K-THEORY

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1. Introduction

1.1. **Preamble.** These are notes for a graduate course in *complex topological K-theory*: the study of spaces through their complex vector bundles. Why might this be a sensible thing to do?

- It's analogous to studying a ring through its modules or a scheme through its sheaves.
- Vector bundles marry algebra and geometry. They have algebraic structure and one can perform operations like direct sums, quotients, and tensor/symmetric/exterior products, but are also spaces in their own right and have various associated spaces like Thom spaces and projectivisations.
- They arise naturally in geometry, for example as tangent/normal bundles and tautological bundles.
- We already have powerful methods to study them and relate them to ordinary cohomology via characteristic classes.

To save much repetition, all vector bundles will be assumed complex and of finite rank.

1.2. **The main point (sort of).** Recall that a monoid is a set with an associative binary operation with two-sided identity, i.e. a 'group without inverses'.

Definition 1.1. For a commutative monoid M, its *Grothendieck group* $K(M)$ is its 'abelian groupification'. In other words, it is an abelian group equipped with a monoid map $M \to K(M)$ such that every monoid map $M \to A$, with A an abelian group, factors uniquely through $K(M)$.

$K(M)$ has an explicit description in terms of formal differences of objects in M:

$$K(M) = \{(m_1, m_2) \in M \times M\}/\sim,$$

where $(m_1, m_2) \sim (m_1', m_2')$ if and only if there exists an m_3 with

$$m_1 + m_2' + m_3 = m_1' + m_2 + m_3 \text{ in } M.$$

Addition is component-wise, and we can write (m_1, m_2) as $m_1 - m_2$.

Remark 1.2. The reason for the m_3 here is that a monoid can have distinct elements m_1 and m_1' which become equal after addition of some m_3, and if we want to adjoin an inverse to m_3 then we had better make m_1 and m_1' equal. For instance, take M to be $(\mathbb{Z}_{\geq 0})^2/\sim$ where $(a, b) \sim (c, d)$ if and only if $a + b = c + d > N$, where N is an arbitrary fixed positive integer. The elements $(N, 0), (N - 1, 1), \ldots, (0, N)$ are pairwise distinct, but all become equal after adding $(1, 0)$.

For a space X let $\mathrm{Vect}(X)$ denote the commutative monoid of vector bundles on X under direct sum. Roughly we will define $K^0(X) = K(\mathrm{Vect}(X))$. Remarkably one can extend the definition to $K^i(X)$ for all integers i so that the functor K^* has all the properties of cohomology (it satisfies the Eilenberg–Steenrod axioms) and hence defines a *generalised cohomology theory*, which can then be used to prove results in topology. For example, as the culmination of these notes we prove:

Theorem. *If S^{2n} admits an almost complex structure then n is 1, 2 or 3.*

Remark 1.3. In fact, we won't define K-theory directly in terms of vector bundles. We will define it using homotopy theory, and see that for nice spaces X our definition of $K^0(X)$ coincides with $K(\mathrm{Vect}(X))$. However, vector bundles are clearly the motivation.

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1.3. **Variants.** There is a similar theory with real vector bundles, which gives a more powerful invariant $KO^*(X)$. The setup is basically the same but a little more complicated.

One can define the Grothendieck group of an abelian category $\mathscr{C}$ to be the abelian group generated by isomorphism classes of objects modulo the relation $[A] + [C] = [B]$ for every short exact sequence

$$0 \to A \to B \to C \to 0.$$

Similarly the Grothendieck group of a triangulated category $\mathscr{T}$, e.g. $D^b\mathscr{C}$, is generated by isomorphism classes of objects modulo $[A] + [B] = [C]$ for every triangle

$$\begin{tikzcd} A \arrow[rr] & & B \arrow[dl] \\ & C \arrow[ul, "{[1]}"'] & \end{tikzcd}$$

The obvious map $K(\mathscr{C}) \to K(D^b\mathscr{C})$ is an isomorphism.

Remark 1.4. The map $D^b\mathscr{C} \to K(\mathscr{C})$ is a kind of 'universal Euler characteristic'.

For a space X we can consider the abelian category $\mathscr{V}(X)$ of vector bundles, with morphisms given by bundle maps of constant rank. If X is paracompact (has partitions of unity) then every vector bundle can be given an inner product, and orthogonal complements provide splittings of short exact sequences in $\mathscr{V}(X)$. This means that $K(\mathrm{Vect}(X))$ coincides with $K(\mathscr{V}(X))$. If X is a complex manifold and we consider only holomorphic bundles, this need no longer be true:

$$K(\mathrm{Vect}_{\mathrm{hol}}(\mathbb{CP}^1)) \cong \mathbb{Z}^{\oplus\mathbb{Z}} \quad \text{whilst} \quad K(\mathscr{V}_{\mathrm{hol}}(\mathbb{CP}^1)) \cong \mathbb{Z}^2.$$

(The $\mathbb{Z}^{\oplus\mathbb{Z}}$ records the multiplicities of the line bundles $\dots, \mathcal{O}(-1), \mathcal{O}, \mathcal{O}(1), \dots$, whilst the $\mathbb{Z}^2$ only records the total rank and degree.)

1.4. **(Un)originality and acknowledgements.** Most of the material is standard, but there are a few places where the presentation is potentially novel (although probably well-known to experts): the use of non-abelian cellular Čech cohomology to relate bundles to classifying spaces; the use of mapping webs and the exact sequence (21) to (re)prove the coexact Puppe sequence and construct the Milnor sequence; the construction of the K-theory product via reduction to finite dimensions using the Milnor sequence and Atiyah–Hirzebruch spectral sequence.

I have borrowed liberally from many sources, most notably Hatcher [5], May [7], and Oscar Randal-Williams's lecture notes on 'Characteristic classes and K-theory'. Thanks are due to Oscar and to Neil Strickland for answering various questions about this material, and to the attendees of the lectures for helping to clarify several points. Any errors are my own and I would be glad to hear of them[1]. I am supported by EPSRC Grant EP/P02095X/1.

2. HOMOTOPY THEORY AND COHOMOLOGY

2.1. **Philosophy.** Cohomology is supposed to give a contravariant functor h from the category

$$\mathsf{hTop} = (\text{spaces, homotopy classes of maps}),$$

and a natural way to construct such a thing is to take homs into some fixed target object Z, which *represents* the functor. We will therefore try to construct cohomology theories as functors of the form

$$h : X \mapsto \mathsf{hTop}(X, Z),$$

and our problem then is to translate the homological-algebraic properties we want h to have into homotopy-theoretic statements about X and Z. Roughly we want to take statements about the familiar singular or cellular chain complexes of spaces and upgrade them to statements (up to homotopy) about the spaces themselves, so that we can then apply our unfamiliar cohomology functor.

Remark 2.1. Let $C_*(X)$ be the singular chain complex of a space X, and let $D^*(X) = C_{-*}(X)$ be its cohomologically graded version. The singular cohomology $H^0(X; A)$ is then precisely the set of homotopy classes of chain maps from $D(X)$ to the complex comprising a single copy of A, in degree 0.

The mantra of derived categories (borrowed from [13]) is

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(co)homology bad, chain complexes good

The perspective of modern algebraic topology, which will be our guiding principle, is

(co)homology bad, chain complexes good, spaces better.

Upgrading algebra to topology not only captures finer information but also makes it easier to understand and remember. An added benefit, which we'll see when we construct the Chern character using the Yoneda lemma, is that you can then apply algebraic topology to itself.

2.2. **Base points.** It is useful in homotopy theory to have base points, so ***from now on all spaces, maps and homotopies are based*** unless stated otherwise. We denote the category of based spaces by Top_* and its homotopy category by hTop_*. Base points will be denoted by $*$, as will the based space comprising a single point, and morphism spaces $\mathsf{hTop}_*(X, Z)$ will be denoted by $[X, Z]$. Given an unbased space X, we can build a canonical based space X_+ by adding in a disjoint base point. Whenever we talk about CW complexes, we will assume that the base point is a 0-cell.

Remark 2.2. One reason that hTop_* is nicer than hTop is that it has a zero object, namely $*$, so all hom sets are based too (the base point is the unique map which factors through the zero object).

What does incorporating base points mean in terms of cohomology theories? Well, recall that for singular cohomology, the reduced complex is obtained by artificially adding a $\mathbb{Z}$ in degree -1

$$\widetilde{C}^*(X) = \mathbb{Z} \xrightarrow{i} C^0(X) \xrightarrow{\mathrm{d}} C^1(X) \xrightarrow{\mathrm{d}} C^2(X) \xrightarrow{\mathrm{d}} \dots,$$

where i is the map which sends 1 in $\mathbb{Z}$ to the 'point counting cocycle'. A choice of base point $*$ in X induces a natural splitting of i, namely

$$\mathrm{ev}_* : \phi \in C^0(X) \mapsto \phi(*),$$

so we can think of $\widetilde{C}^0(X)$ as $\ker \mathrm{ev}_*$, and of $\widetilde{H}^*(X)$ as $H^*(X, *)$. Working with based spaces thus corresponds to doing *reduced cohomology* $\widetilde{h}$. We will therefore work mainly with reduced theories; the unreduced versions can be obtained by setting $h(X) = \widetilde{h}(X_+)$.

Remark 2.3. We know from Remark 2.1 that singular cohomology is homotopy classes of chain maps $D \to A$. A base point $*$ defines an element in $D^0 = C_0$, and reduced cohomology is given by those homotopy classes of maps which send this base point to the base point 0 in A. We denote the homotopy classes of degree i by $[D, A]^i$.

2.3. **Shifting.** Cohomology should provide a graded object, but at the moment we just have a single set $[X, Z]$ for each space X. We can think of this as $\widetilde{h}^0(X)$ and then look for a shift operator [1] on our category hTop_* so that $\widetilde{h}^i(X)$ can be defined as $\widetilde{h}^0(X[-i]) = [X[-i], Z]$.

Remark 2.4. For singular cohomology we have (in the notation of Remark 2.3) that

$$\widetilde{H}^i(X; A) = [D, A]^i = [D[-i], A]^0.$$

How do you shift a space? Loosely, we think of $\widetilde{h}^i(X)$ as seeing the i-dimensional information in X, so we want this to be the same as the 0-dimensional information in $X[-i]$. In other words, we want $X[1]$ to carry the same information as X but one dimension higher (this is consistent with Remark 2.4 in the sense that $D[1]$ is the cohomologically graded version of the *upward* shift of $C_*(X)$). If X is a CW complex then for each n the set of n-cells in X (excluding the base point $*$) should be in bijection with the $(n+1)$-cells in $X[1]$, and the attaching maps should be 'the same'. We therefore want a functorial way to turn n-cells and their boundaries into $(n+1)$-cells and their boundaries. The simplest way to do this is by suspension, and this matches with the familiar fact that suspension induces a shift isomorphism on ordinary reduced cohomology.

It is therefore natural to define $X[1]$ to be the (reduced) suspension

$$\Sigma X = X \times [0,1] / \big(X \times \{0,1\} \cup \{*\} \times [0,1]\big),$$

and hence

$$\widetilde{h}^i(X) = [\Sigma^{-i} X, Z] \tag{1}$$

for all integers i (although we don't yet know what Σ^{-i} means when i is positive).

Remark 2.5. The Eilenberg–Steenrod axioms actually include the property $\widetilde{h}^i(X) = \widetilde{h}^{i+1}(\Sigma X)$—and this is also forced upon us if we want $\widetilde{h}$ to have Mayer–Vietoris sequences—but hopefully the above discussion motivates this somewhat.

Remark 2.6. This definition of shift is also compatible with our existing definition of graded homotopy groups

$$\pi_i(Z) = [S^i, Z],$$

after replacing π_i with p^{-i} to make the grading cohomological: we have

$$[S^i[j], Z] = p^{-i-j}(Z) = [S^{i+j}, Z] = [\Sigma^j S^i, Z].$$

2.4. **Suspending versus looping.** In categories of chain complexes, the shift operator [1] has an inverse [−1] and these are two-sided adjoints to each other: shifting up (or down) on the domain is equivalent to shifting down (respectively up) on the codomain. In homotopy theory there is no inverse to Σ but it would be nice to have a right adjoint so that we can transfer shifts from the domain to the codomain.

The crucial observation is that ΣX can be expressed as $S^1 \wedge X$, where $\wedge$ is the smash product:

$$X \wedge Y = X \times Y / X \vee Y.$$

Here $\vee$ is wedge sum

$$X \vee Y = X \cup_* Y = \{(x, y) \in X \times Y : x = * \text{ or } y = *\},$$

which is the coproduct in the category Top_* of based spaces (and, as such, is associative and commutative up to unique isomorphism). Note that $\wedge$ is not the product in Top_* and in pathological cases is not associative.

The nice property of the smash product is that maps from $X \wedge Y$ to Z are roughly the same thing as families of maps $Y \to Z$ parametrised by X (everything is based, as usual). To make this precise, recall that for unbased spaces X and Y the set $\mathsf{Top}(X, Y)$ carries the compact-open topology, with a subbasis given by sets of the form

$$\{f : f(K) \subset U\},$$

where $K \subset X$ is compact and $U \subset Y$ is open. If Y is locally compact (any neighbourhood of any point contains a compact subneighbourhood) then obvious map

$$\mathsf{Top}(X \times Y, Z) \to \mathsf{Top}(X, \mathsf{Top}(Y, Z)) \tag{2}$$

is a bijection for any unbased space Z. Returning to the world of based spaces, restricting the left-hand side of (2) to maps which send $X \vee Y$ to $*$ in Z, we obtain a bijection

$$\mathsf{Top}_*(X \wedge Y, Z) \to \mathsf{Top}_*(X, \mathsf{Top}_*(Y, Z)) \tag{3}$$

whenever Y is locally compact.

