, α) is a coequaliser in $\mathscr{C}^{\mathbb{T}}$. □

Notice that $(TA, \mu_A) = F^{\mathbb{T}}G^{\mathbb{T}}(A, \alpha)$. So the "primeval" idea behind monadicity theorems is that we recognise a monadic adjunction $\mathscr{C} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathscr{D}$ by the fact that for any $B \in \operatorname{ob} \mathscr{D}$,

$$\begin{tikzcd}
FGFGB \arrow[r, shift left, "\epsilon_{FGB}"] \arrow[r, shift right, "FG\epsilon_B"'] & FGB \arrow[r, "\epsilon_B"] & B
\end{tikzcd}$$

is a coequaliser[4]. This diagram is called the **standard free presentation** of B.

31 Theorem: (Precise Monadicity Theorem)
A functor $G\colon \mathscr{D} \longrightarrow \mathscr{C}$ is monadic if and only if

i) G has a left adjoint and
ii) G creates coequalisers of G-split parallel pairs.

32 Theorem: (Crude Monadicity Theorem)
Consider $G\colon \mathscr{D} \longrightarrow \mathscr{C}$ such that

i) G has a left adjoint,
ii) G reflects isomorphisms,
iii) $\mathscr{D}$ has and G preserves reflexive coequalisers.

Then G is monadic.

Proof. (Precise $\Rightarrow$) If G is monadic, it has a left adjoint by definition. For ii) it is sufficient to show that $G^{\mathbb{T}}\colon \mathscr{C}^{\mathbb{T}} \longrightarrow \mathscr{C}$ creates coequalisers of $G^{\mathbb{T}}$-split pairs. If $(A,\alpha)\underset{g}{\overset{f}{\rightrightarrows}}(B,\beta)$ is a parallel pair in $\mathscr{C}^{\mathbb{T}}$ and $\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] \arrow[l, bend left=50, "t"] & C \arrow[l, bend left=50, "s"] \end{tikzcd}$ is a split coequaliser in $\mathscr{C}$, then $TA\underset{Tg}{\overset{Tf}{\rightrightarrows}}TB\xrightarrow{Th}TC$ is also a coequaliser. So as $h\beta Tf = hf\alpha = hg\alpha = h\beta Tg$, we get a unique $\gamma\colon TC \longrightarrow C$ such that

$$\begin{tikzcd} TB \arrow[r, "Th"] \arrow[d, "\beta"'] \arrow[dr, phantom, "(\dagger)"] & TC \arrow[d, "\gamma"] \\ B \arrow[r, "h"'] & C \end{tikzcd} \qquad (\dagger)$$

commutes.

To show that (C,γ) is a $\mathbb{T}$-algebra, i.e. that $\gamma\eta_C = 1_C$ and $\gamma T\gamma = \gamma\mu_C$, it is enough to show $\gamma\eta_C h = h$ and $\gamma T\gamma TTh = \gamma\mu_C TTh$, as h and TTh are coequalisers. These two equations follow from naturality of η and μ, $(\dagger)$ and the fact that (B,β) is a $\mathbb{T}$-algebra. Then $h\colon (B,\beta) \longrightarrow (C,\gamma)$ is the coequaliser of f and g in $\mathscr{C}^{\mathbb{T}}$ (proof as in previous lemma).

(Precise $\Leftarrow$ and Crude) We have

$$\begin{tikzcd} \mathscr{D} \arrow[rr, "K"] \arrow[dr, shift left, "G"] & & \mathscr{C}^{\mathbb{T}} \arrow[dl, shift left, "G^{\mathbb{T}}"] \\ & \mathscr{C} \arrow[ul, shift left, "F", "\dashv"] \arrow[ur, shift left, "F^{\mathbb{T}}", "\dashv"'] & \end{tikzcd}$$

We will construct a left adjoint $L\colon \mathscr{C}^{\mathbb{T}} \longrightarrow \mathscr{D}$ for K and the unit and counit of $L \dashv K$ and show that they are isos.

Given a $\mathbb{T}$-algebra (A,α), form the coequaliser

$$FGFA \underset{\epsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} FA \xrightarrow{l_{(A,\alpha)}} L(A,\alpha)$$

in $\mathscr{D}$. We can do this as $(F\alpha, \epsilon_{FA})$ is a reflexive pair with common splitting $F\eta_A$, so by (Crude iii) it has a coequaliser, or because it is G-split (see Example 29 "Split coequalisers") so by (Precise ii) it has a coequaliser.

---

[4] It is G-split: see the "split coequalisers" Example 29.

Any algebra homomorphism $F\colon (A,\alpha) \longrightarrow (B,\beta)$ induces two commutative squares

$$\begin{tikzcd}
FGFA \arrow[r, shift left, "F\alpha"] \arrow[r, shift right, "\epsilon_{FA}"'] \arrow[d, "FGFf"'] & FA \arrow[d, "Ff"] \\
FGFB \arrow[r, shift left, "F\beta"] \arrow[r, shift right, "\epsilon_{FB}"'] & FB
\end{tikzcd}$$

and hence a unique morphism $L(f)\colon L(A,\alpha) \longrightarrow L(B,\beta)$. So L is a functor.

To get the counit $\theta\colon LK \longrightarrow 1_{\mathscr{D}}$, consider $B \in \operatorname{ob}\mathscr{D}$. Then $KB = (GB, G\epsilon_B)$, so we have a coequaliser

$$\begin{tikzcd}
FGFGB \arrow[r, shift left, "FG\epsilon_B"] \arrow[r, shift right, "\epsilon_{FGB}"'] & FGB \arrow[r] \arrow[dr, "\epsilon_B"'] & LKB \arrow[d, dotted, "\theta_B"] \\
& & B
\end{tikzcd}$$

But ϵ_B has equal composite with this pair, so we get a morphism $\theta_B\colon LKB \longrightarrow B$.

In fact, $(FG\epsilon_B, \epsilon_{FGB})$ is G-split (see Example 29 "Split coequalisers"), so either by (Precise ii) or by (Crude ii and iii)[5] we deduce that θ_B is an isomorphism (i.e. ϵ_B is also a coequaliser for this pair). Naturality of θ follows from it being an iso and LKf being uniquely determined by the coequaliser property.

For the unit $\theta\colon 1_{\mathscr{C}^{\mathbb{T}}} \longrightarrow KL$, we have $KL(A,\alpha) = (GL(A,\alpha), G\epsilon_{L(A,\alpha)})$ and

$$\begin{tikzcd}
(GFGFA, \mu TA) \arrow[r, shift left, "GF\alpha"] \arrow[r, shift right, "G\epsilon_{FA}"'] & (GFA, \mu_A) \arrow[r, "\alpha"] & (A,\alpha)
\end{tikzcd}$$

is a coequaliser in $\mathscr{C}^{\mathbb{T}}$ by the "$\mathbb{T}$-algebras are coequalisers" Lemma 30. So via

$$\begin{tikzcd}
GFGFA \arrow[r, shift left, "GF\alpha"] \arrow[r, shift right, "G\epsilon_{FA}"'] & GFA \arrow[r, "\alpha"] \arrow[dr, "Gl_{(A,\alpha)}"'] & A \arrow[d, dotted, "\phi_{(A,\alpha)}"] \\
& & GL(A,\alpha)
\end{tikzcd}$$

we get a homomorphism $\phi_{(A,\alpha)}\colon (A,\alpha) \longrightarrow (GL(A,\alpha), G\epsilon_{L(A,\alpha)})$. (Note that $Gl_{(A,\alpha)}$ is an algebra morphism by naturality of $G\epsilon$.) To show that $\phi_{(A,\alpha)}$ is an iso, it is enough to show that

$$\begin{tikzcd}
GFGFA \arrow[r, shift left, "GF\alpha"] \arrow[r, shift right, "G\epsilon_{FA}"'] & GFA \arrow[r, "\alpha"] & A
\end{tikzcd}$$

is a coequaliser in $\mathscr{C}$. But

$$\begin{tikzcd}
FGFA \arrow[r, shift left, "F\alpha"] \arrow[r, shift right, "\epsilon_{FA}"'] & FA \arrow[r, "l_{(A,\alpha)}"] & L(A,\alpha)
\end{tikzcd}$$

is a coequaliser by definition, so using (Precise ii), G creates coequalisers of G-split pairs, so it also preserves them, or using (Crude iii) G preserves reflexive coequalisers. Thus ϕ is a natural iso (naturality follows as for θ). ☐

Exercise: Check that θ and ϕ satisfy the triangular identities[6].

