Exterior algebras and local mirror symmetry

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A toy problem

- V an n-dimensional vector space over $\mathbf{k}$
- Exterior algebra $\Lambda^\bullet V$: $\mathbb{Z}$-graded, unital, associative
- Deform to unital associative algebra A such that:
  - $\mathbb{Z}/2$-grading is preserved
  - deformation terms strictly decrease $\mathbb{Z}$-degree
- Call such a deformation superfiltered

$$* : \Lambda^i V \otimes \Lambda^j V \longrightarrow \Lambda^{i+j} V \oplus \Lambda^{i+j-2} V \oplus \cdots$$

$$\wedge$$

A toy problem

- For $v \in V = \Lambda^1 V$ have

$$v * v = v \wedge v + Q(v) \in \Lambda^2 V \oplus \Lambda^0 V$$

- Obtain quadratic form $Q : V \to \mathbf{k}$
- Get Cifford algebra

$$A \cong C\ell(Q) = TV/(v \otimes v - Q(v))$$

Proof

Natural surjective hom $C\ell(Q) \to A$. Both have dimension 2^n.

A toy problem

Better proof

- Both $C\ell(Q)$ and A are filtered by $\mathbb{Z}$-degree:

$$F^p = \{\text{elements of } \mathbb{Z}\text{-degree} \leq p\} = \text{image of } V^{\otimes \leq p}$$

- The map $f : C\ell(Q) \to A$ respects the filtrations
- f induces an isomorphism

$$\begin{tikzcd} \operatorname{gr} f : \operatorname{gr} C\ell(Q) \arrow[rr, "\cong"] \arrow[dr, equal] & & \operatorname{gr} A \arrow[dl, equal] \\ & \Lambda^\bullet V & \end{tikzcd}$$

so is itself an isomorphism

A toy problem

Remark

- A is determined by $*$ on degree 1
- Described by quadratic $Q : V \to \mathbf{k}$

Question

What about superfiltered deformations of $\Lambda^\bullet V$ as an A_∞-algebra?

Superfiltered A_∞-deformations

Recall

A $\mathbb{Z}$-graded A_∞-algebra $\mathcal{A}$ has:

- product $\mu^2 : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ degree 0
- higher products $\mu^k : \mathcal{A}^{\otimes k} \to \mathcal{A}$ degree $2 - k$

satisfying homotopy-associativity relations

Can make $\Lambda^\bullet V$ into an A_∞-algebra with operations:

- $\mu^2_\Lambda = \wedge$ up to sign
- $\mu^k_\Lambda = 0$ for $k \geq 3$

Superfiltered A_∞-deformations

- $\mathcal{A}$ superfiltered A_∞-deformation of $\Lambda^\bullet V$
- Operations $\mu^k = \mu^k_\Lambda +$ corrections
- For $v \in V$ have

$$\sum_{k\geq 2} \mu^k(v,\ldots,v) = \mu^2_\Lambda(v,v) + \mathfrak{P}(v) \in \Lambda^2 V \oplus \Lambda^0 V$$

- Obtain formal function $\mathfrak{P} : V \to \mathbf{k}$, i.e. $\mathfrak{P} \in \mathbf{k}[\![V]\!]$
- Note $\mathfrak{P} \in \mathfrak{m}^2$ where $\mathfrak{m} \subset \mathbf{k}[\![V]\!]$ is the maximal ideal

Superfiltered A_∞-deformations

Theorem 1

The map $\mathcal{A} \mapsto \mathfrak{P}$ induces a bijection

$$\frac{\text{superfiltered } A_\infty\text{-defs of } \Lambda^\bullet V}{\substack{\text{superfiltered } A_\infty\text{-isom} \\ \text{(inducing identity on } V\text{)}}} \to \frac{\text{formal functions in } \mathfrak{m}^2}{\substack{\text{formal change of variables} \\ \text{(identity to first order)}}}$$