In particular, taking $Y = S^1$, we obtain a bijection between maps $\Sigma X \to Z$ and maps $X \to \Omega Z$, and taking $Y = S^1 \wedge [0, 1]_+$ we obtain a bijection between homotopies of such maps. This gives a natural bijection

$$[\Sigma X, Z] \cong [X, \Omega Z]$$

for any spaces X and Z. In other words, Σ is left adjoint to Ω on hTop_*. Using this, we can rewrite (1) as

$$\widetilde{h}^i(X) = [X, \Omega^{-i} Z].$$

Remark 2.7. The failure of (3) to be a bijection for general spaces Y is annoying, and to get around it most topologists really work in the category of *compactly generated spaces*, where it becomes a homeomorphism.

2.5. **Ω-spectra.** We still have to make sense of Ω^{-i} for all integers i, but we can deal with this in a rather formal way:

Definition 2.8. An Ω-spectrum is a sequence $(Z_i)_{i \in \mathbb{Z}}$ of spaces and for each i a homotopy equivalence $Z_i \to \Omega Z_{i+1}$. These equivalences are part of the data but we will not notate them explicitly.

Remark 2.9. Some people require homeomorphism or weak equivalence instead of (strong) homotopy equivalence.

Given an Ω-spectrum (Z_i), we can view Z_i as $\Omega^{-i}Z_0$, and thus define a contravariant functor

$$\widetilde{h}^* : \mathsf{hTop}_* \to \mathbb{Z}\text{-graded based sets}$$

by setting

$$\widetilde{h}^i(X) = [X, Z_i].$$

The adjunction between Σ and Ω, together with the homotopy equivalences $Z_i \to \Omega Z_{i+1}$, give natural bijections

$$\widetilde{h}^i(X) \cong \widetilde{h}^{i+1}(\Sigma X)$$

for all X and i.

Example 2.10. For an abelian group A, the *Eilenberg–MacLane spectrum* HA is defined by

$$HA_i = \begin{cases} K(A,i) & \text{if } i \geq 0 \\ \quad * & \text{otherwise.} \end{cases}$$

In this case, the functor $\widetilde{h}^*$ is the ordinary reduced cohomology with coefficients in A.

In analogy with chain complexes, where the right adjoint to upward shift is downward shift, it is tempting to think of Ω as a downward shift, and hence as an inverse to Σ, but although there are natural *unit* and *counit* maps

$$X \to \Omega\Sigma X \quad \text{and} \quad \Sigma\Omega X \to X$$

they are not in general be isomorphisms in hTop_*. Intuitively, the space $\Sigma^n\Omega^n X$ is obtained by shifting all cells in $\Omega^n X$ (apart from the basepoint) up n dimensions, so it is $(n-1)$-connected (has trivial homotopy groups π_i for $i < n$), whereas the original space X may not have been. However, there is the following partial result:

Theorem 2.11 (Freudenthal suspension theorem)**.** *If $(X, *)$ is a relative CW complex (or, more generally, 'nondegenerately based') and $(n-1)$-connected then the map*

$$\pi_i(X) \to \pi_i(\Omega\Sigma X) \cong \pi_{i+1}(\Sigma X)$$

induced by the unit is an isomorphism for $i < 2n-1$ and a surjection for $i = 2n-1$.

In particular, this means that as n gets larger, the unit map $\Sigma^n X \to \Omega\Sigma^{n+1}X$ becomes increasingly close to being a weak equivalence (it induces isomorphisms on $\pi_{<2n-1}$). *Stable homotopy theory* is 'homotopy theory after infinitely many suspensions' and is really the correct place to talk about spectra. In the stable homotopy category, Σ and Ω actually are inverse.

2.6. **Group structure.** Cohomology theories defined in the form $\widetilde{h}^i(X) = [X, Z_i]$ seem to produce only graded based sets, whereas really we would like graded abelian groups. Recall, however that for any spaces X and Z the based set $[X, \Omega Z]$ is actually a group: concatenation of loops provides the operation and the base point is the identity. (Alternatively you write $[X, \Omega Z]$ as $[\Sigma X = S^1 \wedge X, Z]$ and use concatentation on the S^1 in the smash product.) Moreover, $[X, \Omega^2 Z]$ is an *abelian* group, via:

Lemma 2.12 (The Eckmann–Hilton trick)**.** *Suppose S is a set equipped with binary operations $\circ$ and $\otimes$, which commute with each other in the sense that*

$$(a \circ b) \otimes (c \circ d) = (a \otimes c) \circ (b \otimes d), \tag{4}$$

and such that each has a two-sided unit ($1_\circ$ and $1_\otimes$ respectively). Then the units and operations coincide, and the common operation is commutative.

Proof. Represent $a \circ b$ and $a \otimes b$ by

$$\boxed{a \quad b} \quad \text{and} \quad \boxed{\begin{array}{c} a \\ b \end{array}}$$

respectively. The condition (4) then just says that a 2×2 box is unambiguous: we can view it as a horizontal box around two vertical boxes or vice versa. Using this, we have

$$1_\circ = \boxed{1_\circ \quad 1_\circ} = \boxed{\begin{array}{cc} 1_\circ & 1_\otimes \\ 1_\otimes & 1_\circ \end{array}} = \boxed{\begin{array}{c} 1_\otimes \\ 1_\otimes \end{array}} = 1_\otimes,$$

so the two units agree—we'll write their common value as 1. We then have

$$\boxed{a \quad b} = \boxed{\begin{matrix} a & 1 \\ 1 & b \end{matrix}} = \boxed{\begin{matrix} a \\ b \end{matrix}} = \boxed{\begin{matrix} 1 & a \\ b & 1 \end{matrix}} = \boxed{b \quad a},$$

for all a and b, so $a \circ b = a \otimes b = b \circ a$. □

In our case, the two operations we want to consider are concatenation on the first and second loop factors of $\Omega^2 Z$.

Remark 2.13. The same argument shows that π_1 of a topological group is abelian, and that the higher homotopy groups of any space are abelian.

Using this we conclude that for any Ω-spectrum (Z_i) and any space X the set

$$\widetilde{h}^i(X) = [X, Z_i] = [X, \Omega^2 Z_{i+2}]$$

is an abelian group for each i. Moreover, for any map $f : X \to Y$ the pullback $f^* : \widetilde{h}^i(Y) \to \widetilde{h}^i(X)$ is a group homomorphism.

2.7. **Digression: iterated loop spaces.** In an Ω-spectrum (Z_i), each space Z_j determines Z_i for all $i < j$ via $Z_i = \Omega^{j-i} Z_j$. It doesn't determine the Z_i with $i > j$ however. For example, the Eilenberg–MacLane spectra for different coefficient groups A all coincide in negative degree. The process of looping is easy (and mechanical), but delooping is not.

It is natural to wonder when a space Z can be delooped, i.e. when it is homotopy equivalent to a loop space $\Omega Z'$. Well, we know that any loop space admits an operation—concatenation—which is associative up to homotopy, so the same must be true of Z. Moreover, for loop concatenation these homotopies themselves satisfy compatibility conditions up to homotopy, which in turn satisfy higher compatibilities up to homotopy, and so on. This makes every loop space into an A_∞*-space*: it carries an action of the A_∞-operad of *little intervals*. Therefore, in order to have a hope of delooping Z, it must be an A_∞-space. In fact, it can be shown that this condition, plus the assumption that Z is *grouplike*, is sufficient as well as necessary. Here grouplike means that the A_∞-product turns $\pi_0(Z)$ into a group; a priori it is just a monoid, and we need it to have inverses. Moreover, the A_∞-structure uniquely determines the delooping Z' up to adding extra components (which are invisible in $\Omega Z'$).

The A_∞-operad generalises to the E_n-operad of little n-cubes, and we have the following:

Theorem 2.14 (May recognition theorem)**.** *A space Z is homotopy equivalent to an n-fold loop space if and only if it is a grouplike E_n-space.*

Back to our spectrum, the space Z_0 can be delooped arbitrarily many times. Boardman–Vogt and May (I'm not sure of the chronology) also showed that such things—*infinite loop spaces*—are precisely grouplike E_∞-spaces, where $E_\infty = \operatorname{colim} E_n$ is the operad of little ∞-cubes which have full width in all but finitely many coordinates. They are the homotopy theoretic version of abelian groups; compare with the Eilenberg–MacLane spectrum, where Z_0 is exactly an abelian group.

2.8. **Short exact sequences.** The other key algebraic property of cohomology is that it should turn short exact sequences into long exact sequences. In an abelian category $\mathscr{C}$ the functor $\mathscr{C}(-, Z)$ is left-exact for any object Z. More precisely, it takes right-exact sequences

$$A \to B \to C \to 0$$

to left-exact sequences

$$0 \to \mathscr{C}(C, Z) \to \mathscr{C}(B, Z) \to \mathscr{C}(A, Z).$$

This suggests that we should consider 'right-exact' sequences of spaces

$$X \xrightarrow{f} Y \to Y/f(X) \tag{5}$$

to get (partial) exactness when we map out. Indeed, maps out of $Y/f(X)$ are (almost by definition) the same as maps out of Y which are constant when pulled back to X.

We are really interested in *homotopy classes* of maps though, so we should think about maps from Y which are *nullhomotopic* when pulled back to X. These are precisely maps from the (reduced) *mapping cone*

$$Cf = \big(X \times [0,1] \cup_{f \times \{1\}} Y\big) / \big(X \times \{0\} \cup \{*\} \times [0,1]\big).$$

This gives us a sequence

$$X \xrightarrow{f} Y \xrightarrow{i} Cf \tag{6}$$

and for any space Z there is an induced exact sequence of based sets

$$[Cf, Z] \xrightarrow{i^*} [Y, Z] \xrightarrow{f^*} [X, Z].$$

The space Cf is the *homotopy cofibre* of f, and its isomorphism class in hTop_* depends only on X, Y and f up to homotopy. Any sequence isomorphic in hTop_* to one of the form (6) is called a *homotopy cofibre sequence.*

In passing from (5) to (6), however, we have destroyed 'exactness at the right': the map $i : Y \hookrightarrow Cf$ is not surjective, but instead its 'cokernel' is $Cf/Y \cong \Sigma X$. This suggests that we should extend the sequence (6) to

$$X \xrightarrow{f} Y \xrightarrow{i} Cf \xrightarrow{q} \Sigma X, \tag{7}$$

where q is the map which crushes $Y \subset Cf$ to a point. On the other hand, we know that to extend exactness we should continue the sequence to

$$X \xrightarrow{f} Y \xrightarrow{i} Cf \to Ci. \tag{8}$$

Happily, (7) and (8) are isomorphic in hTop_* via

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "i"] \arrow[d, equal] & Cf \arrow[r] \arrow[d, equal] & Ci \arrow[d, "p"', "\wr"] \\
X \arrow[r, "f"] & Y \arrow[r, "i"] & Cf \arrow[r, "q"] & \Sigma X
\end{tikzcd}$$

where $p : Ci \to \Sigma X$ is the map which crushes $CY \subset Ci$ to a point. To see that p is indeed a homotopy equivalence note that we can realise it as

$$Ci = CX \cup_f CY \simeq CX \cup_{f'} CY \simeq \Sigma X,$$

where f' is the map $X \to CY$ which sends all of X to the cone point. Here the first homotopy equivalence comes from homotoping f to f' (as maps $X \to CY$) and the second comes from contracting CY to its cone point.

Remark 2.15. The cone Cf—and similar constructions where one glues spaces together by cylinders–is sometimes called a *homotopy colimit*; in this case, of the diagram $* \leftarrow X \xrightarrow{f} Y$. Be warned, however, that such homotopy colimits are *not* generally categorical colimits of the corresponding diagrams in hTop_*.

2.9. **Long exact sequences.** Given the above, there is now an obvious way to extend indefinitely

$$X \xrightarrow{f} Y \xrightarrow{i} Cf \xrightarrow{q} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma i} \Sigma(Cf) \xrightarrow{-\Sigma q} \Sigma^2 X \xrightarrow{\Sigma^2 f} \Sigma^2 Y \to \dots, \tag{9}$$

and this is called the *(coexact) Puppe sequence* associated to f. Okay, maybe the signs—which denote reversal of the suspension directions—are not obvious. They depend on choosing certain sensible identifications of quotients of cones with suspensions (one should draw cones on 'alternate sides') and are needed to guarantee certain naturality properties. Basically they should be thought of as Koszul signs, since Σ has 'odd degree'.

The crucial property of this sequence is:

Lemma 2.16. *Any three consecutive terms in the Puppe sequence are isomorphic in* hTop_* *to a sequence of the form* (6)*, so for any space Z we obtain a long exact sequence of based sets*

$$\cdots \to [\Sigma Y, Z] \xrightarrow{(-\Sigma f)^*} [\Sigma X, Z] \xrightarrow{q^*} [Cf, Z] \xrightarrow{i^*} [Y, Z] \xrightarrow{f^*} [X, Z]. \tag{10}$$

Proof. There's nothing to prove for the first three terms, and we've already dealt with the second three. We can reduce the third three to the same form as the second if we can show that

$$\begin{tikzcd}
Cf \arrow[r] \arrow[d, equal] & Ci \arrow[r] \arrow[d, "p"', "\wr"] & \Sigma Y \arrow[d, equal] \\
Cf \arrow[r, "q"] & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y
\end{tikzcd}$$

commutes up to homotopy. This can be proved by drawing Ci as CY (point up) on top of Cf (point down), and ΣY as CY on top of CY, and varying the height in ΣY to which one maps the 'floor' between the two halves of Ci.

It's now enough to show that the sequences of the form (6) are preserved by suspension, which essentially follows from the fact that there is a natural identification (homeomorphism) between ΣCf and $C\Sigma f$. □

Remark 2.17. In Section 5 we will prove a much more general statement which reduces to this exact sequence in a special case.