33 Examples:

a) For any category $\mathscr{D}$ whose objects are sets A equipped with algebraic operations $A^k \longrightarrow A$ satisfying equations, and whose morphisms are homomorphisms, the forgetful functor $G\colon \mathscr{D} \longrightarrow \mathsf{Set}$ is monadic iff it has a left adjoint. (For infinitary structure, the free functor may not exist, e.g. for complete Boolean algebras; but for finitary structure it does, c.f. Example Sheet 3 Question 6.) This can be proved using the Precise Monadicity Theorem, c.f. Example Sheet 3 Question 9.

---

[5] It is also a reflexive pair.

[6] We don't actually need it for this proof, but they do, and we'll need it later.

Remark: For a finitary algebraic category $\mathscr{C}$, the forgetful functor $\mathscr{C} \longrightarrow \mathsf{Set}$ satisfies the hypotheses of the Crude Monadicity Theorem.

b) Any reflection is monadic: this can be proved directly (see Example Sheet 3 Question 7), but also follows from the Precise Monadicity Theorem. If $G\colon \mathscr{D} \longrightarrow \mathscr{C}$ is the inclusion of a (full) reflective subcategory, and $A \underset{g}{\overset{f}{\rightrightarrows}} B$ is a G-split pair with splitting

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] \arrow[l, bend left, "t"] & C \arrow[l, bend left, "s"] \end{tikzcd}$$

in $\mathscr{C}$, then $t \in \operatorname{mor} \mathscr{D}$ (as $\mathscr{D}$ is a *full* subcategory), and so $ft = sh$ is in $\mathscr{D}$. We have $shsh = sh$, so sh is an idempotent in $\mathscr{D}$. But an idempotent e splits iff the pair $(e, 1_{\mathrm{dom}\, e})$ has an equaliser (Exercise, see Example Sheet 2 Question 2), so the splitting (s, h) in $\mathscr{C}$ can be obtained by the equaliser of $(sh, 1_B)$. But $\mathscr{D}$ is closed under all limits which exist in $\mathscr{C}$, so (up to isomorphism) h, s and C also live in $\mathscr{D}$ (i.e. G creates coequalisers of G-split pairs).

c) Let $\mathscr{C} \subseteq \mathsf{AbGp}$ be the full subcategory of torsion-free abelian groups. The inclusion $\mathscr{C} \longrightarrow \mathsf{AbGp}$ has a left adjoint $A \longmapsto A/A_t$ (where A_t is the subgroup of elements of finite order in A), so it is monadic by b). Also, the forgetful functor $\mathsf{AbGp} \longrightarrow \mathsf{Set}$ is monadic by a).

However, the composite adjunction $\mathscr{C} \rightleftarrows \mathsf{AbGp} \rightleftarrows \mathsf{Set}$ isn't monadic since it induces the same monad on Set as $\mathsf{AbGp} \rightleftarrows \mathsf{Set}$.Thus monadicity is not stable under composition. (Note that the hypotheses of the Crude Monadicity Theorem *are* stable under composition.)

In general, given an adjunction

$$\begin{tikzcd} \mathscr{D} \arrow[d, shift left, "G"] \arrow[d, phantom, "\dashv"] \\ \mathscr{C} \arrow[u, shift left, "F"] \end{tikzcd}$$

where $\mathscr{D}$ has (at least) reflexive coequalisers, we can construct the "monadic tower"

$$\begin{tikzcd} \mathscr{D} \arrow[dddd, shift left, "G"] \arrow[dddd, phantom, "\dashv"'] \arrow[ddr, "K"] \arrow[drr, "K'"] & & \vdots \\ & & (\mathscr{C}^{\mathbb{T}})^{\mathbb{S}} \arrow[ull, "L'"] \arrow[dl, shift left, "G^{\mathbb{S}}"] \\ & \mathscr{C}^{\mathbb{T}} \arrow[uul, "L"] \arrow[ur, shift left, "F^{\mathbb{S}}"] \arrow[ddl, shift left, "G^{\mathbb{T}}"] & \\ & & \\ \mathscr{C} \arrow[uuuu, shift left, "F"] \arrow[uur, shift left, "F^{\mathbb{T}}"] & & \end{tikzcd}$$

where $\mathbb{T}$ is the monad induced by $(F \dashv G)$, K is the comparison functor to the Eilenberg-Moore adjunction, L is the left adjoint of K constructed as in the proof earlier, $\mathbb{S}$ is the monad induced by $(L \dashv K)$, and so on.

Definition: We say $(F \dashv G)$ has **monadic length** n if this process produces an equivalence after n steps.

Examples: a) The forgetful functor $G\colon \mathsf{Top} \longrightarrow \mathsf{Set}$ has a left adjoint D, but has monadic length ∞, since $GD = 1_{\mathsf{Set}}$, $\eta = \mu = 1_{1_{\mathsf{Set}}}$ and so all categories in the monadic tower are isomorphic to Set.

b) An equivalence of categories has monadic length 0, and a monadic adjunction has monadic length 1.

c) The composite adjunction of torsion-free abelian groups in sets from Example 33c) above has monadic length 2. Another example of the same form is given by the reflective subcategory of Stone spaces (compact 0-dimensional spaces) inside the category KHaus

of compact Hausdorff spaces, which itself is monadic over Top (meaning the forgetful functor is monadic). The composite adjunction will again have monadic length 2.

CHAPTER 5

Abelian Categories

A Pointed Categories, Kernels and Cokernels

Definition: A **zero object** is an object 0 in a category $\mathscr{C}$ which is both initial and terminal. A **zero morphism** $0\colon A \longrightarrow B$ is the unique morphism factoring over the zero object $A \longrightarrow 0 \longrightarrow B$.[1] A category with a zero object is called **pointed**.

Examples: The categories of pointed sets Set_*, monoids Mon, groups Gp, abelian groups AbGp, R-modules R-$\mathsf{Mod}$are all pointed.

Notice that when $\mathscr{C}$ is locally small and pointed, the functor $\mathscr{C}(-,-)\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \longrightarrow \mathsf{Set}$ factors over the category of pointed sets. We then say that $\mathscr{C}$ is enriched in Set_*.

Lemma: *If $\mathscr{C}$ is enriched in Set_* and $I \in \operatorname{ob}\mathscr{C}$, the following are equivalent:*

(i) I is initial;
(ii) I is terminal;
(iii) $1_I = 0\colon I \longrightarrow I$.

Proof. Clearly (i) ⇒ (iii) and (ii) ⇒ (iii). Moreover, (iii) implies that for any $f\colon I \longrightarrow A$ we have

$$I \xrightarrow{f} A \;=\; I \xrightarrow{1_I} I \xrightarrow{f} A \;=\; I \xrightarrow{0} I \xrightarrow{f} A \;=\; I \xrightarrow{0} A$$

So (iii) ⇒ (i). Similarly, (iii) ⇒ (ii). □

Definition: Given $f\colon A \longrightarrow B$ in a pointed category $\mathscr{C}$, the **kernel** of f is the pullback of $0 \longrightarrow B$ along f:

$$\begin{tikzcd}
\operatorname{Ker} f \arrow[r, tail, "\ker f"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
0 \arrow[r] & B
\end{tikzcd}$$

The **cokernel** of f is the pushout

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d, two heads, "\operatorname{coker} f"] \\
0 \arrow[r] & \operatorname{Coker} f. \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

(We write arrows which are kernels or cokernels as indicated.)

Notice that when $\mathscr{C}$ is pointed, any morphism $0 \longrightarrow A$ is a (split) mono. So as pullbacks preserve monos, every kernel is a mono. (Similarly every morphism $B \longrightarrow 0$ is a split epi.[2])

Definition: A **normal monomorphism** is a morphism which occurs as the kernel of some morphism. A **normal epimorphism** is a morphism which occurs as the cokernel of some morphism.

---

[1] So composing anything with 0 gives 0.