Corollary

$\mathcal{A}$ is determined by (symmetrised) A_∞-operations on degree 1

Superfiltered A_∞-deformations

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Remark

Associative classification is 'truncation to order 2'

$$Q = \text{quadratic part of } -\mathfrak{P}$$

Superfiltered A_∞-deformations

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Example

$\mathcal{A}$ is formal iff $\mathfrak{P}$ is quadratic up to change of variables, e.g.

$$n = 1,\ \mathfrak{P} = z^2 + z^3 \implies \mathcal{A} \text{ formal iff } \operatorname{char} \mathbf{k} \neq 2$$

Superfiltered A_∞-deformations

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Moreover

Have explicit description $\mathcal{A} \simeq \mathrm{end}(\mathcal{O}_0)$ in $\mathrm{mf}(\widehat{V}, \mathfrak{P})$

Superfiltered deformations in symplectic topology

- (X^{2n}, ω) compact symplectic, monotone (Fano)
- $QH^\bullet(X; \mathbf{k})$ is a superfiltered deformation of $H^\bullet(X; \mathbf{k})$

$$\begin{tikzcd} * : QH^i \otimes QH^j \arrow[r] \arrow[r, bend right, "\smile"'] & QH^{i+j} \oplus QH^{i+j-2} \oplus \cdots \end{tikzcd}$$

- QH^{i+j-2k} term counts curves of Chern number k

Superfiltered deformations in symplectic topology

- $L \subset X$ Lagrangian torus, monotone
- Floer theory of L is a superfiltered def of classical topology
- Have disc-counting potential

$$W_L : H^1(L; \mathbf{k}^\times) \to \mathbf{k}$$

  on space of rank 1 local systems on L

- For $\rho \in H^1(L; \mathbf{k}^\times)$ want to understand $HF^\bullet((L, \rho), (L, \rho))$

Known results

Theorem (Cho, Cho–Oh, Biran–Cornea)

There is an isomorphism of $\mathbb{Z}/2$-graded vector spaces

$$HF^\bullet((L,\rho),(L,\rho)) \cong \begin{cases} H^\bullet(L;\mathbf{k}) & \text{if } \rho \in \operatorname{crit} W_L \\ 0 & \text{otherwise} \end{cases}$$

From now on:

- Fix a choice of $\rho \in \operatorname{crit} W_L$
- Let $V = H^1(L;\mathbf{k})$ so $H^\bullet(L;\mathbf{k}) = \Lambda^\bullet V$

Known results

- $HF^\bullet$ is an associative algebra with Floer product
- It's a superfiltered associative deformation of $\Lambda^\bullet V$
- So determined by a quadratic form $Q : V \to \mathbf{k}$

Theorem (Cho, Biran–Cornea)

$$Q = -\operatorname{Hess}_\rho W_L$$

as quadratic forms on $T_\rho H^1(L; \mathbf{k}^\times) \cong H^1(L; \mathbf{k})$, so

$$HF^\bullet \cong C\ell(-\operatorname{Hess}_\rho W_L)$$

The A_∞-structure

- $HF^\bullet$ carries Fukaya A_∞-structure
- It's a superfiltered A_∞-deformation of $\Lambda^\bullet V$
- So determined by a formal function $\mathfrak{P} : V \to \mathbf{k}$
- Know quadratic part of $\mathfrak{P}$ is $\mathrm{Hess}_\rho W_L$

Theorem 2

$\mathfrak{P}$ is the formal expansion of $W_L - W_L(\rho)$ about ρ, so

$$\begin{aligned} HF^\bullet &\simeq \mathrm{end}(\mathcal{O}_0) \text{ in } \mathrm{mf}(\widehat{V}, \mathfrak{P}) \\ &\simeq \mathrm{end}(\mathcal{O}_\rho) \text{ in } \mathrm{mf}(H^1(L; \mathbf{k}^\times), W_L - W_L(\rho)) \end{aligned}$$