We already saw that Σ plays the role of shift [1] in hTop_* so the Puppe sequence is like an exact triangle:

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \arrow[dl] \\ & Cf \arrow[ul, "{[1]}"] & \end{tikzcd}$$

Passing from Top_* to hTop_*, short exact sequences automatically became exact triangles! Moreover the thing which completes $f : X \to Y$ to an exact triangle really is the geometric cone on f. The sequence (10) is the associated long exact sequence in cohomology, with q^* (and its suspensions) providing the connecting map.

2.10. Eilenberg–Steenrod axioms.

We have seen that for any Ω-spectrum (Z_i)—meaning a sequence of (based) spaces equipped with homotopy equivalences $Z_i \to \Omega Z_{i+1}$—the assignment

$$\widetilde{h}^* : X \mapsto [X, Z_*]$$

defines a contravariant functor from hTop_* to graded abelian groups. Moreover:

- There is a natural isomorphism $\widetilde{h}^* \to \widetilde{h}^{*+1} \circ \Sigma$ (given by the adjunction of Σ and Ω).
- It takes wedge sums of spaces to direct products of graded abelian groups.
- It takes homotopy cofibre sequences to long exact sequences.

A functor with these properties is precisely (according to the Eilenberg–Steenrod axioms) a *generalised reduced cohomology theory*. Using the *Brown representability theorem* one can show that all such theories arise from mapping into spectra.

To define the corresponding unreduced theory we simply include a disjoint basepoint:

$$h^i(X) := \widetilde{h}^i(X_+).$$

Note that $\widetilde{h}^i(*) = 0$ for all i, whilst $h^i(*) = [S^0, Z_i] = \pi_0(Z_i)$. From our long exact sequence, the reduced cohomology of a pair $f : X \to Y$ should be the reduced cohomology of the reduced mapping cone, whilst the unreduced cohomology of the pair should be the reduced cohomology of the unreduced mapping cone (with base point at the cone point).

Remark 2.18. We said back in Section 2.1 that the reduced cohomology of a based space X should be thought of as the cohomology of the pair $(X, *)$ but from our perspective, in which the reduced theory is fundamental, this is only approximately correct. It *is* true if the unreduced mapping cone of $* \hookrightarrow X$ is homotopy equivalent to X *as based spaces*, but there are pathological examples where this fails—for example if $X = \{1, 1/2, 1/3, \ldots, 0\}$ and $* = 0$.

3. BUNDLES AND CLASSIFYING SPACES

3.1. Recap: fibrations.

Recall that a *(Hurewicz) fibration* is a map $\pi : E \to B$ which has the homotopy lifting property: for any space X, any homotopy

$$b^t : X \times [0,1]_t \to B,$$

(we'll write b^t for the whole homotopy and also its restriction to the time t slice) and any lift e^0 of b^0 to E, there exists a lift e^t of b^t

$$\begin{tikzcd} X \times \{0\} \arrow[r, "e^0"] \arrow[d] & E \arrow[d, "\pi"] \\ X \times [0,1] \arrow[r, "b^t"] \arrow[ur, dashed, "e^t"] & B \end{tikzcd}$$

This property is preserved under pullback. Let F denote the fibre $\pi^{-1}(*)$ and $i : F \to E$ the inclusion.

Lemma 3.1. *If B is contractible then i induces a homotopy equivalence $F \simeq E$.*

Proof. Take $X = E$ above, let $e^0 = \mathrm{id}_E$ and let b^t be the composition of π with a contraction of B to its base point (so $b^0 = \pi$ and b^1 sends all of E to $* \in B$). By homotopy lifting there exists a lift e^t of b^t, and e^1 lands in F so we can write it as $i \circ j$ for a unique map $j : E \to F$. By definition of j, e^t is a homotopy from id_E to $j \circ i$. On the other hand, $e^t \circ i$ lands in F so can be written as $i \circ f^t$ for a unique homotopy $f^t : F \times [0,1]_t \to F$, and then f^t is a homotopy from id_F to $i \circ j$. Hence i and j are homotopy inverses. ☐

Corollary 3.2. *All fibres of a fibration $\pi : E \to B$ over a path-connected base are homotopy equivalent.*

Proof. Given two points x and y in B, pick a path $\gamma : [0,1] \to B$ between them and consider the pullback $\gamma^*\pi$. The base $[0,1]$ is contractible, so by Lemma 3.1 the inclusions of the fibres over 0 and 1 are both homotopy equivalences. ☐

It therefore makes sense to talk about *the* fibre of a fibration, meaning the homotopy type of the fibres over the path component containing the base point. We'll often write the fibration $E \to B$ with fibre F as $F \hookrightarrow E \to B$.

Using homotopy lifting, there is a map $m : \Omega B \to F$, which sends a loop γ to the 'monodromy' of π around γ. Explicitly, there is a homotopy $b^t : \Omega B \times [0,1]_t \to B$ given by

$$b^t(\gamma \in \Omega B) = \gamma(t),$$

and an obvious lift e^0 which collapses ΩB to the base point in E, and m is defined to be e^1.

Remark 3.3. Although m depends on the choice of lift e^t, its homotopy class is independent of this choice. Indeed, if $e^{0,t}$ and $e^{1,t}$ are two different lifts then we can extend them to a map

$$e^{s,t} : E \times (\{0,1\}_s \times [0,1]_t \cup [0,1]_s \times \{0\}_t) \to E$$

by defining $e^{s,0}$ to collapse ΩB to the base point in E. This lifts the map

$$b^{s,t} : \Omega B \times [0,1]_s \times [0,1]_t \to B,$$

given by $(\gamma, s, t) \mapsto \gamma(t)$, over three edges of the square $[0,1]_s \times [0,1]_t$, and by homotopy lifting we can extend $e^{s,t}$ to a lift over the whole square. Then $e^{s,1}$ is a homotopy from $e^{0,1}$ to $e^{1,1}$.

We thus have maps

$$\Omega B \xrightarrow{m} F \xrightarrow{i} E \xrightarrow{\pi} B$$

and these extend to a sequence

$$\dots \xrightarrow{\Omega^2\pi} \Omega^2 B \xrightarrow{-\Omega m} \Omega F \xrightarrow{-\Omega i} \Omega E \xrightarrow{-\Omega\pi} \Omega B \xrightarrow{m} F \xrightarrow{i} E \xrightarrow{\pi} B$$

'dual' to the Puppe sequence (9). In particular, for any space W we obtain a long exact sequence of based sets

$$\dots \xrightarrow{(-\Omega\pi)_*} [W, \Omega E] \xrightarrow{(-\Omega\pi)_*} [W, \Omega B] \xrightarrow{m_*} [W, F] \xrightarrow{i_*} [W, E] \xrightarrow{\pi_*} [W, B]$$

Taking $W = S^0$ gives the long exact sequence in homotopy groups for the fibration.

If B is a connected CW complex and E has a compatible CW structure (meaning that $\pi^{-1}(B_n)$ is a subcomplex of E for each n) then there is the *Serre spectral sequence*

$$E_2 = H^*(B; H^*(F)) \implies H^*(E).$$

A technical result of Hurewicz and Hübsch (see [11, Section 2.7] and [5, Proposition 4.48]) states that if B is paracompact (admits partitions of unity; this includes CW complexes) then a map $\pi : E \to B$ is a fibration if and only if it's locally a fibration.

3.2. **Principal bundles.** ***In this subsection we drop all base points.***

Fix a topological group G.

Definition 3.4. An *principal G-bundle* is a triple (E, B, π) where:

- E is a space carrying a (continuous) G-action.
- π is a map $E \to B$.
- B is a space covered by open sets U_α such that for each α there is a G-equivariant homeomorphism
$$\phi_\alpha : \pi^{-1}(U_\alpha) \to G \times U_\alpha.$$

E is the *total space*, B is the *base*, and the ϕ_α are *local trivialisations.* A morphism from $\pi_1 : E_1 \to B_1$ to $\pi_2 : E_2 \to B_2$ comprises a G-equivariant map $f_E : E_1 \to E_2$ covering a map $f_B : B_1 \to B_2$. When working with bundles over a fixed B we will restrict to $f_B = \mathrm{id}_B$. We'll often drop the word *principal* for brevity.

There is a correspondence between trivialisations and sections over each open set U

$$\text{trivialisation } \phi \mapsto \text{section } \big(x \in U \mapsto \phi^{-1}(1, x)\big)$$
$$\text{trivialisation } \big(g \cdot s(x) \mapsto (g, x)\big) \mapsfrom \text{section } s$$

A principal G-bundle over a cube (or equivalently a disc) can always be globally trivialised by cutting the cube into subcubes, each contained in a local trivialisation, and gluing together the local trivialisations in a sensible order (e.g. so that at each stage the cube you're gluing on meets the union of the cubes you've already dealt with in a contractible set). Using this we can classify G-bundles over CW complexes.

First we need a definition:

Definition 3.5. Given a CW complex X, denote the cells—meaning the discs we glued together to make X, *not* their images in$X$—by D_α, and the *characteristic maps* $D_\alpha \to X$ by Φ_α. For tuples $(\alpha_0, \dots, \alpha_k)$ of distinct α_i, let $D_{\alpha_0 \dots \alpha_k}$ be the fibre product of $D_{\alpha_0}, \dots, D_{\alpha_k}$ over X, i.e.
$$D_{\alpha_0 \dots \alpha_k} = \{(x_0, \dots, x_k) \in D_{\alpha_0} \times \cdots \times D_{\alpha_k} : \Phi_{\alpha_0}(x_0) = \cdots = \Phi_{\alpha_k}(x_k)\}.$$
Define Čech-type cochain groups
$$\check{C}^k(X; G) = \prod_{(\alpha_0, \dots, \alpha_k)} \{\text{continuous maps } \psi_{\alpha_0 \dots \alpha_k} : D_{\alpha_0 \dots \alpha_k} \to G\}.$$
Each of these groups contains a canonical element 1, each of whose components is the constant map to the identity in G. We define a 'differential' $\delta^0 : \check{C}^0(X; G) \to \check{C}^1(X; G)$ by
$$(\delta^0 \psi)\big((x_0, x_1) \in D_{\alpha_0 \alpha_1}\big) = \psi(x_1) \cdot \psi(x_0)^{-1};$$
more generally there is an action of $\check{C}^0(X; G)$ on $\check{C}^1(X; G)$ by
$$(\psi \cdot \chi)(x_0, x_1) = \psi(x_1) \cdot \chi(x_0, x_1) \cdot \psi(x_0)^{-1}$$
and δ^0 is simply action on 1. Similarly we define a 'differential' $\delta^1 : \check{C}^1(X; G) \to \check{C}^2(X; G)$ by
$$(\delta^1 \psi)(x_0, x_1, x_2) = \psi(x_0, x_1) \cdot \psi(x_0, x_2)^{-1} \cdot \psi(x_1, x_2).$$
Note that $(\delta^1)^{-1}(1)$ is preserved by the action of $\check{C}^0(X; G)$. Finally we define
$$\check{H}^0(X; G) = (\delta^0)^{-1}(1) = \text{'kernel of } \delta^0\text{'}$$
and
$$\check{H}^1(X; G) = (\delta^1)^{-1}(1)/\text{action of } \check{C}^0(X; G) = \text{'kernel of } \delta^1 \text{ modulo image of } \delta^0\text{'}.$$

Remark 3.6. In contrast with usual Čech cohomology, the sets $D_{\alpha_0 \dots \alpha_k}$ do not form an open cover of X: they are subsets of the products of cells, rather than of X itself, and their images in X are closed.

Remark 3.7. If G is non-abelian then there is no obvious way to define $\check{H}^k(X; G)$ for $k > 2$, and $\check{H}^1(X; G)$ does not carry an obvious group operation (it's just a based set).

Lemma 3.8. *For any CW complex X there is a (sensible) based bijection between $\check{H}^1(X; G)$ and the based set* $\mathrm{Bund}_G(X)$ *of isomorphism classes of principal G-bundles over X.*

Remark 3.9. This shows that $\check{H}^1(X; G)$ is independent of the choice of CW structure on X, which is not obvious a priori.

Proof. Fix a bundle $\pi : E \to X$, and for each cell D_α choose a section s_α of $\Phi_\alpha^* \pi$; we can do this since bundles over discs are trivial. For each pair α_0, α_1, there is a unique continuous function $\psi_{\alpha_0\alpha_1} : D_{\alpha_0\alpha_1} \to G$ which satisfies

$$s_{\alpha_1}(x_1) = \psi_{\alpha_0\alpha_1}(x_0, x_1) \cdot s_{\alpha_0}(x_0) \text{ in the fibre } \pi^{-1}\big(\Phi_{\alpha_0}(x_0) = \Phi_{\alpha_1}(x_1)\big)$$

for all (x_0, x_1) in $D_{\alpha_0\alpha_1}$. These $\psi_{\alpha_0\alpha_1}$ fit together to form a Čech 1-cochain ψ and it is easy to check (not Čech!) that $\delta^1\psi = 1$.

Changing the sections s_α corresponds to modifying ψ by the action of $\check{C}^0(X; G)$, so our bundle π gives rise to a well-defined class $[\psi]$ in $\check{H}^1(X; G)$. Sections over the cells can be pulled back under isomorphisms of bundles, and so $[\psi]$ actually only depends on the isomorphism class of π. We obtain a map

$$\Psi : \mathrm{Bund}_G(X) \to \check{H}^1(X; G)$$

and the trivial bundle clearly maps to the class [1].

To construct a map the other way, suppose ψ is a 1-cocycle. Let the dimension of the cell D_α be denoted by $|\alpha|$, and let the skeleta of X be $X_0 \subset X_1 \subset \ldots$. Just as X_{n+1} is obtained from X_n by gluing on those D_α with $|\alpha| = n+1$, we shall construct a G-bundle $\pi : E \to X$ inductively over skeleta $\pi_n : E_n \to X_n$ by gluing on trivial bundles $G \times D_\alpha$ using ψ to define the fibre gluings.