[2] We won't do all the dual results in what follows, you can supply them yourself.

Lemma: *Any normal mono is a regular mono.*

Proof. The morphism $k\colon K \longrightarrow A$ is the kernel of $f\colon A \longrightarrow B$ if and only if it is the equaliser of $A \underset{0}{\overset{f}{\rightrightarrows}} B$. ☐

Examples: In Gp, every mono is regular, but a mono $K \longrightarrow G$ is normal iff K is a normal subgroup of G. But every epimorphism $f\colon G \longrightarrow H$ is normal, since if f is surjective then $H \cong G/\operatorname{Ker} f$.

In Set_*, every mono is normal, since if $f\colon A \longrightarrow B$ is injective, then it is the kernel of $B \longrightarrow B/{\sim}$ (where $b_1 \sim b_2 \Leftrightarrow b_1 = b_2$ or $\{b_1, b_2\} \subset \operatorname{Im} f$). But not every epi in Set_* is normal.

34 Lemma: ("A normal mono is the kernel of its cokernel.")
Let $\mathscr{C}$ be pointed with cokernels. Then $f\colon A \longrightarrow B$ is a normal mono in $\mathscr{C}$ iff $f = \ker(\operatorname{coker} f)$.

Proof. $\Leftarrow$ trivial. $\Rightarrow$ Suppose $f = \ker(g\colon B \longrightarrow C)$. Let $q = \operatorname{coker} f$. Then as $gf = 0$, g factors as hq.

$$\begin{tikzcd}
& E \arrow[dl, dotted, "l"'] \arrow[dr, "e"] & & & \\
A \arrow[rr, tail, "f"] & & B \arrow[rr, "g"] \arrow[dr, two heads, "q"'] & & C \\
& & & D \arrow[ur, dotted, "h"'] &
\end{tikzcd}$$

Given $e\colon E \longrightarrow B$ with $qe = 0$, then also $ge = hqe = 0$, so e factors uniquely as fl. Then (as $qf = 0$) this implies that $f = \ker q$. ☐

Lemma: *Let $\mathscr{C}$ be pointed. Then any mono has a kernel, and that kernel is $0\colon 0 \rightarrowtail A$.*

Proof. If $f\colon A \rightarrowtail B$ has a kernel, then we see from

$$\begin{tikzcd}
\operatorname{Ker} f \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, tail, "f"] \\
0 \arrow[r, tail] & B
\end{tikzcd}$$

that $\operatorname{Ker} f \rightarrowtail 0$ is a mono since pullbacks preserve monos. But $\operatorname{Ker} f \rightarrowtail 0$ is always split epic, so here it is an isomorphism.

Moreover, for any mono f, if for any $g\colon C \longrightarrow A$ we have $fg = 0$, then as f is monic, $g = 0$, so indeed

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, tail, "f"] \\
0 \arrow[r] & B
\end{tikzcd}$$

is a pullback. ☐

35 Lemma: ("kernel of zero")
The kernel of $0\colon A \longrightarrow B$ is 1_A.

Proof. Exercise. ☐

Recall that the kernel pair of $f\colon A \longrightarrow B$ is the pullback of f along itself:

$$\begin{tikzcd}
R(f) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

Lemma: *Let $\mathscr{C}$ be pointed with pullbacks. Then given $f\colon A \longrightarrow B$, we have* $\ker f\colon \operatorname{Ker} f \longrightarrow A = \operatorname{Ker}\pi_1 \xrightarrow{\ker\pi_1} R(f) \xrightarrow{\pi_2} A$.

Proof. Use "pullback composition" (Question 10 on Sheet 1) on the two pullbacks

$$\begin{tikzcd}
\operatorname{Ker}\pi_1 \arrow[r, "\ker\pi_1"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & R(f) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
0 \arrow[r] & A \arrow[r, "f"'] & B.
\end{tikzcd}$$

$\square$

36 Lemma: ("Kernels and pullbacks")
Let $\mathscr{C}$ be pointed with kernels. Consider

$$\begin{tikzcd}
K \arrow[r, tail, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "(1)"] & A \arrow[r, "g"] \arrow[d, "a"] \arrow[dr, phantom, "(2)"] & B \arrow[d, "b"] \\
K' \arrow[r, tail, "f'"'] & A' \arrow[r, "g'"'] & B'
\end{tikzcd}$$

where $f = \ker g$ and $f' = \ker g'$.

(i) If b is a mono, then (1) is a pullback.
(ii) If (2) is a pullback, then k is an iso.

Proof. (i) Consider $h_1\colon D \longrightarrow A$ and $h_2\colon D \longrightarrow K'$ such that $ah_1 = f'h_2$. Then $bgh_1 = g'ah_1 = g'f'h_2 = 0$, so as b is a mono, $gh_1 = 0$. So h_1 factors uniquely over the kernel of g: $$\begin{tikzcd} D \arrow[rr, "h_1"] \arrow[dr, "l"'] & & A \\ & K \arrow[ur, tail, "f"'] & \end{tikzcd}$$. As f' is monic, also $kl = h_2$, so (1) is a pullback.

(ii) $f'\colon K' \longrightarrow A'$ satisfies $g'f' = 0 = b\circ 0$. So as (2) is a pullback, there is a unique $h\colon K' \longrightarrow A$ such that $ah = f'$ and $gh = 0$. Then there is a unique $l\colon K' \longrightarrow K$ such that $fl = h$, as $f = \ker g$. Then $f' = ah = afl = f'kl$, so as f' is monic, $kl = 1_{K'}$. It remains to show $lk = 1_K$. For this, consider $gflk = 0 = gf1_K$ and $aflk = ahk = f'k = af1_K$. So as (2) is a pullback, $flk = f1_K$, but f is monic, so $lk = 1_K$.

$\square$

Alternative proof. Consider the cube

$$\begin{tikzcd}
 & 0 \arrow[rr] \arrow[dd, dotted] & & B \arrow[dd, "b"] \\
K \arrow[ur] \arrow[rr, tail, "f"] \arrow[dd, "k"'] & & A \arrow[ur, "g"] \arrow[dd, "a"] & \\
 & 0 \arrow[rr, dotted] & & B' \\
K' \arrow[ur, dotted] \arrow[rr, tail, "f'"'] & & A \arrow[ur, "g'"'] &
\end{tikzcd}$$

in which the top and bottom side are pullbacks, as $f = \ker g$ and $f' = \ker g'$. If b is a mono, the back is also a pullback, so as "pullbacks of pullbacks are pullbacks" (see Example Sheet 2), the front square (which is (1)) is also a pullback, which prove (i). If (ii) is a pullback, then "pullbacks of pullbacks are pullbacks" implies that the left-hand square is a pullback too, which makes k an isomorphism. □

B Additive Categories

Consider two morphisms $A \overset{f}{\underset{g}{\rightrightarrows}} B$ between abelian groups A and B. We can define the "pointwise sum" $f + g\colon A \longrightarrow B$ by $(f + g)(a) = f(a) + g(a)$. Then as B is abelian, $f + g$ is also a group homomorphism. So the homset $\mathsf{AbGp}(A, B)$ has an abelian group structure.

Definition: A locally small category $\mathscr{A}$ is **enriched in abelian groups** if the functor

$$\mathscr{A}(-,-)\colon \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \longrightarrow \mathsf{Set}$$

factors through the forgetful functor $\mathsf{AbGp} \longrightarrow \mathsf{Set}$.

I.e. $\mathscr{A}$ is enriched in abelian groups if each homset $\mathscr{A}(A, B)$ is an abelian group, and composition

$$\begin{aligned} \mathscr{A}(A,B) \times \mathscr{A}(B,C) &\longrightarrow \mathscr{A}(A,C) \\ (f,g) &\longmapsto gf \end{aligned}$$

is "a group homomorphism in each variable", i.e.

$$\begin{aligned} g(f_1 + f_2) &= gf_1 + gf_2 \qquad \text{and} \\ (g_1 + g_2)f &= g_1f + g_2f. \end{aligned}$$

Some people call such an $\mathscr{A}$ **preadditive** or an Ab-**category**.