The A_∞-structure: example

- Take $X = S^2$, $L =$ equator
- Have $W_L = z + \frac{1}{z}$
- Critical points
  - ρ_1: $z = 1$, trivial local system
  - ρ_{-1}: $z = -1$, monodromy -1 around equator
- $\lambda(\rho_{\pm 1}) = W_L(\pm 1) = \pm 2$
- Theorem 2 says

$$HF^\bullet((L, \rho_{\pm 1}), (L, \rho_{\pm 1})) \simeq \mathrm{end}(\mathcal{O}_{\pm 1})$$

in $\mathrm{mf}\left(\mathbf{k}^\times, z + \dfrac{1}{z} \mp 2\right)$

The A_∞-structure: example

$\operatorname{char}\mathbf{k} \neq 2$

- $\mathcal{O}_1 \neq \mathcal{O}_{-1}$
- Expansion of $z + \frac{1}{z} \mp 2$ about $z = \pm 1$ is

$$\pm\left((1+x) + \frac{1}{1+x} - 2\right) = \pm(x^2 - x^3 + x^4 - x^5 + \dots)$$

- Can be made quadratic so $\mathcal{O}_1$ and $\mathcal{O}_{-1}$ are formal

$\operatorname{char}\mathbf{k} = 2$

- $\mathcal{O}_1 = \mathcal{O}_{-1}$
- $z + \frac{1}{z} \mp 2$ cannot be made quadratic so $\mathcal{O}_1$ is not formal

Mirror symmetry expectation

X Fano

LG model $\begin{cases} Y \text{ variety} \\ W : Y \to \mathbf{k} \end{cases}$

$\mathrm{Fuk}(X)_\lambda$

$\mathrm{mf}(Y, W - \lambda)$

$L \subset X$ monotone torus

$U = H^1(L; \mathbf{k}^\times)$ chart on Y
with $W|_U = W_L$

$(L, \rho) \in \mathrm{Fuk}(X)_{W_L(\rho)}$

$\mathcal{O}_\rho \in \begin{cases} \mathrm{mf}(U, W_L - W_L(\rho)) \\ \mathrm{mf}(Y, W - W_L(\rho)) \end{cases}$

Theorem 2

(L, ρ) and $\mathcal{O}_\rho$ have isomorphic endomorphism A_∞-algebras

Localised mirror functor

Theorem (Cho–Hong–Lau)

Associated to a monotone torus L is a localised mirror functor

$$\mathcal{LM}_\lambda^L : \mathrm{Fuk}(X)_\lambda \to \mathrm{mf}(H^1(L;\mathbf{k}^\times), W_L - \lambda),$$

an A_∞-functor defined for each $\lambda \in \mathbf{k}$

Localised mirror functor

Theorem 3 ($\implies$ Theorem 2)

For each ρ with $W_L(\rho) = \lambda$, $\mathcal{LM}^L_\lambda$ sends (L, ρ) to $\mathcal{O}_\rho$

Corollary

For each λ, $\mathcal{LM}^L_\lambda$ induces a quasi-equivalence

$$\mathrm{Fuk}_L(X)_\lambda \simeq \mathrm{mf}(H^1(L; \mathbf{k}^\times), W_L - \lambda)$$

if W_L has isolated critical points

Key idea of Theorem 3: filtered versions of mf and $\mathcal{LM}^L_\lambda$

Recap of mf

- Y variety, $W : Y \to \mathbf{k}$ regular function
- $\mathrm{mf}(Y, W)$ is $\mathbb{Z}/2$-graded dg-category over $\mathbf{k}$
- Objects are matrix factorisations $\mathcal{E} \in \mathrm{mf}(Y, W)$:
  - 2-periodic 'complexes' of vector bundles on Y

$$\cdots \to E^0 \to E^1 \to E^0 \to E^1 \to \cdots$$

  - 'differential' $\delta : E^i \to E^{i+1}$ satisfies $\delta^2 = W \cdot \mathrm{id}$
- $\hom^i(\mathcal{E}, \mathcal{F})$ comprises maps $f : \begin{cases} E^0 \to F^i \\ E^1 \to F^{i+1} \end{cases}$ with

$$\mathrm{d}f = \delta_{\mathcal{F}} \circ f - (-1)^i f \circ \delta_{\mathcal{E}}$$