To make this precise, suppose we have constructed a bundle $\pi_n : E_n \to X_n$ with a choice of section s_α over each cell D_α with $|\alpha| \leq n$ such that

$$s_{\alpha_1}(x_1) = \psi_{\alpha_0\alpha_1}(x_0, x_1) \cdot s_{\alpha_0}(x_0) \tag{11}$$

holds for all pairs (α_0, α_1) of such α. We want to extend this construction from n to $n + 1$, so take an arbitrary $(n + 1)$-cell D_α and consider the trivial bundle $G \times D_\alpha \to D_\alpha$ with the obvious identity section. The cell D_α is attached to X_n by taking $X_n \amalg D_\alpha$ and identifying those points $x \in D_\alpha$ and $\Phi_{\alpha'}(x') \in X_n$ for which (x, x') lies in $D_{\alpha\alpha'}$. To attach $G \times D_\alpha$ to E_n instead, we take $E_n \amalg (G \times D_\alpha)$ and identify those $(g, x) \in G \times D_\alpha$ and $g \cdot \psi_{\alpha'\alpha}(x', x) \cdot s_{\alpha'}(x') \in E_n$ for which (x, x') lies in $D_{\alpha\alpha'}$. The cocycle condition ensures that this identification of $G \times \{x\}$ with $\pi_n^{-1}(\Phi_\alpha(x))$ is independent of the choice of α'.

We have now glued $G \times D_\alpha$ onto E_n in a way which respects the projection to the base and the G-action on the total space. Doing this for all $(n + 1)$-cells D_α gives the bundle $\pi_{n+1} : E_{n+1} \to X_{n+1}$ we want. The identity sections of the $G \times D_\alpha$ define sections of π_{n+1} over the D_α and by construction these satisfy (11). Taking the colimit of the E_n we obtain a bundle $\pi : E \to X$ with $\Psi(\pi) = [\psi]$.

This proves that Ψ is surjective. To see that it is injective, suppose that π and π' are bundles with $\Psi(\pi) = \Psi(\pi')$. The definition of Ψ involves choosing sections over the cells, and by choosing these sections s_α and s'_α appropriately we can ensure that the equality $\Psi(\pi) = \Psi(\pi')$ holds at the chain level. We can then define an isomorphism from π to π' by sending the point $g \cdot s_\alpha(x)$ to $g \cdot s'_\alpha(x)$ for each α and each x in D_α. ☐

Remark 3.10. When constructing the bundle by gluing we were a little sloppy about local triviality: whilst our construction clearly produces a map $E \to X$ which is trivial when pulled back to each cell, it is not obviously trivial over small open sets. This is easy to fix though, since any section defined on an open subset U of X_n can be extended to an open subset of X_{n+1} by extending into a collar neighbourhood of $\Phi_\alpha^{-1}(U) \subset \partial D_\alpha$ in D_α for each $(n+1)$-cell D_α. Such an extension exists by triviality over cells.

Remark 3.11. We are repeatedly making implicit use of the fact that defining a continuous map from a space—for example ∂D_α—is equivalent to defining continuous maps from each piece of a finite closed cover—for example the preimages of the $\Phi_{\alpha'}(D'_\alpha) \subset X$ in D_α—which agree on overlaps. Similarly, continuity of a map from a CW complex can be check by pulling it back to each cell.

3.3. **Classifying spaces.** ***In this subsection we drop all base points.***

We've classified principal bundles in terms of a set which looks like non-abelian Čech cohomology. In the spirit of homotopy theory, we'd like to represent this cohomology set as the set of homotopy classes of maps to some fixed space. Such a space is basically what is meant by a classifying space for G. In formalising this there seem to be different definitions in different places, but we shall use the following:

Definition 3.12. A *classifying space* BG for G is the base of a principal G-bundle $\pi : EG \to BG$ whose total space EG is contractible. This bundle is called a *universal* principal G-bundle.

Lemma 3.13. *For any CW complex X and any classifying space BG for G there is a (sensible) based bijection between $\check{H}^1(X;G)$ and the based set $\mathsf{hTop}(X,BG)$ of (unbased—recall that throughout this subsection we are ignoring base points) homotopy classes of maps from X to BG.*

Proof. Given a map $f : X \to BG$ we can pull back the universal G-bundle $\pi : EG \to BG$ to get a principal G-bundle $f^*\pi$ over X, and then apply the (chain level) map Ψ from Lemma 3.8 to obtain a Čech cocycle ψ. Explicitly, for each α we choose a lift $f_\alpha : D_\alpha \to EG$ of $f \circ \Phi_\alpha : D_\alpha \to BG$—which is the same thing as a section of the pullback bundle—and then define ψ by

$$f_{\alpha_1}(x_1) = \psi_{\alpha_0\alpha_1}(x_0, x_1) \cdot f_{\alpha_0}(x_0) \tag{12}$$

for all (x_0, x_1) in each $D_{\alpha_0\alpha_1}$. As in Lemma 3.8, changing the lifts (i.e. the sections of the pullback) corresponds to modifying this cocycle by the action of $\check{C}^0(X;G)$.

Given an unbased homotopy f^t from $f^0 = f$ to f^1 we can pull back the universal bundle to obtain a G-bundle over $X \times [0,1]$, and extend each lift $f^0_\alpha = f_\alpha$ of f over D_α to a lift f^t_α of f^t over $D_\alpha \times [0,1]$. This gives a path ψ^t of cocycles defined by the obvious t-parametrised version of (12), and we need to show that ψ^1 is cohomologous to ψ^0. In fact, we'll show that the extensions f^t_α can be chosen so that ψ^t is actually constant.

We'll construct these extensions f^t_α cellwise by induction on the dimension, so focus on a particular $(n+1)$-cell D_α and assume we have chosen extensions $f^t_{\alpha'}$ for all cells $D_{\alpha'}$ with $|\alpha'| \leq n$. We already have a lift f^0_α of f^0 over D_α, and a cocycle ψ^0 at time 0. Using this cocycle, the expression

$$f^t_\alpha(x) = \psi^0_{\alpha'\alpha}(x', x) \cdot f^t_{\alpha'}(x')$$

defines a lift f^t_α of f^t over the boundary ∂D_α. Here $D_{\alpha'}$ ranges over all cells of dimension at most n, whose images in X cover the boundary $\Phi_\alpha(\partial D_\alpha)$, and the cocycle condition ensures consistency between the different expressions for $f^t_\alpha(x)$ corresponding to different choices of α' and x'.

We now just need to extend this partial definition of f^t_α from $(D_\alpha \times \{0\}) \cup (\partial D_\alpha \times [0,1])$. To see that this is possible, just observe that the pair

$$\big(D_\alpha \times [0,1], (D_\alpha \times \{0\}) \cup (\partial D_\alpha \times [0,1])\big)$$

is homeomorphic to $(D_\alpha \times [0,1], D_\alpha \times \{0\})$, and for this pair the extension problem is easy (pick a trivialisation over $D_\alpha \times [0,1]$ and then extend 'constantly' in the $[0,1]$-direction). This completes the proof that an unbased homotopy class of map $f : X \to BG$ gives rise to a well-defined class $\Psi(f^*\pi)$ in $\check{H}^1(X;G)$.

Conversely, suppose we are given a Čech 1-cocycle ψ. We need to construct a map $f : X \to BG$ with $\Psi(f^*\pi) = [\psi]$. What we'll actually construct is a map $f_\alpha : D_\alpha \to EG$ for each α such that for all (α_0, α_1) and all $(x_0, x_1) \in D_{\alpha_0\alpha_1}$ the equation (12) holds. This ensures that the maps $\pi \circ f_\alpha$ glue to give a continuous map $f : X \to BG$, and that $\Psi(f^*\pi) = [\psi]$.

Again, we'll define the f_α by induction on the dimension, so fix an $(n+1)$-cell D_α and assume we've constructed $f_{\alpha'}$ for all cells $D_{\alpha'}$ with $|\alpha'| \leq n$. The condition (12) applied with $\alpha_1 = \alpha$ and $\alpha_0 = \alpha'$ defines f_α on the boundary ∂D_α, and now we use contractibility of EG to extend arbitrarily over the interior of the disc. This gives f_α with the desired properties and we conclude that the assignment $f \mapsto \Psi(f^*\pi)$ is surjective.

Finally, we need to check that it is injective, i.e. that if two maps $X \to BG$ define the same Čech cohomology class $[\psi]$ then they are (unbased) homotopic. So suppose we're given two such maps, f^0 and f^1. The fact that they define the same class means that we can choose cellwise lifts $f^i_\alpha : D_\alpha \to EG$ of f^i such that for each i we have

$$f^i_{\alpha_1}(x_1) = \psi_{\alpha_0\alpha_1}(x_0, x_1) \cdot f^i_{\alpha_0}(x_0)$$

for all $(x_0, x_1) \in D_{\alpha_0\alpha_1}$. We can now construct cellwise homotopies between the f^i_α, by induction on dimension, so that the t-parametrised version of this condition holds but with ψ kept constant. More explicitly, we have maps

$$f^i_\alpha : D_\alpha \times \{i\} \subset D_\alpha \times [0,1] \to EG$$

and assuming we have already chosen lifts $f^t_{\alpha'} : D_\alpha \times [0,1] \to EG$ over lower-dimensional cells $D_{\alpha'}$ the ψ condition defines f^t_α over $\partial D_\alpha \times [0,1]$. We thus have a definition over the boundary $\partial(D_\alpha \times [0,1])$, and we can use contractibility of EG to extend over the interior.

This proves that the assignment

$$f \in \mathsf{hTop}(X, BG) \mapsto \Psi(f^*\pi) \in \check{H}^1(X;G)$$

is injective, completing the proof of the lemma. ☐

The upshot of this section so far is that for any CW complex X and any classifying space BG for G we have the following commuting diagram of isomorphisms of based sets

$$\begin{tikzcd} \mathsf{hTop}(X,BG) \arrow[rr, "\text{pull back universal bundle}"] \arrow[dr, "\text{Lemma 3.13}"'] & & \mathrm{Bund}_G(X) \arrow[dl, "\Psi"', "\text{Lemma 3.8}"] \\ & \check{H}^1(X;G) & \end{tikzcd} \tag{13}$$

Note that a priori it is not obvious that the horizontal arrow is even well-defined, i.e. that pullbacks of the universal bundle by homotopic maps are isomorphic. In fact, we obtain:

Corollary 3.14. *Assume that a classifying space BG exists. If X and Y are unbased CW complexes, and f_0 and f_1 are maps $X \to Y$ which are (unbased) homotopic, then the pullback maps $f_i^* : \mathrm{Bund}_G(Y) \to \mathrm{Bund}_G(X)$ are equal.*

Proof. By Eq. (13) it suffices to show that the pullback maps

$$f_i^* : \mathsf{hTop}(Y,BG) \to \mathsf{hTop}(X,BG)$$

are equal and this is clearly true if f_0 and f_1 are unbased homotopic. ☐

We won't say anything general about existence of classifying spaces, but results in this direction can be proved using *Brown's representability theorem.* We can prove a uniqueness result though:

Corollary 3.15. *If BG and BG' are classifying spaces for G which are both CW complexes then they are homotopy equivalent.*

Proof. Let their universal bundles be $\pi : EG \to BG$ and $\pi' : EG' \to BG'$ respectively. By Eq. (13) there exist unique unbased homotopy classes of map $f : BG \to BG'$ and $f : BG' \to BG$ such that $f^*\pi' \cong \pi$ and $(f')^*\pi \cong \pi'$. We then have $(f' \circ f)^*\pi \cong \pi \cong \mathrm{id}_{BG}^*\pi$, so $f' \circ f$ is (unbased) homotopic to $\mathrm{id}_B\, G$, and similarly $f \circ f'$ is (unbased) homotopic to $\mathrm{id}_{BG'}$. Thus f and f' are mutually inverse homotopy equivalences. ☐

In light of this ***from now on we add the requirement that classifying spaces must be CW complexes***. It then makes sense to talk of BG as a unique space up to homotopy.

3.4. **Base points.** Earlier we were religiously keeping track of base points, but in our discussion of bundles and classifying spaces this seems to have gone out of the window. We'll now reinstate base points and explain what changes. Our standing convention—that ***all spaces, maps and homotopies are based*** unless stated otherwise—is back in force. We'll always take 1 as the base point in G.

Definition 3.16. A *based principal G-bundle* is a principal G-bundle (E, B, π) in which E, B π are all based. Maps of based bundles are required to respect base points, and we write $\mathrm{Bund}_{G,*}(X)$ for the based set of isomorphism classes of principal G-bundles over a space X. Classifying spaces and their universal bundles will all be assumed to be based.

Remark 3.17. To make an unbased principal bundle into a based one, you just have to choose a base point $*$ in B and then a base point in the fibre $\pi^{-1}(*)$. Since each fibre is a torsor for G (meaning it carries a transitive free G-action), a choice of base point in $\pi^{-1}(*)$ is equivalent to a choice of equivariant identification of this fibre with G. In a slight abuse of notation, we might call such an identification an *trivialisation over* $*$. Isomorphisms of based bundles over a fixed space X are then just isomorphisms of unbased bundles which respect this identification.