Examples: AbGp, R-Mod, $\mathsf{AbGp}_{\text{t.f.}}$ (torsion free abelian groups). Also "abelian topological groups" Ab(Top). But not Gp! A ring R is a preadditive category with just one object.

37 Lemma: ("preadditive ⇒ product=biproduct")
If $\mathscr{A}$ is enriched in abelian groups, and $A, B, C \in \operatorname{ob}\mathscr{A}$, the following are equivalent:

(i) There exists $$\begin{tikzcd} & C \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ A & & B \end{tikzcd}$$ *making C into a product $A \times B$.*

(ii) There exists $$\begin{tikzcd} A \arrow[dr, "\iota_1"'] & & B \arrow[dl, "\iota_2"] \\ & C & \end{tikzcd}$$ *making C into a coproduct $A + B$.*

(iii) There exist morphisms $A \underset{\iota_1}{\overset{\pi_1}{\leftrightarrows}} C \underset{\iota_2}{\overset{\pi_2}{\rightleftarrows}} B$ *satisfying $\pi_1\iota_1 = 1_A$, $\pi_2\iota_2 = 1_B$, $\pi_2\iota_1 = 0$, $\pi_1\iota_2 = 0$ and $\iota_1\pi_1 + \iota_2\pi_2 = 1_C$.*

Proof. (i) ⇒ (iii): Take π_1, π_2 to be the given projections, and take ι_1 and ι_2 to be the morphisms defined by the first four equations. To verify that $\iota_1\pi_1 + \iota_2\pi_2 = 1_C$, it is enough to show they have the same composite with π_1 and π_2. Now $\pi_1(\iota_1\pi_1 + \iota_2\pi_2) = \pi_1\iota_1\pi_1 + \pi_1\iota_2\pi_2 = \pi_1 + 0 = \pi_1 1_C$, and $\pi_2(\iota_1\pi_1 + \iota_2\pi_2) = 0 + \pi_2 = \pi_2 1_C$.

(iii) ⇒ (i): We want to show that $A \xleftarrow{\pi_1} C \xrightarrow{\pi_2} B$ is a product, i.e. given $f\colon D \longrightarrow A$ and $g\colon D \longrightarrow B$, we want to find a unique $h\colon D \longrightarrow C$ such that $\pi_1 h = f$ and $\pi_2 h = g$.

If such an h exists, then $h = 1_C h = (\iota_1\pi_1 + \iota_2\pi_2)h = \iota_1 f + \iota_2 g$, so then it is unique. Moreover, $\pi_1(\iota_1 f + \iota_2 g) = f + 0g = f$ and $\pi_2(\iota_1 f + \iota_2 g) = 0 + g = g$, so such an h exists.

Dually (ii) ⇔ (iii). □

Notice that the conditions in (iii) make ι_1, ι_2 split monic and π_1, π_2 split epic.

Definition: Given A, B in a category $\mathscr{A}$ enriched in abelian groups, we call $(C, \pi_1, \pi_2, \iota_1, \iota_2)$ satisfying the conditions in the previous lemma the **biproduct** of A and B. We usually write $C = A \oplus B$.

38 Remark: ("zero morphism")
If $\mathscr{A}$ is enriched in abelian groups and pointed, the composite $A \longrightarrow 0 \longrightarrow B$ must be the additive $0 \in \mathscr{A}(A, B)$, as $\mathscr{A}(A, 0) \times \mathscr{A}(0, B) \longrightarrow \mathscr{A}(A, B)$ is a group homomorphism in each variable.

Lemma: *If $(A \oplus B, \pi_1, \pi_2, \iota_1, \iota_2)$ is a biproduct and $\mathscr{A}$ is pointed, then $\iota_1 = \ker \pi_2$, $\iota_2 = \ker \pi_1$, $\pi_1 = \operatorname{coker} \iota_2$ and $\pi_2 = \operatorname{coker} \iota_1$.*

Proof. We already know $\pi_2 \iota_1 = 0$ and $\pi_1 \iota_2 = 0$. Consider

$$\begin{tikzcd} A \arrow[r, "\iota_1"] & A \oplus B \arrow[r, "\pi_2"] & B \\ D \arrow[ru, "f"'] & & \end{tikzcd}$$

with $\pi_2 f = 0$. Then $f = (\iota_1 \pi_1 + \iota_2 \pi_2) f = \iota_1 \pi_1 f + 0$, so setting $h = \pi_1 f$ we have $f = \iota_1 h$. For uniqueness consider $h \colon D \longrightarrow A$ such that $f = \iota_1 h$. Then $\iota_1 h = f = \iota_1 \pi_1 f$, but ι_1 is (split) monic, so $h = \pi_1 f$.

The other statements are similar or dual. ☐

Definition: An **additive category** is a pointed category $\mathscr{A}$ which is enriched in abelian groups and has biproducts.

Notice that this definition is self-dual, i.e. $\mathscr{A}$ is additive iff $\mathscr{A}^{\mathrm{op}}$ is.

Examples: AbGp, R-Mod, $\mathsf{AbGp}_{\text{t.f.}}$, $\mathrm{Ab}(\mathsf{Top})$.

We will write $A \xrightarrow{\binom{f}{g}} B \times C$ for the morphism induced by $f \colon A \longrightarrow B$ and $g \colon A \longrightarrow C$, and $B + C \xrightarrow{(h,k)} D$ for the morphism induced by $h \colon B \longrightarrow D$ and $k \colon C \longrightarrow D$.

39 Proposition: ("Additive structures are unique.")
Suppose $\mathscr{A}$ is locally small, pointed and has binary products. Then any additive structure on (the homsets of) $\mathscr{A}$ is unique.

Proof. As soon as $\mathscr{A}$ has an additive structure, any product $A \times B$ becomes a biproduct $A \oplus B$ by the "products=biproducts" Lemma 37, so $1_{A \oplus B} = \iota_1 \pi_1 + \iota_2 \pi_2$.

Now consider $\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$. Then we have

$$A \xrightarrow{\binom{1}{1}} A \oplus A \xrightarrow{(f,g)} B \;=\; A \xrightarrow{\binom{1}{1}} A \oplus A \xrightarrow{1} A \oplus A \xrightarrow{(f,g)} B,$$

so $(f, g) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (f, g)(\iota_1 \pi_1 + \iota_2 \pi_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (f, g) \iota_1 \pi_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (f, g) \iota_2 \pi_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = f + g$. So addition in the homsets is completely determined by the "product-coproduct" structure of $\mathscr{A}$. Since the 0 must be $A \longrightarrow 0 \longrightarrow B$ (see Remark 38 "zero morphism") and if an inverse $-f$ of f exists, it is unique, the additive structure on $\mathscr{A}$ is unique. ☐

Notation: In an additive category, any morphism $A \oplus B \longrightarrow C \oplus D$ is determined by four morphisms $f \colon A \longrightarrow B$, $g \colon A \longrightarrow D$, $h \colon B \longrightarrow C$ and $k \colon B \longrightarrow D$. We write

$$A \oplus B \xrightarrow{\begin{pmatrix} f & h \\ g & k \end{pmatrix}} C \oplus D.$$

Then composition of such morphisms is matrix multiplication:

$$A \xrightarrow{\binom{f}{g}} B \oplus C \xrightarrow{(h,k)} D \;=\; (h,k)(\iota_1\pi_1 + \iota_2\pi_2)\begin{pmatrix} f \\ g \end{pmatrix} = hf + kg.$$

It now makes sense to look at functors which preserve this additive structure:

Definition: Let $\mathscr{A}$, $\mathscr{B}$ be additive categories. A functor $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ is **additive** if its action on each homset

$$\begin{aligned} \mathscr{A}(A,B) &\longrightarrow \mathscr{B}(FA,FB) \\ f &\longmapsto Ff \end{aligned}$$

is a group homomorphism.

Remark: Any additive functor preserves the zero object, which is very closely intertwined with the additive structure (recall "zero morphism" Remark 38). To show this, notice that the zero object is the only object whose identity morphism is the zero morphism. So as an additive functor F preserves identities (as it is a functor) and the zero morphism (as it is additive), $F(0)$ is also a zero object.

40 Proposition: ("Additive functors preserve biproducts.")
Let $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ be a functor between additive categories. The following are equivalent:

(i) F is additive.
(ii) F preserves biproducts.
(iii) F preserves finite products.
(iv) F preserves finite coproducts.