Recap of mf

Theorem (Orlov)

If Y smooth and $Z = W^{-1}(0) \subset Y$ then

$$H\mathrm{mf}(Y, W) \simeq D^b\,\mathrm{Sing}(Z) = D^b\,\mathrm{Coh}(Z) \big/ \mathrm{Perf}(Z)$$

Corollary

Can view (complexes of) sheaves on Z as matrix factorisations, e.g.

$$\mathcal{O}_\rho \in \mathrm{mf}(H^1(L; \mathbf{k}^\times), W_L - \lambda) \text{ if } W_L(\rho) = \lambda$$

Recap of mf

If Y affine then $\mathrm{mf}(Y, W)$ is a superfiltered def of $\mathrm{Perf}_Z(Y)$

- given complex $(\mathcal{E}, \mathrm{d})$ of v.b.s on Y with $H\mathcal{E}$ supported on Z
- $\times W$ acts as 0 on $H\mathcal{E}$ so is nullhomotopic on $\mathcal{E}$
- pick a nullhomotopy h (degree -1)
- $(\mathcal{E}, \delta = \mathrm{d} + h)$ is (almost!) the associated matrix factorisation

Construction of $\mathcal{LM}_\lambda^L$

- Given monotone torus L
- Fix co-oriented hypertori $H_1, \ldots, H_n$, basis for $H^1(L; \mathbb{Z})$
- Obtain identification $H^1(L; \mathbf{k}^\times) \cong \operatorname{Spec} \mathbf{k}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$
  - z_i is monodromy around loop dual to H_i
  - can think of $W_L \in \mathbf{k}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$
- For $K \in \mathrm{Fuk}(X)_\lambda$ need to construct

$$\mathcal{LM}_\lambda^L(K) \in \mathrm{mf}(H^1(L; \mathbf{k}^\times), W_L - \lambda)$$

Construction of $\mathcal{LM}_\lambda^L$

- Want 2-periodic 'complex' over $\mathbf{k}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$
- 'Differential' δ should square to $W_L - \lambda$

Define

$$\mathcal{LM}_\lambda^L(K) = CF^\bullet(L, K) \otimes_{\mathbf{k}} \mathbf{k}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$$

$\delta =$ Floer differential weighted by monomials

Weight attached to holomorphic strip u is $z_1^{m_1} \ldots z_n^{m_n}$ where

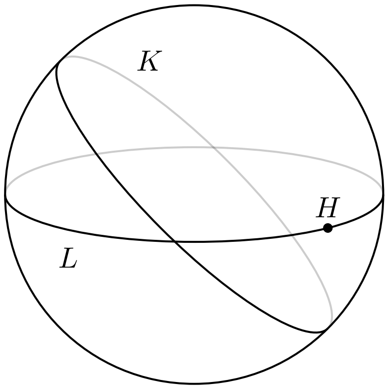
$$m_i = \partial_L u \cdot H_i.$$

Standard bubbling argument gives $\delta^2 = W_L - \lambda$.

Morally: equip L with universal local system.

Construction of $\mathcal{LM}_\lambda^L$: example

- Take $X = S^2$, $L =$ equator, $H =$ point
- $K =$ pushoff of L with trivial local system
- Recall $W_L = z + \frac{1}{z}$ and $\lambda = 2$



Construction of $\mathcal{L}\mathcal{M}_\lambda^L$: example

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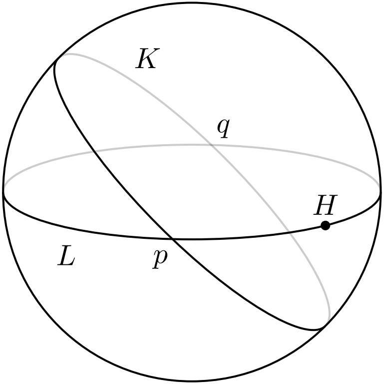