Now let X be a (based) CW complex. Recall that we assume that the base point $*$ is a 0-cell, which we'll denote by D_*. The group $\check{C}^0(X;G)$, under pointwise multiplication in G, carries a surjective homomorphism ev_* to G which sends a cochain ψ to the value of ψ_*. This is split by the obvious homomorphism i in the opposite direction given by

$$i : g \mapsto \psi \text{ where } \psi_\alpha = \begin{cases} g & \text{if } \alpha = * \\ 1 & \text{otherwise.} \end{cases}$$

There is a reduced version of the Čech cohomology $\check{H}^1(X;G)$, which we denote by $\check{H}^1(X,*;G)$, defined as in the unreduced case but with $\check{C}^0(X;G)$ replaced by $\check{C}^0(X,*;G) := \ker \mathrm{ev}_*$. Note that $\check{H}^1(X,*;G)$ is the quotient of $\check{H}^1(X;G)$ by the action of G via the splitting map i.

The translation of (13) to the based world is then as follows:

Proposition 3.18. *Assume a classifying space BG exists. For any CW complex X the maps defined in Lemma 3.8 and Lemma 3.13 can be refined to give a commutative diagram of isomorphisms of based sets*

$$\begin{tikzcd}
{[X,BG]} \arrow[rr, "\textit{pull back universal bundle}"] \arrow[dr] & & \mathrm{Bund}_{G,*}(X) \arrow[dl, "\Psi"] \\
& \check{H}^1(X,*;G) &
\end{tikzcd}$$

Proof. Take the constructions from Lemma 3.8 and Lemma 3.13 but each time you have to choose a section or lift over D_* you simply choose the base point. ☐

Example 3.19. When $X = S^1$ we can understand everything concretely:

- $[S^1, BG]$ is $\pi_1(BG)$, whilst $\mathsf{hTop}(S^1, BG)$ is the set $\mathrm{ccl}(\pi_1(BG))$ of conjugacy classes of elements in $\pi_1(BG)$.
- If you cut the circle open then a based G-bundle becomes canonically trivialised and the gluing map which tells you how to stick it back together is simply an element of G. Elements of G which can be connected by a path define isomorphic bundles. Starting instead with an unbased bundle, the cut-open bundle is trivial but not canonically trivialised; an element of G determines a gluing map, but conjugate elements correspond to the same gluing map after changing trivialisation, and hence give rise to isomorphic unbased bundles. In other words, $\mathrm{Bund}_{G,*}(S^1)$ is canonically identified with $\pi_0(G)$, and $\mathrm{Bund}_G(S^1)$ with $\mathrm{ccl}(\pi_0(G))$.
- Taking the obvious minimal CW structure on S^1, with a single 0-cell and a single 1-cell, we have
$$\big((\delta^1)^{-1}(1) \text{ in } \check{C}^1(X,*;G)\big) = \big((\delta^1)^{-1}(1) \text{ in } \check{C}^1(X;G)\big) = G \times G$$
and
$$\check{C}^0(X,*;G) = \{1\} \times \mathsf{Top}([0,1],G) \subset C^0(X;G) = G \times \mathsf{Top}([0,1],G),$$
and $(g,\gamma) \in G \times \mathsf{Top}([0,1],G)$ acts on $G \times G$ by
$$(g,\gamma)\cdot(g_1,g_2) = (g\cdot g_1 \cdot \gamma(0)^{-1}, g\cdot g_2\cdot\gamma(1)^{-1}).$$
Therefore $\check{H}^1(X,*;G) = \pi_0(G)$ and $\check{H}^1(X;G) = \mathrm{ccl}(\pi_0(G))$.

Remark 3.20. We would like to say that for any space X there is a fibration

$$\mathsf{Top}(X,BG) \to BG,$$

given by sending a map f to the image $f(*)$ of the base point, with fibre $\mathsf{Top}_*(X,BG)$. We would then like to to say that the long exact sequence of homotopy groups for this fibration yields an exact sequence

$$\pi_1(BG) \to [X,BG] \to \mathsf{hTop}(X,BG) \to \pi_0(BG),$$

so that if BG is simply-connected—i.e. G is path-connected—then it makes no difference whether our fibrations are base or not.

This argument can be made to work, at least when X is a CW complex, as follows. As discussed in May [7, Chapter 5] there is a category $\mathscr{U}$ of compactly generated spaces and unbased continuous maps between them (warning: what May calls 'compactly generated', many authors call 'CGWH'—'compactly generated weak Hausdorff'). This includes all CW complexes. Given objects X and Y in $\mathscr{U}$ there is a topology on $\mathscr{U}(X,Y)$, the *k-ification* of the compact-open topology, which makes it into a compactly generated space, denoted Y^X. For any X, Y and Z in $\mathscr{U}$ we then have a homeomorphism

$$Z^{k(X\times Y)} \cong (Z^X)^Y, \tag{14}$$

where $k(X \times Y)$ denotes the k-ification of the standard product topology. We will only be interested in locally compact Y, for which $k(X\times Y)$ is homeomorphic to $X \times Y$.

Using (14) with $Y = [0,1]$ you can show that for any X and Z in $\mathscr{U}$ we have

$$\pi_0(Z^X) = \mathsf{hTop}(X,Z) \quad \text{and} \quad \pi_0((Z^X)_*) = [X,Z],$$

where $(Z^X)_*$ denotes the subspace of Z^X comprising *based* maps. You can also show that if X is well-based (the inclusion of the base point is a cofibration) then the map

$$Z^X \xrightarrow{\mathrm{ev}_*} Z$$

which sends $f : X \to Z$ to $f(*)$ is a fibration, with fibre $(Z^Y)_*$. Considering the long exact sequence in homotopy groups, we conclude that for all CW complexes X and Z there is indeed an exact sequence

$$\pi_1(Z) \to [Y, Z] \to \mathsf{hTop}(Y, Z) \to \pi_0(Z).$$

Now take $Z = BG$ to get what we want.

3.5. B **as delooping.** Recall that a principal G-bundle $\pi : E \to B$ over a CW complex is a fibration so there is a 'monodromy' map $\Omega B \to G$, which fits into the Puppe sequence

$$\cdots \to \Omega E \to \Omega B \to G \to E \to B.$$

Lemma 3.21. *The monodromy map $m : \Omega BG \to G$ for the universal bundle defines a homotopy equivalence*

$$\Omega BG \to G$$

which intertwines loop concatenation on ΩBG with the group product on G up to homotopy. In other words, BG is a delooping of G.

Proof. The definition of m in terms of monodromy shows that it respects the products up to homotopy so it's left to show that it's a homotopy equivalence.

Applying the general machinery above to the universal bundle $EG \to BG$, the spaces EG and ΩEG are contractible so m_* induces a bijection

$$[X, \Omega BG] \xrightarrow{\cong} [X, G]$$

for all spaces X. Taking $X = G$, surjectivity of m_* implies that there exists a map $f : G \to \Omega BG$ such that $m \circ f \simeq \mathrm{id}_G$, and it's now enough to prove that $f \circ m \simeq \mathrm{id}_{\Omega BG}$.

To see that this is indeed the case, take $X = \Omega BG$. The map m_* sends both $f \circ m$ and $\mathrm{id}_{\Omega BG}$ to m, so its injectivity gives the result. ☐

One way to think about this is that $[X, \Omega BG] = [\Sigma X, BG]$ classifies bundles over ΣX, and by the clutching construction (cutting ΣX into two copies of the cone CX, glued together along X) such bundles are also classified by $[X, G]$.

Remark 3.22. Without invoking any fibration technology you can prove that $\pi_n(BG) \cong \pi_{n-1}(G)$ for all n by generalising Example 3.19 to higher-dimensional spheres with the minimal cell structure.

Remark 3.23. We saw in Section 2.7 that a space is deloopable if and only if it has an A_∞-structure, meaning a product which is associative up to coherent homotopies, and that further deloopability is determined by the commutativity of this product. Roughly speaking then, we expect every group to have a delooping BG, but only the abelian groups to have higher deloopings $B^2G, B^3G, \ldots$ (e.g. Eilenberg–MacLane spaces). This connection between commutativity and higher deloopings has a nice interpretation in (our CW version of) Čech cohomology, where $\check{H}^0(X, *; G)$ and $\check{H}^1(X, *; G)$ can be defined for any G, and are computed by $\mathsf{Top}_*(X, G)$ and $[X, BG]$ respectively. The obvious algebraic definition of $\check{H}^i(X, *; G)$ for $i \geq 2$ only makes sense when G is abelian and it is precisely in this case that we expect the representing objects B^iG to exist.

3.6. **Examples.** The simplest example of a classifying space is $B\mathbb{Z} \simeq S^1$, coming from the principal $\mathbb{Z}$-bundle $\mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1$. This shows that

$$\widetilde{H}^1(X; \mathbb{Z}) = [X, S^1]$$

for any space X (the Eilenberg–MacLane spectrum HA representing $\widetilde{H}^*(-; A)$ is defined by repeatedly taking *connected* deloopings of A). To build more complicated classifying spaces, one of the basic building blocks is:

Lemma 3.24. *The sphere $S^\infty = \operatorname{colim} S^n$ is contractible.*

Sketch proof. For each n the inclusion $S^n \hookrightarrow S^{n+1}$, and hence the induced inclusion $i_n : S^n \hookrightarrow S^\infty$, is nullhomotopic. The i_n glue together to give the identity map $S^\infty \to S^\infty$, and the nullhomotopies can be chosen so that they also glue. ☐

Example 3.25. The group $\mathrm{U}(1)$ acts freely on $S^{2n-1} \subset \mathbb{C}^{2n}$ for each n and this induces an action on S^∞. The projection $S^\infty \to S^\infty/\mathrm{U}(1) = \mathbb{CP}^\infty$ is a principal $\mathrm{U}(1)$-bundle, and contractibility of S^∞ implies that $B\mathrm{U}(1) \simeq \mathbb{CP}^\infty$. For any space X we thus have

$$\widetilde{H}^2(X;\mathbb{Z}) = [X, \mathbb{CP}^\infty].$$

If X is a CW complex then $[X, \mathbb{CP}^\infty] = \mathrm{Bund}_{\mathrm{U}(1),*}(X)$, so we have a natural bijection

$$\mathrm{Bund}_{\mathrm{U}(1),*}(X) \to \widetilde{H}^2(X;\mathbb{Z})$$

and the first Chern class c_1 is defined to be exactly this map (or maybe this is $-c_1$; I think it's a matter of convention). It is immediate that c_1 is natural (i.e. commutes with pullbacks) and is a complete invariant of based principal $\mathrm{U}(1)$ bundles. Note that we haven't defined c_1 for higher rank bundles or proved anything about its interaction with $\oplus$ or $\otimes$.

Remark 3.26. We have arrived at the following description of the low degree integral cohomology groups of a CW complex X:

	Set	Operation
$H^0(X;\mathbb{Z})$	unbased homotopy classes of map $X \to \mathbb{Z}$	addition on $\mathbb{Z}$
$H^1(X;\mathbb{Z})$	unbased homotopy classes of map $X \to S^1$	addition on S^1
$H^2(X;\mathbb{Z})$	isomorphism classes of complex line bundle	tensor product

The operation on complex line bundles only needs to be checked for $X = \mathbb{CP}^\infty$.

Example 3.27. Any finite cyclic group $\mathbb{Z}/N$ sits inside $\mathrm{U}(1)$ and we have $B(\mathbb{Z}/N) \simeq S^\infty/(\mathbb{Z}/N)$. For instace $B(\mathbb{Z}/2) = \mathbb{RP}^\infty$, so

$$\widetilde{H}^1(X;\mathbb{Z}/2) = [X, \mathbb{RP}^\infty] = \mathrm{Bund}_{\mathbb{Z}/2,*}(X)$$

for any CW complex X. The map from bundles to $\widetilde{H}^1$ is the first Stiefel–Whitney class, w_1.

We can put S^∞'s together to construct $E\mathrm{U}(n)$ for higher n:

Definition 3.28. For positive integers $n \leq N$ the *Stiefel manifold* $V_n(\mathbb{C}^N)$ is the space of orthonormal n-frames in $\mathbb{C}^N$ (with the standard Hermitian inner product)

$$V_n(\mathbb{C}^N) = \{(v_1, \dots, v_n) \in (\mathbb{C}^N)^n : \langle v_i, v_j \rangle = \delta_{i,j}\}.$$

The *infinite Stiefel manifold* V_n is defined to be $\mathrm{colim}_N\, V_n(\mathbb{C}^N)$. For example $V_1(\mathbb{C}^N) = S^{2N-1}$ and $V_1 = S^\infty$.

Note that $\mathrm{U}(n)$ acts freely on $V_n(\mathbb{C}^N)$, and the quotient is exactly the Grassmannian $\mathrm{Gr}_n(\mathbb{C}^N)$ of n-planes in $\mathbb{C}^N$. Write Gr_n for $V_n/\mathrm{U}(n) = \mathrm{colim}\, \mathrm{Gr}_n(\mathbb{C}^N)$.

Lemma 3.29. *For each n, Gr_n is a model for $B\mathrm{U}(n)$ with universal bundle*

$$V_n \to V_n/\mathrm{U}(n) = \mathrm{Gr}_n.$$

Sketch proof. By forgetting the nth vector of our n-frame, there is a map $V_n \to V_{n-1}$. This is a fibration with fibre S^∞. Mapping V_n into the corresponding exact Puppe sequence, we see inductively that V_n is contractible.

For our definition of classifying space we should also check that $V_n \to \mathrm{Gr}_n$ is principal bundle (the only non-obvious part is that it is locally trivial), but we won't do this here. □

3.7. Splittings.

If $\theta : G \to H$ is a homomorphism of topological groups then for any space X there is a map

$$\theta_* : \mathrm{Bund}_{G,*}(X) \to \mathrm{Bund}_{H,*}(X).$$

given by

$$\theta_*(E \to X) = (H \times_G E \to X),$$

where G acts on H via θ.

Definition 3.30. A lift of a bundle E in $\mathrm{Bund}_{H,*}(X)$ to $\mathrm{Bund}_{G,*}(X)$ is called a *reduction of the structure group* of E, from H to G. If H is a Lie group and θ is the inclusion of a maximal torus then this is also called a *splitting* of E.