PROOF. (i) ⇒ (ii): By the definition of biproducts, we see that if F preserves $+$ and 0,

$$\begin{tikzcd} FA \arrow[r, shift right, "F(\iota_1)"'] & F(A\oplus B) \arrow[l, shift right, "F(\pi_1)"'] \arrow[r, shift left, "F(\pi_2)"] & F(B) \arrow[l, shift left, "F(\iota_2)"] \end{tikzcd}$$

satisfies the conditions making $F(A\oplus B)$ into a biproduct of $F(A)$ and $F(B)$.

(ii) ⇒ (i): Given $\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$, the sum $f+g$ is $A \xrightarrow{\binom{1}{1}} A\oplus A \xrightarrow{(f,g)} B$. So

$$F(f)+F(g) \;=\; FA \xrightarrow{\binom{1}{1}} FA \oplus FA \cong F(A\oplus B) \xrightarrow[F(f,g)]{\overset{(Ff,Fg)}{\cong}} FB \;=\; F(f+g).$$

(iii) ⇒(ii) and (iv) ⇒ (ii) by the "products=biproducts" Lemma 37.

For (ii) ⇒(iii) and (ii) ⇒(iv), we just need to show that F preserves the zero object (i.e. the product of the empty family and the coproduct of the empty family). For this it suffices to show that $F(0)$ is terminal in $\mathscr{B}$. For any $B \in \operatorname{ob}\mathscr{B}$, there is always at least $0\colon B \longrightarrow F(0)$, as $\mathscr{B}$ is additive. Given $\begin{tikzcd} B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & F(0) \end{tikzcd}$, we have

$$f - g = B \xrightarrow{\binom{f}{g}} F(0)\oplus F(0) \xrightarrow{\pi_1-\pi_2} F(0).$$

But $F(0)\oplus F(0) \cong F(0\oplus 0)$ with $\pi_1 \cong F(\pi_1)$ and $\pi_2 \cong F(\pi_2)$. As 0 is terminal in $\mathscr{A}$, we have $\pi_1 = \pi_2\colon 0\oplus 0 \longrightarrow 0$. So $f-g = (\pi_1-\pi_2)\begin{pmatrix} f\\ g\end{pmatrix} = (F(\pi_1)-F(\pi_2))\begin{pmatrix} f\\ g\end{pmatrix} = 0\begin{pmatrix} f\\ g\end{pmatrix} = 0$. So $f = g$ and $F(0)$ is terminal. □

C Abelian Categories

Definition: A category $\mathscr{A}$ is **abelian** when it is additive, has kernels and cokernels and every mono is normal and every epi is normal.

This definition is self-dual.

Examples: AbGp, R-Mod, $\mathsf{AbGp}_{\mathrm{fin}}$ of finite abelian groups. The functor category $[\mathscr{C},\mathscr{A}]$ if ($\mathscr{C}$ is small and) $\mathscr{A}$ is abelian. If $\mathscr{C}$ is preadditive and $\mathscr{A}$ abelian, the full subcategory $\mathrm{Add}(\mathscr{C},\mathscr{A}) \subseteq [\mathscr{C},\mathscr{A}]$ of additive functors $\mathscr{C} \longrightarrow \mathscr{A}$ is abelian (see Example Sheet 4).

The category of abelian compact Hausdorff groups $\mathrm{Ab}(\mathsf{Haus})$ is abelian.

Gp, $\mathsf{AbGp}_{\mathrm{t.f.}}$ and $\mathrm{Ab}(\mathsf{Top})$ are not abelian.

Lemma: *In an abelian category every mono is the kernel of its cokernel and every epi is the cokernel of its kernel.*

Proof. Every mono is normal, and every normal mono is the kernel of its cokernel (Lemma 34). □

Corollary: *An abelian category is balanced.*

Proof. As any mono is normal, it is in particular regular monic. So if f is a mono and an epi, it is a regular mono and an epi and so an iso (Proposition 8 in Section 2C). □

41 Lemma: ("Preadditive equalisers via kernels")

Let $\mathscr{A}$ be preadditive. Then the pair $A \underset{g}{\overset{f}{\rightrightarrows}} B$ has an equaliser iff the kernel of $f-g$ exists, and then they coincide.

Proof. The equaliser of f and g and the kernel of $f-g$ have the same universal property: given $h\colon C \longrightarrow A$, we have $fh = gh \Leftrightarrow (f-g)h = 0$. □

Notice that in general normal $\Rightarrow$ regular $\Rightarrow$ strong $\Rightarrow$ mono. This lemma shows that in a preadditive category, normal $\Leftrightarrow$ regular; and in an abelian category we have normal $\Leftrightarrow$ mono, so all steps coincide.

Corollary: *Any abelian category is finitely complete and cocomplete.*

Proof. As an abelian category $\mathscr{A}$ has biproducts and a zero object, it has all finite products. So, using the "constructing limits" Theorem 7 from Section 2B, it suffices to show that $\mathscr{A}$ has equalisers. Given $A \underset{g}{\overset{f}{\rightrightarrows}} B$, the kernel of $f-g$ exists as $\mathscr{A}$ has kernels, so $\mathscr{A}$ has equalisers. □

42 Proposition: ("abelian: zero kernel implies mono")

Let $f\colon A \longrightarrow B$ be a morphism in an abelian category $\mathscr{A}$. The following are equivalent:

- *(i) f is a mono;*
- *(ii)* $\mathrm{Ker}\, f = 0$*;*
- *(iii) for all $g\colon C \longrightarrow A$ in $\mathscr{A}$ with $fg = 0$, we have $g = 0$.*

Proof. We have seen (i) $\Rightarrow$ (ii), and (i) $\Rightarrow$ (iii) is obvious. For (ii) $\Rightarrow$ (iii), suppose $fg = 0$. Then g factors through the kernel of f, which is 0, so $g = 0$ by the definition of a zero morphism.

Finally we prove (iii) $\Rightarrow$ (i): Suppose $fg = fh$. Then $f(g-h) = fg - fh = 0$, so by (iii) $g - h = 0$, giving $g = h$, and f is monic. □

43 Corollary:
In an abelian category, pullbacks reflect monos.

Proof. Consider a pullback square and take kernels to the left.

$$\begin{tikzcd}
0 = \operatorname{Ker} m \arrow[r, tail] \arrow[d] & P \arrow[r, tail, "m"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\
\operatorname{Ker} f \arrow[r, tail] & A \arrow[r, "f"'] & C
\end{tikzcd}$$

By "kernels and pullbacks" Lemma 36(ii), $\operatorname{Ker} m \cong \operatorname{Ker} f$, so $\operatorname{Ker} f = 0$. So by the previous lemma, f is also a mono. ☐

Dually, in an abelian category g is epic ⇔ $\operatorname{coker} g = 0$, and pushouts reflect epis.

Lemma: *Given a square*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\
C \arrow[r, "k"'] & D
\end{tikzcd}$$

in an abelian category, consider

$$A \xrightarrow{\binom{f}{-g}} B \oplus C \xrightarrow{(h,k)} D.$$

Then

(i) $(h,k)\begin{pmatrix} f \\ -g \end{pmatrix} = 0$ *iff the square commutes.*

(ii) $\begin{pmatrix} f \\ -g \end{pmatrix} = \ker(h,k)$ *iff the square is a pullback.*

(iii) $(h,k) = \operatorname{coker}\begin{pmatrix} f \\ -g \end{pmatrix}$ *iff the square is a pushout.*

Proof. Exercise. ☐

Lemma: *In an abelian category, pullbacks preserve epis.*

Proof. Consider a pullback square

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, two heads, "h"] \\
C \arrow[r, "k"'] & D
\end{tikzcd}$$

with h epic. Then $\binom{f}{-g} = \ker(h,k)$, but as $h = (h,k)\binom{1}{0}$ is epic, (h,k) is epic, so it is the cokernel of its kernel. So the square is a pushout, and pushouts reflect epis, so g is epic. ☐