$$CF^0(L, K) = \mathbf{k} \cdot p$$
$$CF^1(L, K) = \mathbf{k} \cdot q$$

Construction of $\mathcal{LM}_\lambda^L$: example

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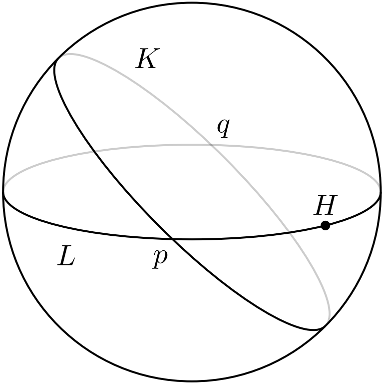

$$CF^0(L, K) = \mathbf{k} \cdot p$$
$$CF^1(L, K) = \mathbf{k} \cdot q$$

$$\delta p = (1 - z)q$$
$$\delta q = \left(\frac{1}{z} - 1\right)p$$

$$\delta^2 = \left(z + \frac{1}{z}\right) - 2$$

Idea of Theorem 3

Theorem 3

For each ρ with $W_L(\rho) = \lambda$, $\mathcal{LM}^L_\lambda$ sends (L, ρ) to $\mathcal{O}_\rho$

Proof sketch

- $\mathcal{E} = \mathcal{LM}^L_\lambda(L, \rho)$ is filtered geometrically
- Matrix factorisation $\mathcal{E}_0$ representing $\mathcal{O}_\rho$ is filtered algebraically
- Get spectral sequence $\hom(\operatorname{gr}\mathcal{E}, \operatorname{gr}\mathcal{E}_0) \implies \hom(\mathcal{E}, \mathcal{E}_0)$
- There's an obvious isomorphism $\overline{f} : \operatorname{gr}\mathcal{E} \to \operatorname{gr}\mathcal{E}_0$
- $\overline{f}$ survives spec seq so lifts to isomorphism $f : \mathcal{E} \to \mathcal{E}_0$

Idea of Theorem 3

More details

- $\mathcal{O}_\rho$ has Koszul resolution of shape $(\Omega^\bullet V, \eta \lrcorner -)$
  - V is (formal) neighbourhood of ρ, η is Euler vector field on V
- $\mathcal{E}_0$ looks like $(\Omega^\bullet V, (\eta \lrcorner -) + (\gamma \wedge -))$ where $\langle \eta, \gamma \rangle = W_L - \lambda$
- $\mathcal{E}$ looks like $(\Omega^\bullet V, (\eta \lrcorner -) + \dots)$
- $\hom(\operatorname{gr} \mathcal{E}, \operatorname{gr} \mathcal{E}_0)$ calculates $\operatorname{Ext}(\mathcal{O}_\rho, \mathcal{O}_\rho) = \Lambda^\bullet V$
- $\mathrm{id}_{\mathcal{O}_\rho}$ is the desired isomorphism $\overline{f} \in \hom(\operatorname{gr} \mathcal{E}, \operatorname{gr} \mathcal{E}_0)$
- It's in degree 0 in $\Lambda^\bullet V$ so survives spectral sequence

Idea of Theorem 1

Key step in Theorem 1

If $\mathcal{A}$ superfiltered A_∞-deformation of $\Lambda^\bullet V$ then

$$\mathcal{A} \simeq \mathrm{end}(\mathcal{O}_0) \text{ in } \mathrm{mf}(\widehat{V}, \mathfrak{P})$$

where $\mathfrak{P} : V \to \mathbf{k}$ is $\sum_{k \geq 2} \mu^k$

Proof sketch

- Apply algebraic version of LMF (CHL) to $\mathcal{A}$
- Get $\mathcal{A} \simeq \mathrm{end}(\mathcal{E})$ for some $\mathcal{E} \in \mathrm{mf}(\widehat{V}, \mathfrak{P})$; want $\mathcal{E} \cong \mathcal{E}_0$
- $\mathcal{E}$ is filtered since $\mathcal{A}$ superfiltered
- Same spectral sequence argument shows $\mathcal{E} \cong \mathcal{E}_0$

Thanks for listening!