Example 3.31. Principal $\mathrm{O}(n)$-bundles correspond to real vector bundles and a reduction of the structure group to $\mathrm{SO}(n) \hookrightarrow \mathrm{O}(n)$ is a choice of orientation. A further reduction of the structure group to $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ is a choice of spin structure.

Remark 3.32. Even though it's called a *reduction* of the structure group, G could actually be much larger than H. For instance G could be U, H could be $\mathrm{U}(1)$ and θ could be the determinant. Reductions may not exist, for example there are vector bundles which are not orientable and orientable bundles which are not spin (in the $\mathrm{U} \to \mathrm{U}(1)$ example they do always exist because det has a right-inverse, namely the inclusion $\mathrm{U}(1) \hookrightarrow \mathrm{U}$). If they do exist then they need not be unique, for example the orientations on an orientable vector bundle over X form a torsor for $H^0(X; \mathbb{Z}/2)$ and the spin structures on a spin bundle form a torsor for $H^1(X; \mathbb{Z}/2)$.

Suppose $(EG \to)BG$ and $(EH \to)BH$ are classifying spaces for G and H. There is a quotient map

$$EG \times EH \to EG \times_G EH,$$

where G acts on EG in the usual way and on EH via θ, and this is a principal G-bundle. To see this note that $EG \times EH$ is looks locally like $(G \times H) \to U$, for U an open set in $BG \times BH$, so it's enough to show that the projection $G \times H \to G \times_G H$ is a G-equivariantly trivial. And to prove this, observe that the quotient $G \times_G H$ is homeomorphic to H via the maps

$$G \times_G H \to H \quad \text{and} \quad H \to G \times H \to G \times_G H$$

given by $(g, h) \mapsto \theta(g^{-1})h$ and $h \mapsto (1, h)$, and then that the map

$$G \times H \to G \times (G \times_G H) = G \times H$$

given by $(g, h) \mapsto (g, \theta(g^{-1})h)$ is a trivialisation (i.e. a G-equivariant homeomorphism).

This means that $EG \times_G EH$ is a classifying space for G (assuming that it's a CW complex). This space carries an obvious projection to $EH/H = BH$, so we deduce that our homomorphism $\theta : G \to H$ induces a map $B\theta : BG \to BH$ which is a fibration with fibre $EG \times_G H$ (the homotopy quotient of H by G). For any CW complex X we then have a commuting diagram

$$\begin{tikzcd}
\mathrm{Bund}_{G,*}(X) \arrow[r, "\theta_*"] \arrow[d, equal] & \mathrm{Bund}_{H,*}(X) \arrow[d, equal] \\
{[X, BG]} \arrow[r, "B\theta"] & {[X, BH]}
\end{tikzcd}$$

and a reduction of the structure group of a bundle classified by $f : X \to BH$ is a lift $F : X \to BG$ such that

$$\begin{tikzcd}
 & BG \arrow[d, "B\theta"] \\
X \arrow[ur, dashed, "F"] \arrow[r, "f"] & BH
\end{tikzcd}$$

commutes up to homotopy. Existence of such lifts can be studied by obstruction theory.

Example 3.33. A fairly crude obstruction to existence of a lift comes from the fact that if $f = B\theta \circ F$ then the map $f^* : H^*(BH) \to H^*(X)$ must factor through $B\theta^*$. You can show that

$$H^*(B\mathrm{O}(n); \mathbb{Z}/2) = \mathbb{Z}/2[w_1, \ldots, w_n],$$

where w_i has degree i (it's the ith Stiefel–Whitney class), and

$$H^*(B\mathrm{SO}(n); \mathbb{Z}/2) = \mathbb{Z}/2[w_2, \ldots, w_n],$$

so we immediately see that any vector bundle which admits an orientation has vanishing w_1. Similarly, $B\mathrm{Spin}(n)$ is 2-connected (since $\mathrm{Spin}(n)$ is simply-connected) so has vanishing H^1 and H^2, and we see that any orientable bundle which admits a spin structure must have vanishing w_2. These conditions turn out to be sufficient as well as necessary (it's obvious for orientations, less so for spin structures).

4. Bott periodicity and the definition of K-theory

4.1. U **and** BU**.** For each n there are natural *stabilisation* maps

$$\begin{aligned} \mathrm{U}(n) &\to \mathrm{U}(n+1) & A &\mapsto \begin{pmatrix} 1 & \\ & A \end{pmatrix} \\ V_n &\to V_{n+1} & (v_1, \dots, v_n) \in (\mathbb{C}^\infty)^n &\mapsto \big((1,0),(0,v_1),\dots,(0,v_n)\big) \in (\mathbb{C} \oplus \mathbb{C}^\infty)^{n+1} \\ \mathrm{Gr}_n &\to \mathrm{Gr}_{n+1} & n\text{-plane } \Pi \subset \mathbb{C}^\infty &\mapsto \mathbb{C} \oplus \Pi \subset \mathbb{C} \oplus \mathbb{C}^\infty, \end{aligned}$$

making use of the fact that $\mathbb{C} \oplus \mathbb{C}^\infty \cong \mathbb{C}^\infty$. These intertwine with the action of $\mathrm{U}(n)$ on V_n.

Definition 4.1. The spaces U, V_∞ and Gr_∞ are the colimits of $\mathrm{U}(n)$, V_n and Gr_n respectively under stabilisation. The $\mathrm{U}(n)$, V_n and Gr_n carry CW structures with countably many cells (finitely many for $\mathrm{U}(n)$) such that the stabilisation maps are all inclusions of subcomplexes, so U, V_∞ and Gr_∞ all inherit cell structures and the CW topology.

Remark 4.2. The group U is *not* the unitary group of $\mathbb{C}^\infty$; each element of U must act as the identity on a subspace of $\mathbb{C}^\infty$ of finite codimension.

The group and bundle structures glue to give:

Lemma 4.3. *The obvious group operation on* U*, and the obvious action of* U *on* V_∞*, are both continuous (note that the inversion map on* U *is trivially continuous). The projection* $V_\infty \to \mathrm{Gr}_\infty$ *is a principal* U*-bundle.*

Proof. The multiplication maps

$$\mathrm{U}(n) \times \mathrm{U}(n) \to \mathrm{U}(n) \hookrightarrow \mathrm{U}$$

induce a map $\mathrm{colim}(\mathrm{U}(n) \times \mathrm{U}(n)) \to \mathrm{U}$, and we'd like to say that

$$\mathrm{colim}(\mathrm{U}(n) \times \mathrm{U}(n)) \cong (\mathrm{colim}\,\mathrm{U}(n)) \times (\mathrm{colim}\,\mathrm{U}(n)) = \mathrm{U} \times \mathrm{U} \tag{15}$$

so that the group operation on U is continuous. The homeomorphism (15) *does* hold but is not totally trivial. One way to prove it is to note that the colimit of a sequence of CW complexes, included in each other as subcomplexes, carries the CW topology, whilst the product of two CW complexes with countably many cells also carries the CW topology (proved in Hatcher [5, Theorem A.6]). A similar argument shows that the actions of the $\mathrm{U}(n)$ on the V_n induce a continuous action of U on V_∞. It's left to deal with the U-bundle structure.

For each cell $\Phi_\alpha : D_\alpha \to \mathrm{Gr}_n$, we can pick a section $s_n : D_\alpha \to \Phi_\alpha^* V_n$ of $\pi_n : V_n \to \mathrm{Gr}_n$. This then defines sections $s_m : D_\alpha \to \Phi_\alpha^* V_m$ for all $m \geq n$, and hence a compatible sequence of trivialisations

$$\Phi_\alpha^* V_m \cong \mathrm{U}(m) \times D_\alpha.$$

Taking colimits we obtain a trivialisation (meaning a U-equivariant homeomorphism)

$$\mathrm{colim}\,\Phi_\alpha^* V_m \cong \mathrm{U} \times D_\alpha,$$

so if we can show that the topologies on $\mathrm{colim}\,\Phi_\alpha^* V_m$ and $\Phi_\alpha^*\,\mathrm{colim}\,V_m$ coincide then we in fact have a trivialisation of $\Phi_\alpha^* V_\infty$. From this we deduce that $V_\infty \to \mathrm{Gr}_\infty$ is a principal U-bundle, using Remark 3.11 and the construction of trivialisations over open sets from trivialisations over cells described in the proof of Lemma 3.8.

To prove the necessary equality, note that each $\Phi_\alpha^* V_m$ is a closed subset of $D_\alpha \times V_m$ and carries the subspace topology. This means that a subset of $\Phi_\alpha^* V_m$ is closed if and only if it is closed in $D_\alpha \times V_m$, so a subset of $\mathrm{colim}\,\Phi_\alpha^* V_m$ is closed if and only if it is closed in $\mathrm{colim}(D_\alpha \times V_m)$. But the latter is equal to $D_\alpha \times V_\infty$, and we conclude that $\mathrm{colim}\,\Phi_\alpha^* V_m$ and $\Phi_\alpha^* V_\infty$ both carry the subspace topology from $D_\alpha \times V_\infty$. ☐

In fact, this is a universal U-bundle:

Lemma 4.4. *The space* V_∞ *is contractible so* Gr_∞ *is a model for* BU*.*

Sketch proof. Consider the *mapping telescope* T of the inclusions $V_1 \to V_2 \to \dots$. The space V_∞ has the structure of a CW complex such that the V_i are all subcomplexes. In this setting Hatcher [5, pages 138–139] describes a homotopy equivalence $T \simeq V_\infty$, so it's enough to show that T is contractible. This follows from contractibility of each of the telescopic sections $V_n \times [0,1]$. ☐

Remark 4.5. By using the *reduced* mapping telescope instead you can ensure that the contraction of V_∞ fixes the base point.

Remark 4.6. Recall that the classifying space $B\mathrm{U}(n)$ is uniquely defined up to homotopy, and we can describe $B\mathrm{U}$ in a homotopy-invariant way as the homotopy colimit (i.e. reduced mapping telescope) of the $B\mathrm{U}(n)$.

4.2. **The definition, at last.** Recall our strategy for producing generalised cohomology theories: find a space Z_i with a product which is almost commutative (E_∞), choose successive deloopings $Z_{i+1}, Z_{i+2}, \ldots$ to obtain an Ω-spectrum (defining Z_j to be $\Omega^{j-i} Z_i$ for $j < i$), and then take homotopy classes of maps into it. The key to K-theory is that the group multiplication makes U into just such an E_∞-space, which we place in degree 1 (i.e. we set $Z_1 = \mathrm{U}$). To see that this is not unreasonable, observe that for each n the multiplication map

$$\mathrm{U}(n) \times \mathrm{U}(n) \to \mathrm{U}(n) \hookrightarrow \mathrm{U}(2n)$$

is homotopy-commutative, by 'homotoping the second $\mathrm{U}(n)$ into the last n components of $\mathrm{U}(2n)$'. By including into $\mathrm{U}(kn)$ for sufficiently large k, one can build compatible homotopies between these homotopies to arbitrarily high degree.

As an aside, the $2n$ here is actually sharp:

Theorem 4.7 (James–Thomas [6]). *For $n \geq 2$, $\mathrm{U}(n)$ is not homotopy-commutative in $\mathrm{U}(2n-1)$.*

Now we just need to choose deloopings. In fact there is a particularly natural choice due to the following remarkable result of Bott:

Theorem 4.8 (Bott periodicity [2] [3]).

$$\Omega \mathrm{U} \simeq \mathbb{Z} \times B\mathrm{U}.$$

Note that this immediately shows that U is an infinite loop spaces without recourse to any E_∞-technology.

Definition 4.9. The *K-theory spectrum* $K\mathrm{U}$ is given by

$$K\mathrm{U}_i = \begin{cases} \mathbb{Z} \times B\mathrm{U} & \text{if } i \text{ is even} \\ \mathrm{U} & \text{if } i \text{ is odd.} \end{cases}$$

The homotopy equivalence $K\mathrm{U}_i \simeq \Omega KU_{i+1}$ is given by Bott when i is even and by the standard identification of B with delooping when i is odd. The *reduced* and *unreduced (complex topological) K-theory functors*

$$\widetilde{K}^* : \mathsf{hTop} \to \mathsf{GrAbGp} \quad \text{and} \quad K^* : \mathsf{Top} \to \mathsf{GrAbGp}$$

are then given by

$$\widetilde{K}^i(X) = [X, K\mathrm{U}_i] \quad \text{and} \quad K^i(X) = \widetilde{K}^i(X_+),$$

where GrAbGp is the category of graded abelian groups.

By the general arguments from Section 2, $\widetilde{K}^*$ and K^* are reduced and unreduced generalised cohomology theories. Bott periodicity immediately gives another property:

Lemma 4.10. *K-theory is 2-periodic in its grading.*

Let's finally compute some K-groups.

Lemma 4.11. *For all non-negative integers n we have*

$$\widetilde{K}^i(S^n) \cong \begin{cases} \mathbb{Z} & \text{if } i \equiv n \mod 2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For any such n we have

$$\widetilde{K}^i(S^n) = \widetilde{K}^i(\Sigma^n S^0) = \widetilde{K}^{i-n}(S^0) = \pi_0(K\mathrm{U}_{i-n}).$$

If $i - n$ is even then this is group is $\pi_0(\mathbb{Z} \times B\mathrm{U}) = \mathbb{Z}$, whilst if $i - n$ is odd then the group is $\pi_0(\mathrm{U}) = 0$. □

Corollary 4.12. *We have*

$$K^i(*) \cong \begin{cases} \mathbb{Z} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Thus the K-groups of a point are not concentrated in degree 0, and we say that K-theory is an *extraordinary* cohomology theory: you can't get it just by taking ordinary cohomology with a funny coefficient ring.