44 Proposition: (Image factorisation)
In an abelian category, any morphism factors as an epi followed by a mono.

Proof. Let $f\colon A \longrightarrow B$ be a morphism in $\mathscr{A}$. Let $k\colon K \rightarrowtail A$ be the kernel of f and $p\colon A \twoheadrightarrow I$ be the cokernel of k. Then as $fk = 0$, we have

$$\begin{tikzcd}
K \arrow[r, tail, "k"] & A \arrow[rr, "f"] \arrow[dr, two heads, "p"'] & & B \\
& & I \arrow[ur, "i"'] &
\end{tikzcd}$$

We will show that i is monic by showing that $ix = 0$ implies $x = 0$. So consider $x\colon X \longrightarrow I$ with $ix = 0$.

$$\begin{tikzcd}
 & \bullet \arrow[dl, dotted, "l"'] \arrow[d, "h"] & \\
K \arrow[r, tail, "k"] & A \arrow[r, "f"] \arrow[d, twoheadrightarrow, "p"'] & B \\
X \arrow[r, "x"] & I \arrow[ur, "i"] \arrow[r, dotted, twoheadrightarrow, "c"] & \bullet \arrow[u, dotted, "r"'] \arrow[l, dotted, bend left, "s"]
\end{tikzcd}$$

We get a unique r such that $r \operatorname{coker} x = i$. Now as both p and $c = \operatorname{coker} x$ are epis, cp is an epi and so the cokernel of some h. Then $fh = iph = rcph = 0$, so h factors over the kernel of f by $h = kl$.

Finally $ph = pkl = 0$, so $\exists! s$ such that $s(cp) = p$. But as p is epic, this implies $sc = 1$, so c is (split) monic. Then $cx = 0$ implies $x = 0$, so the kernel of i is zero and i is a mono.

Thus f factors as an epi followed by a mono. ☐

Proposition: *Image factorisation is unique (up to iso) and functorial.*

Proof. Suppose $\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, twoheadrightarrow, "p"'] & & B \\ & I \arrow[ur, "i"'] & \end{tikzcd}$ with i monic. Using "Kernels and pullbacks" Lemma 36(i) on

$$\begin{tikzcd}
\operatorname{Ker} p \arrow[r, tail] \arrow[d] & A \arrow[r, "p"] \arrow[d, equal] & I \arrow[d, tail, "i"] \\
\operatorname{Ker} f \arrow[r, tail] & A \arrow[r, "f"'] & B
\end{tikzcd}$$

we see that the first square is a pullback and therefore $\operatorname{Ker} p \cong \operatorname{Ker} f$. So if p is epic, it is the cokernel of its kernel, i.e. $p = \operatorname{coker}(\ker f)$. So the factorisation is unique.

Given $\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$, the kernel property of $\operatorname{Ker} f'$ and the cokernel property of p induce

$$\begin{tikzcd}
A \arrow[r, twoheadrightarrow, "p"] \arrow[d, "a"'] & I \arrow[r, tail, "i"] \arrow[d, dotted] & B \arrow[d, "b"] \\
A' \arrow[r, twoheadrightarrow, "p'"'] & I' \arrow[r, tail, "i'"'] & B'
\end{tikzcd}$$

making both squares commute. ☐

We saw that $p = \operatorname{coker}(\ker f)$; dually $i = \ker(\operatorname{coker} f)$.

Definition: Given $f\colon A \longrightarrow B$ in an abelian category, we call $\ker(\operatorname{coker} f) = i\colon\ I \rightarrowtail B$ the **image of** f. Write $I = \operatorname{Im} f$, $i = \operatorname{im} f$.

So we can view $\operatorname{Im}\colon \operatorname{Arr}\mathscr{A} \longrightarrow \mathscr{A}$ as a functor, with natural transformations dom $\longrightarrow$ Im and Im $\longrightarrow$ cod. ("Can view" means here that we'd have to actually *choose* a particular factorisation out of the isomorphic possibilities.)

D Exact Sequences

Definition: A **short exact sequence** in an abelian category $\mathscr{A}$ is

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}$$

where $f = \ker g$ and $g = \operatorname{coker} f$.

In general a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is **exact at** B if $\operatorname{im} f = \ker g$. We say

$$\cdots \longrightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \longrightarrow \cdots$$

is **exact** if it is exact at every (internal) A_n.

Lemma: *Let*

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, two heads, "p"'] & & B \arrow[rr, "g"] \arrow[dr, two heads, "q"'] & & C \\
& I \arrow[ur, tail, "i"'] & & J \arrow[ur, tail, "j"'] &
\end{tikzcd}$$

be the image factorisation of f and g. Then $A \xrightarrow{f} B \xrightarrow{g} C$ is exact iff $0 \longrightarrow I \rightarrowtail^{i} B \overset{q}{\twoheadrightarrow} J \longrightarrow 0$ is a short exact sequence.

Proof. Exercise. ☐

Examples:

- $0 \longrightarrow A \xrightarrow{f} B$ is exact (at A) iff f is monic.
- $B \xrightarrow{g} C \longrightarrow 0$ is exact (at C) iff g is epic.
- $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$ is exact iff $f = \ker g$.
- $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact iff $g = \operatorname{coker} f$.
- $0 \longrightarrow A \xrightarrow{\iota_1} A \oplus B \xrightarrow{\pi_2} B \longrightarrow 0$ is a short exact sequence.

  In fact, it is split: $$\begin{tikzcd} A \arrow[r, "\iota_1"] & A \oplus B \arrow[l, bend left, "\pi_1"] \arrow[r, "\pi_2"] & B \arrow[l, bend left, "\iota_2"] \end{tikzcd}$$

Definition: A short exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is **split** when g is split epic.

45 Lemma: ("abelian: split SES=biproduct")

In an abelian $\mathscr{A}$, if $$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[l, bend left, "s"] \arrow[r] & 0 \end{tikzcd}$$ *is a split short exact sequence then $B \cong A \oplus C$.*

Proof. Consider $1_B - sg \colon B \longrightarrow B$. Then $g(1_B - sg) = g - gsg = 0$, so $1_B - sg$ factors over the kernel of g. I.e. $\exists r \colon B \longrightarrow A$ such that $fr = 1_B - sg$. We will prove that

$$\begin{tikzcd} A \arrow[r, shift right, "f"'] & B \arrow[l, shift right, "r"'] \arrow[r, shift left, "g"] & C \arrow[l, shift left, "s"] \end{tikzcd}$$

satisfies the conditions of a biproduct.

We already know $gf = 0$, $gs = 1_C$ and $fr + sg = 1_B$. Now $frf = (1_B - sg)f = f - sgf = f$, so as f is monic, $rf = 1_A$. Finally $frs = (1_B - sg)s = s - sgs = 0$, so $rs = 0$. Thus $B \cong A \oplus C$. ☐

Corollary: *The notions of exact sequence and split short exact sequence in an abelian category are self-dual.* ☐

Definition: A functor $F \colon \mathscr{A} \longrightarrow \mathscr{B}$ between abelian categories is **exact** if it preserves short exact sequences.

F is **left exact** if it preserves exact sequences of the form $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$.

F is **right exact** if it preserves exact sequences of the form $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$.

Lemma: *Any left (or right) exact functor is additive.*

Proof. Consider the (split) short exact sequence $0\longrightarrow A\xrightarrow{\iota_1}A\oplus B\xrightarrow{\pi_2}B\longrightarrow 0$. Then the sequence $0\longrightarrow FA\xrightarrow{F(\iota_1)}F(A\oplus B)\xrightarrow{F(\pi_2)}FB$ is exact, but as $F(\pi_2)$ is split epic, we in fact get a split SES, i.e. $F(A\oplus B)\cong FA\oplus FB$. ☐

Lemma: *(i) F is left exact $\Leftrightarrow$ F is additive and preserves kernels $\Leftrightarrow$ F preserves finite limits.*

(ii) F is right exact $\Leftrightarrow$ F is additive and preserves cokernels $\Leftrightarrow$ F preserves finite colimits.

(iii) F is exact $\Leftrightarrow$ F is additive and perserves kernels and cokernels $\Leftrightarrow$ F preserves finite limits and colimits.

Proof. Use "additive functors preserve biproduct" Proposition 40, "preadditive equalisers via kernels" Lemma 41 and the "constructing limits" Theorem 7 (Section 2B). ☐

Corollary: *A left exact functor between abelian categories is exact iff it preserves epimorphisms.*

Proof. Exercise. ☐

E Diagram Lemmas

46 Theorem: (Short Five Lemma)
Let $\mathscr{A}$ be abelian. Consider a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, tail, "f"] \arrow[d, "k"', "\cong"] & A \arrow[r, two heads, "g"] \arrow[d, "a"] & B \arrow[r] \arrow[d, "\cong"', "b"] & 0 \\
0 \arrow[r] & K' \arrow[r, tail, "f'"'] & A' \arrow[r, two heads, "g'"'] & B' \arrow[r] & 0
\end{tikzcd}$$

where both rows are exact, and k and b are isos. Then a is also an iso.

Proof. By "Kernels and pullbacks" Lemma 36(i), we see that the first square is a pullback. As, in an abelian category, pullbacks reflect monos (Corollary 43), a is a mono.

Dually a is an epi, so a is an iso. ☐

47 Corollary: (Five Lemma)
In an abelian category $\mathscr{A}$, consider the commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "a"'] & B \arrow[r] \arrow[d, "b"'] & C \arrow[r] \arrow[d, "c"'] & D \arrow[r] \arrow[d, "d"] & E \arrow[d, "e"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

with exact rows, a epic, b and d isos and e monic. Then c is an iso.

Proof. We write out the image factorisation of all horizontal morphisms:

$$\begin{tikzcd}
A \arrow[r, two heads] \arrow[d, two heads] & I_1 \arrow[r, tail] \arrow[d] & B \arrow[r, two heads] \arrow[d, tail, two heads, "\cong"'] & I_2 \arrow[r, tail] \arrow[d] & C \arrow[r, two heads] \arrow[d] & I_3 \arrow[r, tail] \arrow[d] & D \arrow[r, two heads] \arrow[d, tail, two heads, "\cong"] & I_4 \arrow[r, tail] \arrow[d] & E \arrow[d, tail] \\
A' \arrow[r, two heads] & I_1' \arrow[r, tail] & B' \arrow[r, two heads] & I_2' \arrow[r, tail] & C' \arrow[r, two heads] & I_3' \arrow[r, tail] & D' \arrow[r, two heads] & I_4' \arrow[r, tail] & E'
\end{tikzcd}$$

Looking at the first square, we see that $I_1\longrightarrow I_1'$ has to be an epi, as it is the second part of a composite which is an epi. Similarly, looking at the second square, we see that it must be a mono. So we find that $I_1\longrightarrow I_1'$ and $I_4\longrightarrow I_4'$ are isos are they are epis and monos, and so $I_2\longrightarrow I_2'$ and $I_3\longrightarrow I_3'$ are isos as they are induced morphisms between cokernels resp. kernels of isomorphic morphisms. So we can use the Short Five Lemma to see that c is an iso. ☐

Corollary: *In an abelian category, given a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "f"] \arrow[d, "k"'] & A \arrow[r, "g"] \arrow[d, "a"'] \arrow[dr, phantom, "(2)"] & B \arrow[r] \arrow[d, "b"] & 0 \\
0 \arrow[r] & K' \arrow[r, "f'"'] & A' \arrow[r, "g'"'] & B' &
\end{tikzcd}$$

where both rows are exact, k is an iso iff (2) is a pullback.

Proof. *Proof not examinable as bookwork. It is on the example sheet however, so I would expect you to have looked at it in the same way as for other example sheet questions.*

We've already seen $\Leftarrow$. For $\Rightarrow$, form the pullback of g' and b and consider

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[rr, tail, "f"] \arrow[dr] \arrow[dd, "\cong"'] & & A \arrow[rr, two heads, "g"] \arrow[dr] \arrow[dd] & & B \arrow[r] \arrow[dr, equal] \arrow[dd] & 0 \\
& & \operatorname{Ker}\pi_2 \arrow[rr, tail] \arrow[dl] & & P \arrow[rr, "\pi_2"] \arrow[dl] \arrow[drr, phantom, "\lrcorner", very near start] & & B \arrow[dl, "b"] \\
0 \arrow[r] & K' \arrow[rr, tail, "f'"'] & & A' \arrow[rr, "g'"'] & & B' &
\end{tikzcd}$$

We know that $\operatorname{Ker}\pi_2 \longrightarrow K'$ is an iso by "$\Leftarrow$", and the front triangle commutes as f' is monic. So $K \longrightarrow \operatorname{Ker}\pi_2$ is also an iso. Now π_2 is an epi as g is, so we can use the Short Five Lemma to see that $A \longrightarrow P$ is an iso, i.e. (2) is a pullback. □

Remark: We could have used this together with the fact that pullbacks reflect monos in the image factorsation proof to show that i is monic.

48 Lemma: (Pullback cancellation (on the left))

In an abelian category, consider

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\lrcorner\ (1)", very near start] & B \arrow[r, "g"] \arrow[d, two heads, "b"] \arrow[dr, phantom, "(2)"] & C \arrow[d, "c"] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C'
\end{tikzcd}$$

where the rectangle (1,2) and the square (1) are pullbacks and b is an epi. Then (2) is also a pullback.

Proof. *Proof not examinable as bookwork. It is on the example sheet however, so I would expect you to have looked at it in the same way as for other example sheet questions.*

Consider the kernels of a, b and c:

$$\begin{tikzcd}
\operatorname{Ker} a \arrow[r, "\overline{f}"] \arrow[d, tail] & \operatorname{Ker} b \arrow[r, "\overline{g}"] \arrow[d, tail] & \operatorname{Ker} c \arrow[d, tail] \\
A \arrow[r, "f"] \arrow[d] & B \arrow[r, "g"] \arrow[d, two heads, "b"] & C \arrow[d] \\
A' \arrow[r] & B' \arrow[r] & C'
\end{tikzcd}$$

Then by "Kernels and pullbacks" Lemma 36(i), $\overline{f}$ and $\overline{g}\,\overline{f}$ are isomorphisms, as (1) and (1,2) are pullbacks. So $\overline{g}$ is also an isomorphism, so by the previous result, (2) is a pullback (this needs b to be epic). □

From here onwards everything is extra material which was not lectured. It will not be on the exams.

49 Theorem: (Nine Lemma)
Consider

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, "b"] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d, "b'"] & C' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\arrow[from=2-2, to=3-3, phantom, "(1)"]
\arrow[from=3-3, to=4-4, phantom, "(2)"]
\end{tikzcd}$$

where all rows are exact and $b'b = 0$. Then if any two columns are exact, the third column is also exact. In that case (1) is a pullback and (2) is a pushout.

Proof. Not in this course. □

50 Theorem: (Snake Lemma)
A commutative diagram with exact rows as the solid one below induces a six-term exact sequence between the kernels and cokernels as indicated.

$$\begin{tikzcd}
 & \operatorname{Ker} a \arrow[r, dotted] \arrow[d, dotted] & \operatorname{Ker} b \arrow[r, dotted] \arrow[d, dotted] & \operatorname{Ker} c \arrow[d, dotted] \arrow[dddll, dotted, rounded corners, "\delta"] & \\
 & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"'] \arrow[d, dotted] & B' \arrow[r, "g'"'] \arrow[d, dotted] & C' \arrow[d, dotted] & \\
 & \operatorname{Coker} a \arrow[r, dotted] & \operatorname{Coker} b \arrow[r, dotted] & \operatorname{Coker} c &
\end{tikzcd}$$

Proof. (non-examinable) Consider the kernels and cokernels with the induced maps between them. For shortness of notation we will write $\operatorname{Ker} a = K_1$, $\operatorname{Ker} b = K_2$ and $\operatorname{Ker} c = K_3$, similarly we will call the cokernels Q_i.

$$\begin{tikzcd}
 & K_1 \arrow[r, "\overline{f}"] \arrow[d, "k_1"] & K_2 \arrow[r, "\overline{g}"] \arrow[d, "k_2"] & K_3 \arrow[d, "k_3"] & \\
 & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"'] \arrow[d, "q_1"] & B' \arrow[r, "g'"'] \arrow[d, "q_2"] & C' \arrow[d, "q_3"] & \\
 & Q_1 \arrow[r, "\hat{f}"'] & Q_2 \arrow[r, "\hat{g}"'] & Q_3 &
\end{tikzcd}$$