4.3. **Sketch proof of Bott periodicity.** Bott proved the periodicity theorem using Morse theory, and this is explained in Milnor's book [10]—in fact the MathSciNet review of this book gives a nice summary of the argument. I'll outline an alternative approach proposed by McDuff [8]. A more detailed, streamlined account is given by Behrens [1].

First we'll modify our picture of $B\mathrm{U}$. We originally defined it as the colimit of the Grassmannians Gr_n under stabilisation, but this involves awkwardly keeping track of a string of embeddings

$$\mathbb{C}^\infty \xrightarrow{0\oplus\mathrm{id}} \mathbb{C} \oplus \mathbb{C}^\infty \cong \mathbb{C}^\infty \xrightarrow{0\oplus\mathrm{id}} \mathbb{C} \oplus \mathbb{C}^\infty \cong \mathbb{C}^\infty \to \dots.$$

It is more convenient to introduce another copy of $\mathbb{C}^\infty$, 'where the stabilisation directions live'. We'll now make this precise.

Let $\mathbb{C}^\infty_+$ and $\mathbb{C}^\infty_-$ be copies of the standard vector space $\mathbb{C}^\infty$, with bases $e_{\pm 1}, e_{\pm 2}, \dots$. We defined V_n as

$$\{\text{othonormal } n\text{-frames in } \mathbb{C}^\infty\},$$

which we can obviously identify with

$$\{\text{othonormal } n\text{-frames in } \langle e_{-n}, \dots, e_{-1}\rangle \oplus \mathbb{C}^\infty_+\},$$

and then view this as

$$\{\text{inner-product-preserving embeddings } f : \mathbb{C}^\infty_- \hookrightarrow \mathbb{C}^\infty_- \oplus \mathbb{C}^\infty_+ \\ \text{which are the inclusion on } \dots, e_{-n-2}, e_{-n-1}\}.$$

Similarly, instead of thinking of Gr_n as

$$\{\text{rank } n \text{ subspaces of } \mathbb{C}^\infty\},$$

we think of it as

$$\{\text{rank } n \text{ subspaces of } \langle e_{-n}, \dots, e_{-1}\rangle \oplus \mathbb{C}^\infty_+\},$$

and hence as

$$\{\text{subspaces of } \mathbb{C}^\infty_- \oplus \mathbb{C}^\infty_+ \text{ which contain } \langle \dots, e_{-n-2}, e_{-n-1}\rangle \\ \text{and whose projection orthogonal to this has rank } n\}.$$

In this picture, the space $B\mathrm{U}$—which is the union of the Gr_n—is given by

$$\{\text{subspaces } \Pi \subset \mathbb{C}^\infty_- \oplus \mathbb{C}^\infty_+ : \mathrm{codim}_{\mathbb{C}^\infty_-}(\Pi \cap \mathbb{C}^\infty_-) = \mathrm{codim}_\Pi(\Pi \cap \mathbb{C}^\infty_-) < \infty\}$$

(I came across this description on MathOverflow). Similarly, we can describe $\mathbb{Z} \times B\mathrm{U}$ as

$$\{\text{subspaces } \Pi \subset \mathbb{C}^\infty_- \oplus \mathbb{C}^\infty_+ : \mathrm{codim}_{\mathbb{C}^\infty_-}(\Pi \cap \mathbb{C}^\infty_-) \text{ and } \mathrm{codim}_\Pi(\Pi \cap \mathbb{C}^\infty_-) < \infty\}. \tag{16}$$

The group U, meanwhile, is the subgroup of the unitary group $U(\mathbb{C}^\infty_- \oplus \mathbb{C}^\infty_+)$ comprising those operators which act trivially on a subspace of finite codimension.

McDuff's idea is as follows. Let H denote the set of Hermitian operators A on $\mathbb{C}^\infty_- \oplus \mathbb{C}^\infty_+$ with eigenvalues in $[0, 1]$ such that:

- $\ker A$ contains a subspace of $\mathbb{C}^\infty_-$ of finite codimension.
- $\ker(A - I)$ contains a subspace of $\mathbb{C}^\infty_+$ of finite codimension.

This can be linearly contracted to the operator $0_{\mathbb{C}^\infty_-} \oplus I_{\mathbb{C}^\infty_+}$.

Now consider the map $e : \mathrm{H} \to \mathrm{U}$ given by $A \mapsto \exp(2\pi i A)$. Operators in the fibre over a point u are uniquely determined by their 0-eigenspace as a subspace of $\ker u$, so the fibre can be identified with $\mathbb{Z} \times B\mathrm{U}$ in the form (16) via

$$e^{-1}(u) = \{\Pi := \ker A \subset \ker u : \mathrm{codim}_{\mathbb{C}^\infty_-}(\Pi \cap \mathbb{C}^\infty_-) \text{ and } \mathrm{codim}_\Pi(\Pi \cap \mathbb{C}^\infty_-) < \infty\}.$$

The map e is not a fibration, but it's close enough (it's a *quasifibration*) that we still have a long exact sequence, and we conclude that $\mathbb{Z} \times B\mathrm{U} \simeq \Omega\mathrm{U}$.

4.4. **Um... where are the vector bundles?** Right at the start we said that K-theory was about vector bundles, but the definition makes no reference to them at all, so we'll now connect the two pictures. Throughout this subsection X will denote a connected CW complex unless stated otherwise.

First note that any rank n vector bundle E over X can be given an inner product (using a partition of unity), and then the bundle of orthonormal frames becomes a principal $\mathrm{U}(n)$-bundle P. The isomorphism class of P depends only on the isomorphism class of E. Conversely, a principal $\mathrm{U}(n)$-bundle P can be made into a rank n vector bundle $E = P \times_{\mathrm{U}(n)} \mathbb{C}^n$, and again this induces a well-defined map on isomorphism classes. These two constructions are mutually inverse and we conclude that there is a natural bijection

$$\mathrm{Vect}_n(X) \to [X, B\mathrm{U}(n)] \tag{17}$$

where $\mathrm{Vect}_n(X)$ denotes the set of isomorphism classes of rank n vector bundles over X. Recall that since $\mathrm{U}(n)$ is path-connected there is no distinction between based and unbased bundles. Putting all of the maps (17) together we obtain a bijection

$$\mathrm{Vect}(X) \to [X_+, \coprod_n B\mathrm{U}(n)]. \tag{18}$$

We added an extra base point to X so that its original base point is not constrained to lie in any specific $B\mathrm{U}(n)$.

The map $\coprod B\mathrm{U}(n) \to \mathbb{Z}$ which sends each $B\mathrm{U}(n)$ to n combines with (18) to give a map

$$\mathrm{Vect}(X) \to \mathbb{Z} \times \mathrm{colim}[X_+, B\mathrm{U}(n)].$$

To interpret this in terms of vector bundles, note that the stabilisation map $B\mathrm{U}(n) \to B\mathrm{U}(n+1)$ corresponds to the map $\mathrm{Vect}_n(X) \to \mathrm{Vect}_{n+1}(X)$ given by direct sum with the trivial line bundle $\underline{\mathbb{C}}$.

Definition 4.13. Vector bundles E_1 and E_2 over a space X are *stably isomorphic* if there exist non-negative integers n_1 and n_2 such that

$$E_1 \oplus \underline{\mathbb{C}}^{n_1} \cong E_2 \oplus \underline{\mathbb{C}}^{n_2}.$$

We'll write $\mathrm{Vect}_{\mathrm{stab}}(X)$ for the set of stable isomorphism classes of vector bundles, and $E \oplus \underline{\mathbb{C}}^\infty$ for the class of a vector bundle E.

Example 4.14. For each n the real vector bundle TS^n is stably trivial, i.e. stably isomorphic to zero. To see this note that under the standard embedding $i : S^n \hookrightarrow \mathbb{R}^{n+1}$ the normal bundle N is trivial so we have

$$TS^n \cong_{\mathrm{stably}} TS^n \oplus N \cong i^* T\mathbb{R}^{n+1} \cong_{\mathrm{stably}} 0.$$

With this terminology in place, we can write $\mathbb{Z} \times \mathrm{colim}[X_+, B\mathrm{U}(n)]$ as $\mathbb{Z} \times \mathrm{Vect}_{\mathrm{stab}}(X)$, and the map $\mathrm{Vect}(X) \to \mathbb{Z} \times \mathrm{colim}[X_+, B\mathrm{U}(n)]$ as the rank and stabilisation map $E \mapsto (\mathrm{rk}\, E, E \oplus \underline{\mathbb{C}}^\infty)$. The obvious map $\mathbb{Z} \times \mathrm{colim}[X_+, B\mathrm{U}(n)] \to [X_+, \mathbb{Z} \times B\mathrm{U}]$ then induces a map $\mathbb{Z} \times \mathrm{Vect}_{\mathrm{stab}}(X) \to K^0(X)$, and we obtain a commuting diagram

$$\begin{tikzcd}
\mathrm{Vect}(X) \arrow[r, "{(\mathrm{rk}, -\oplus\underline{\mathbb{C}}^\infty)}"] \arrow[d, equal] & \mathbb{Z} \times \mathrm{Vect}_{\mathrm{stab}}(X) \arrow[r] \arrow[d, equal] & K^0(X) \arrow[d, equal] \\
{[X_+, \coprod_n B\mathrm{U}(n)]} \arrow[r] & \mathbb{Z} \times \mathrm{colim}[X_+, B\mathrm{U}(n)] \arrow[r] & \mathbb{Z} \times [X_+, B\mathrm{U}]
\end{tikzcd}$$

Note that $\mathbb{Z} \times \mathrm{Vect}_{\mathrm{stab}}(X)$ is a monoid under addition on $\mathbb{Z}$ and direct sum of vector bundles, and the top left-hand horizontal map is a monoid homomorphism. By Remark 4.21 the top right-hand map is also a monoid homomorphism, so we can factor through Grothendieck groups:

$$\begin{tikzcd}
K(\mathrm{Vect}(X)) \arrow[r, dashed] & \mathbb{Z} \times K(\mathrm{Vect}_{\mathrm{stab}}(X)) \arrow[dr, dashed] & \\
\mathrm{Vect}(X) \arrow[u] \arrow[r, "{(\mathrm{rk}, -\oplus\underline{\mathbb{C}}^\infty)}"] \arrow[d, equal] & \mathbb{Z} \times \mathrm{Vect}_{\mathrm{stab}}(X) \arrow[u] \arrow[r] \arrow[d, equal] & K^0(X) \arrow[d, equal] \\
{[X_+, \coprod_n B\mathrm{U}(n)]} \arrow[r] & \mathbb{Z} \times \mathrm{colim}[X_+, B\mathrm{U}(n)] \arrow[r] & \mathbb{Z} \times [X_+, B\mathrm{U}]
\end{tikzcd} \tag{19}$$

Lemma 4.15. *For any connected CW complex X the map*

$$K(\mathrm{rk}, - \oplus \underline{\mathbb{C}}^\infty) : K(\mathrm{Vect}(X)) \to \mathbb{Z} \times K(\mathrm{Vect}_{\mathrm{stab}}(X))$$

is an isomorphism.

Proof. Recall that the Grothendieck group has an explicit description in terms of formal differences. Suppose $E - F$ is an element of $K(\mathrm{Vect}(X))$ which is sent to 0. Then $\mathrm{rk}\, E = \mathrm{rk}\, F$ and $(E \oplus \underline{\mathbb{C}}^\infty) - (F \oplus \underline{\mathbb{C}}^\infty) = 0$, so there exists a vector bundle G and positive integers a and b such that

$$E \oplus G \oplus \underline{\mathbb{C}}^a \cong F \oplus G \oplus \underline{\mathbb{C}}^b$$

as vector bundles on X. The condition $\mathrm{rk}\, E = \mathrm{rk}\, F$ ensures that $a = b$ and hence that

$$E \oplus (G \oplus \underline{\mathbb{C}}^a) \cong F \oplus (G \oplus \underline{\mathbb{C}}^a)$$

as vector bundles on X, so we conclude that $E - F = 0$ in $K(\mathrm{Vect}(X))$. This shows that the map is injective.

Now suppose that $(n, (E \oplus \underline{\mathbb{C}}^\infty) - (F \oplus \underline{\mathbb{C}}^\infty))$ is an element of $\mathbb{Z} \times K(\mathrm{Vect}_{\mathrm{stab}}(X))$. We need to find an element of $K(\mathrm{Vect}(X))$ which maps to this, and

$$(E \oplus \underline{\mathbb{C}}^{\mathrm{rk}\, F + \max(0,n)}) - (F \oplus \underline{\mathbb{C}}^{\mathrm{rk}\, E + \max(0,-n)})$$

will do. □

If X is compact then any map from X into $B\mathrm{U}$, and any homotopy of such maps, is contained in some $B\mathrm{U}(n)$. This is analogous to the fact that a compact subset of a CW complex is contained in a finite subcomplex. In this case we see that the bottom right-hand horizontal map in (19) is a bijection and hence that all of the maps on the right-hand half of the diagram are bijections/isomorphisms.

Remark 4.16. One consequence is that $\mathrm{Vect}_{\mathrm{stab}}(X)$ already has additive inverses when X is compact. This fact can be shown directly by taking a vector bundle E, covering X with a finite number of trivialisations, and gluing them together using a partition of unity to obtain an embedding of E into some trivial bundle $\underline{\mathbb{C}}^n$. The complement $E^\perp \subset \underline{\mathbb{C}}^n$ then acts as $-E$.

The conclusion is therefore:

Proposition 4.17. *For any connected CW complex X there is a homomorphism*

$$K(\mathrm{Vect}(X)) \to K^0(X),$$

which is an isomorphism if X is compact.