We give a proof which maximises the use of the Duality Principle (borrowed from Peter Johnstone).
1. Construction of δ Form the diagram

$$\begin{tikzcd}
E \arrow[r, tail, "e"] & P \arrow[r, two heads, "p"] \arrow[d, tail, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & K_3 \arrow[d, tail, "k_3"] \\
& B \arrow[r, two heads, "g"'] \arrow[d, "b"] & C \\
A' \arrow[r, tail, "f'"] \arrow[d, two heads, "q_1"'] & B' \arrow[d, two heads, "r"] & \\
Q_1 \arrow[r, tail, "t"'] & T \arrow[r, two heads, "d"'] \arrow[ul, phantom, "\ulcorner", very near start] & D
\end{tikzcd}$$

where the upper square is a pullback, the lower square is a pushout, $e = \ker p$ and $d = \operatorname{coker} t$. Remember that pullbacks and pushout preserve both monos and epis (as we are in an abelian category), so p and r are epis and q and t are monos. So as any epi is the cokernel of its kernel, we have $p = \operatorname{coker} e$ and dually $t = \ker d$. To construct $\delta \colon K_3 \longrightarrow C_1$, it is enough to factor the composite rbq through p and through t. For this we just have to show that $rbqe = 0$ and that $drbq = 0$, which are dual to each other, so showing the first is enough.

To prove the first, note that $gqe = k_3pe = 0$, so qe factors through $\ker g = \operatorname{im} f$. So if we form the pullback

$$\begin{tikzcd}
L \arrow[r, "l"] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, tail, "qe"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

then its top edge l is epic. This is because it is the same as the pullback:

$$\begin{tikzcd}
L \arrow[r, two heads, "l"] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, tail] \arrow[ddr, tail, "qe"] & \\
A \arrow[r, two heads] \arrow[drr, "f"'] & \operatorname{Ker} g \arrow[dr, tail, "\operatorname{im} f"] & \\
& & B
\end{tikzcd}$$

But $rbqel = rbfm = rf'am = tq_1am = 0$ (as q_1 is the cokernel of a),

$$\begin{tikzcd}
L \arrow[r, "el"] \arrow[d, "m"'] & P \arrow[r, two heads, "p"] \arrow[d, tail, "q"] \arrow[dr, phantom, "\lrcorner", very near start] & K_3 \arrow[d, tail, "k_3"] \\
A \arrow[r, "f"'] \arrow[d, "a"'] & B \arrow[r, two heads, "g"'] \arrow[d, "b"] & C \\
A' \arrow[r, tail, "f'"'] \arrow[d, two heads, "q_1"'] & B' \arrow[d, two heads, "r"] & \\
Q_1 \arrow[r, tail, "t"'] & T \arrow[r, two heads, "d"'] \arrow[ul, phantom, "\ulcorner", very near start] & D
\end{tikzcd}$$

so we may deduce $rbqe = 0$ as required. So we get $\delta \colon K_3 \longrightarrow Q_1$ such that $t\delta p = rbq$.

Exactness at K_2 We have $k_3\overline{g}\overline{f} = gk_2\overline{f} = gfk_1 = 0$ and k_3 is monic, so $\overline{g}\overline{f} = 0$. Let $e' \colon E' \longrightarrow K_2$ be the kernel of $\overline{g}$; then the composite k_2e' factors through $\ker g = \operatorname{im} f$, so as before we get an epi $l' \colon L' \longrightarrow E'$ and a morphism $m' \colon L' \longrightarrow A$ such that $fm' = k_2e'l'$. Now $f'am' = bfm' = bk_2e'l' = 0$ and f' is monic, so $am' = 0$, i.e. m' factors through $\ker a = k_1$, say by

$s\colon L' \longrightarrow K_1$. Now $k_2\overline{f}s = fk_1s = fm' = k_2e'l'$ and k_2 is monic, so $\overline{f}s = e'l'$, i.e. s is a morphism $e'l' \longrightarrow \overline{f}$ in $\mathscr{A}/K_2$. But this implies that $\operatorname{im}\overline{f} \geqslant \operatorname{im} e'l' = e' = \ker\overline{g}$ in $\operatorname{Sub}(K_2)$ (by naturality of image factorisation).

$$\begin{tikzcd}
L' \arrow[r, two heads, "l"] \arrow[d, "s"'] & \operatorname{Ker}\overline{g} \arrow[r, tail, "e'"] \arrow[d, dotted] & K_2 \arrow[d, equal] \\
K_1 \arrow[r, two heads] & \operatorname{Im}\overline{f} \arrow[r, tail, "\operatorname{im}\overline{f}"'] & K_2
\end{tikzcd}$$

The reverse inequality follows from $\overline{g}\overline{f} = 0$, so we get exactness at K_2.

Exactness at K_3 The pair $(k_2, \overline{g})$ factors through the pullback P, say by $u\colon K_2 \longrightarrow P$. So to prove that $\delta\overline{g} = 0$, it suffices (since t is monic) to prove that $t\delta pu = 0$, i.e. that $rbqu = 0$ (since δ was induced by $t\delta p = rbq$). But this composite equals rbk_2, which is of course 0.

Now let $h\colon K_3 \longrightarrow H$ be the cokernel of $\overline{g}$, and form the pushout (the right-hand square)

$$\begin{tikzcd}
K_2 \arrow[r, "\overline{g}"] \arrow[d, tail, "k_2"'] & K_3 \arrow[r, two heads, "h"] \arrow[d, tail, "k_3"'] & H \arrow[d, tail, "m"] \\
B \arrow[r, "g"'] & C \arrow[r, two heads, "o"'] & M \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

where m is monic as k_3 is. Then $ogk_2 = ok_3\overline{g} = mh\overline{g} = 0$, so og factors through $\operatorname{coker} k_2 = \operatorname{coim} b$. So (as before with l) if we form another pushout (the right-hand square)

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "og"] \arrow[d, "b"'] & M \arrow[d, tail, "m'"] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "o'"'] & N \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

then m' is monic. Then $o'f'a = o'bf = m'ogf = 0$, so $o'f'$ factors through $\operatorname{coker} a = q_1$, say by $n\colon Q_1 \longrightarrow N$. Then the pair (o', n) factors through the pushout T, say by $x\colon T \longrightarrow N$.

$$\begin{tikzcd}
A' \arrow[r, "f'"] \arrow[d, "q_1"'] & B' \arrow[d, "r"'] \arrow[ddr, tail, "o'"] & \\
Q_1 \arrow[r, "t"'] \arrow[drr, "n"'] & T \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, "x"] & \\
 & & N
\end{tikzcd}$$

Then

$$n\delta p = xt\delta p = xrbq = o'bq = m'ogq = m'ok_3p = m'mhp$$

and as p is epic, we have $n\delta = m'mh$, i.e. n is a morphism $\delta \longrightarrow m'mh$ in the coslice category $K_3\backslash\mathscr{A}$, so $\operatorname{coim}\delta \geqslant \operatorname{coim} m'mh = h = \operatorname{coker}\overline{g}$ in the preorder of quotients of K_3.

$$\begin{tikzcd}
K_3 \arrow[r, two heads] \arrow[d, equal] & \operatorname{Coim}\delta \arrow[r, tail] \arrow[d, dotted] & Q_1 \arrow[d, "n"] \\
K_3 \arrow[r, two heads, "h"'] & \operatorname{Coker}\overline{g} \arrow[r, tail, "m'm"'] & N
\end{tikzcd}$$

The reverse inequality follows from $\delta\overline{g} = 0$. So we have exactness at K_3.

Exactness at Q_1 **and** Q_2 These proofs are dual to those at K_3 and K_2 respectively. □

Notice that when f is a mono, then so is the induced $\operatorname{Ker} a \longrightarrow \operatorname{Ker} b$, and when g' is an epi, so is $\operatorname{Coker} b \longrightarrow \operatorname{Coker} c$.

Fact: Every small abelian category has a full, faithful and exact embedding into a category R-Mod of modules over a ring R. This allows us to prove results about exact sequences, monos, epis, images etc. using elements. But the result is not easy!

(This may not be used in exams!)