To extend to possibly disconnected complexes, note that K^0 sends disjoint unions to products. This is not true of Vect, since vector bundles must have constant rank and so the separate components are not indepedent, but it *is* true of Vect', defined to be the set of isomorphism classes of vector bundles of possibly non-constant rank (the rank must still be locally constant though). We get:

Corollary 4.18. *For any CW complex X there is a homomorphism*

$$K(\mathrm{Vect}'(X)) \to K^0(X),$$

which is an isomorphism if every component of X is compact. The preimage of $\widetilde{K}^0(X) \subset K^0(X)$ comprises those formal differences $E - F$ such that E and F have equal rank over the base point.

4.5. **Addition.**

From our general discussion of spectra we know that each set

$$\widetilde{K}^i(X) = [X, K\mathrm{U}_i]$$

carries the structure of an abelian group, with the group operation arising from loop concatenation on $K\mathrm{U}_i \simeq \Omega K\mathrm{U}_{i+1}$. Our goal now is to understand these operations more explicitly.

We have already seen that for any group G with a classifying space BG the monodromy homotopy equivalence $\Omega BG \to G$ intertwines loop concatentation and the group operation up to homotopy. Applying this to $G = \mathrm{U}$ we see that for odd i the operation on $\widetilde{K}^i(X) = [X, \mathrm{U}]$ is induced by the group operation on U. The case of even i is harder since it involves the Bott periodicity homotopy equivalence

$$\alpha : \Omega\mathrm{U} \to \mathbb{Z} \times B\mathrm{U},$$

and in order to deal with this we introduce some new operations.

Fix isometric identifications $\sigma : \mathbb{C}^\infty_+ \to \mathbb{C}^\infty_-$ and $s_+ : \mathbb{C}^\infty_+ \oplus \mathbb{C}^\infty_+ \to \mathbb{C}^\infty_+$, and then define s_- by $\sigma \circ s_+ \circ (\sigma^{-1}, \sigma^{-1})$. Recall that the group U is the subgroup of the unitary group $U(\mathbb{C}^\infty_- \oplus \mathbb{C}^\infty_+)$ comprising those transformations which act as the identity on a subspace of finite codimension. Given elements A_1 and A_2 in $U(\mathbb{C}^\infty_- \oplus \mathbb{C}^\infty_+)$ we can combine them to give an element $A_1 \oplus A_2$ of

$$U\big((\mathbb{C}^\infty_- \oplus \mathbb{C}^\infty_+) \oplus (\mathbb{C}^\infty_- \oplus \mathbb{C}^\infty_+)\big) = U\big((\mathbb{C}^\infty_- \oplus \mathbb{C}^\infty_-) \oplus (\mathbb{C}^\infty_+ \oplus \mathbb{C}^\infty_+)\big)$$

and using $s_- \oplus s_+$ we can view $A_1 \oplus A_2$ as an element $A_1 \oplus_{\sigma,s} A_2$ of $U(\mathbb{C}^\infty_- \oplus \mathbb{C}^\infty_+)$. Moreover, if A_1 and A_2 came from U then $A_1 \oplus_s A_2$ also lies in U.

Lemma 4.19. *The map* $\oplus_{\sigma,s} : \mathrm{U} \times \mathrm{U} \to \mathrm{U}$ *is homotopic to the group operation* $\cdot : \mathrm{U} \times \mathrm{U} \to \mathrm{U}$.

Proof. For elements $A_1, \dots, A_4$ in U we have

$$(A_1 \oplus_{\sigma,s} A_2) \cdot (A_3 \oplus_{\sigma,s} A_4) = (A_1 \cdot A_3) \oplus_{\sigma,s} (A_2 \cdot A_4).$$

We can thus apply the Eckmann–Hilton trick to the set $[\mathrm{U}, \mathrm{U}]$ with the operations

$$[\mathrm{U}, \mathrm{U}] \times [\mathrm{U}, \mathrm{U}] = [\mathrm{U}, \mathrm{U} \times \mathrm{U}] \to [\mathrm{U}, \mathrm{U}]$$

given by composition with $\oplus_{\sigma,s}$ and with $\cdot$, to see that for all homotopy classes f_1 and f_2 in $[\mathrm{U}, \mathrm{U}]$ we have

$$f_1 \oplus_{\sigma,s} f_2 \simeq f_1 \cdot f_2 : \mathrm{U} \times \mathrm{U} \to \mathrm{U}.$$

Taking $f_1 = f_2 = \mathrm{id}_\mathrm{U}$ gives the result. □

Remark 4.20. This argument also shows that both operations are homotopy-commutative (which we already knew since the group operation is homotopic to loop concatenation on the double loop space $\mathrm{U} \simeq \Omega^2\mathrm{U}$), and that up to homotopy $\oplus_{\sigma,s}$ is independent of the choices of σ and s. We therefore drop the subscripts from the $\oplus$.

By expressing $\mathbb{Z} \times B\mathrm{U}$ in the form (16) we can define a similar operation

$$\oplus : (\mathbb{Z} \times B\mathrm{U}) \times (\mathbb{Z} \times B\mathrm{U}) \to \mathbb{Z} \times B\mathrm{U},$$

by sending a pair of subspaces Π_1 and Π_2 of $\mathbb{C}^\infty_- \oplus \mathbb{C}^\infty_+$ to

$$\Pi_1 \oplus \Pi_2 \subset (\mathbb{C}^\infty_- \oplus \mathbb{C}^\infty_+) \oplus (\mathbb{C}^\infty_- \oplus \mathbb{C}^\infty_+) = (\mathbb{C}^\infty_- \oplus \mathbb{C}^\infty_-) \oplus (\mathbb{C}^\infty_+ \oplus \mathbb{C}^\infty_+) \xrightarrow{s_- \oplus s_+} \mathbb{C}^\infty_- \oplus \mathbb{C}^\infty_+.$$

There's also a map $\oplus : \mathrm{H} \times \mathrm{H} \to \mathrm{H}$, where H is McDuff's space of Hermitian operators. We obtain a map of quasifibrations

$$\begin{tikzcd}
(\mathbb{Z} \times \Omega B\mathrm{U}) \times (\mathbb{Z} \times \Omega B\mathrm{U}) \arrow[r] \arrow[d, "\oplus"] & \mathrm{H} \times \mathrm{H} \arrow[r] \arrow[d, "\oplus"] & \mathrm{U} \times \mathrm{U} \arrow[d, "\oplus"] \\
\mathbb{Z} \times B\mathrm{U} \arrow[r] & H \arrow[r] & \mathrm{U}
\end{tikzcd}$$

and (I think!) this proves that the induced Bott periodicity map $\alpha : \Omega\mathrm{U} \to \mathbb{Z} \times B\mathrm{U}$ intertwines $\oplus$ on $\mathbb{Z} \times B\mathrm{U}$ with $\Omega\oplus$ on $\Omega\mathrm{U}$. By Eckmann–Hilton again, $\Omega\oplus$ is homotopic to loop concatenation on $\Omega\mathrm{U}$, and we conclude that the group operation on $\widetilde{K}^{\mathrm{even}}$ is induced by $\oplus$ on $\mathbb{Z} \times B\mathrm{U}$.

Remark 4.21. If we choose σ to send e_i to e_{-i} and s to send $(e_i, 0)$ to e_{2i-1} and $(0, e_i)$ to e_{2i}, then $\oplus$ is the colimit of the direct sum maps $B\mathrm{U}(n) \times B\mathrm{U}(n) \to B\mathrm{U}(2n)$. This means that the map $K(\mathrm{Vect}'(X)) \to K^0(X)$ intertwines direct sum of bundles with addition on K^0.

4.6. **Subtraction.** What about additive inverses? Again the story is easy for odd K-groups—we simply us inversion on U—but the even case is more interesting. Let τ be the endomorphism of $\mathbb{C}^\infty_- \oplus \mathbb{C}^\infty_+$ given in block matrix form by

$$\begin{pmatrix} 0 & \sigma \\ \sigma^{-1} & 0 \end{pmatrix}$$

and define a map $i : \mathbb{Z} \times B\mathrm{U} \to \mathbb{Z} \times B\mathrm{U}$ by

$$\Pi \subset (\mathbb{C}^\infty_- \oplus \mathbb{C}^\infty_+) \mapsto \tau(\Pi^\perp).$$

This is clearly an involution.

Lemma 4.22. *The map i defines additive inverses on* $\mathbb{Z} \times B\mathrm{U}$.

Proof. We need to show that the composition

$$\mathbb{Z} \times B\mathrm{U} \xrightarrow{(\mathrm{id},i)} (\mathbb{Z} \times B\mathrm{U}) \times (\mathbb{Z} \times B\mathrm{U}) \xrightarrow{\oplus} \mathbb{Z} \times B\mathrm{U}$$

is nullhomotopic, or in other words that the map

$$\Pi \mapsto \Pi \oplus \tau(\Pi^{\perp}) \subset (\mathbb{C}_-^{\infty} \oplus \mathbb{C}_+^{\infty}) \oplus (\mathbb{C}_-^{\infty} \oplus \mathbb{C}_+^{\infty})$$

is homotopic to the map

$$\Pi \mapsto \mathbb{C}_-^{\infty} \oplus \mathbb{C}_-^{\infty} \subset (\mathbb{C}_-^{\infty} \oplus \mathbb{C}_+^{\infty}) \oplus (\mathbb{C}_-^{\infty} \oplus \mathbb{C}_+^{\infty}).$$

An explicit homotopy can be constructed using the path of endomorphisms of $(\mathbb{C}_-^{\infty} \oplus \mathbb{C}_+^{\infty}) \oplus (\mathbb{C}_-^{\infty} \oplus \mathbb{C}_+^{\infty})$ given in block matrix form by

$$\begin{pmatrix} 1 & 0 & 0 & t\sigma \\ 0 & 1-t & 0 & 0 \\ 0 & t\sigma & 1 & 0 \\ 0 & 0 & 0 & 1-t \end{pmatrix} \qquad \square$$

4.7. **Where next?** *K*-groups carry much more algebraic structure, for instance a product operation and a ring homomorphism (the Chern character) to rational cohomology. In order to get our hands on these we need some extra algebraic tools in our armoury, so in the next two sections we return to general cohomology theories and construct some useful exact sequences and the Atiyah–Hirzebruch spectral sequence. We will use these as both computational and theoretical tools.

If you want to skip these sections, the key result we will use later is the following:

Proposition 4.23. *Suppose X is a CW complex and $X_0 \subset X_1 \subset \dots$ is an exhausting sequence of finite-dimensional subcomplexes such that $H^*(X_i;\mathbb{Z})$ is concentrated in even degree for all i. Suppose also that h^* is generalised cohomology theory such that $h^*(*)$ is concentrated in even degree (for instance, ordinary cohomology, or K-theory). Then in even degrees the restriction map*

$$h^*(X) \to \lim_i h^*(X_i)$$

is an isomorphism.

We will also need the Milnor sequence (23) but you can just take this as a black box if you like.

5. More exact sequences

5.1. **Mapping webs.** Spaces are often built up from simpler subspaces, e.g. a CW complex from its skeleta, and it would be nice to have a way to recover the generalised cohomology of the whole space from that of the pieces. To formalise this, suppose we have a diagram in Top_* with objects W_v indexed by vertices $v \in V$ and morphisms f_a indexed by arrows $a \in A$. Write s and t for source and target, so that for each a we have

$$f_a : W_{s(a)} \to W_{t(a)}.$$

Definition 5.1 (Non-standard). The *mapping web W* of the diagram is the space obtained from

$$\Big(\coprod_v W_v\Big) \amalg \Big(\coprod_a (W_{s(a)} \wedge [0,1]_+)\Big)$$

by gluing each $W_{s(a)} \wedge \{0\}$ to $W_{s(a)}$ by the identity and $W_{s(a)} \wedge \{1\}$ to $W_{t(a)}$ by f_a. This is topologised by taking the colimit over all finite subdiagrams.

Example 5.2. Many mapping webs are familar:

$X \to *$	CX	cone
$* \leftarrow X \to *$	ΣX	suspension
$X \xrightarrow{f} Y$	$\mathrm{Cyl}(f)$	mapping cylinder
$X \circlearrowleft f$	Tf	mapping torus
$* \leftarrow X \xrightarrow{f} Y$	Cf	mapping cone
$X \leftarrow Z \to Y$		homotopy pushout
$W_0 \xrightarrow{f_0} W_1 \xrightarrow{f_1} W_2 \xrightarrow{f_2} \dots$	$\mathrm{Tel}(W_*)$	mapping telescope

Can we recover the cohomology of W from that of the W_v? To connect this to our original example of recovering the cohomology of a CW complex X from that of its skeleta X_i, recall from the proof of Lemma 4.4 that the mapping telescope $\mathrm{Tel}(X_*)$ of successive inclusions of skeleta is homotopy equivalent to X itself.

5.2. **The homotopy exact sequence of a mapping web.** First note that by restricting maps $W \to Z$ to the vertices of the web, for any space Z we obtain a surjection

$$[W, Z] \to \lim[W_v, Z], \tag{20}$$

but there is no reason to expect this to be injective in general: there may be maps $W \to Z$ which are not nullhomotopic but which become nullhomotopic when restricted to each W_v.

Example 5.3. Take W to be the mapping web $\Sigma S^1 = S^2$ of $* \leftarrow S^1 \to *$ and Z to be S^2. Then $[W, Z] = \pi_2(S^2) \cong \mathbb{Z}$ but $[W_v, Z] = *$ for each v.

Example 5.4. Take W to be the mapping telescope of the $W_i = S^1$ with $W_i \to W_{i+1}$ a 2 : 1 cover. We have

$$\lim[W_i, \mathbb{CP}^\infty] = \lim H^2(W_i; \mathbb{Z}) = \lim(0 \leftarrow 0 \leftarrow \dots) = 0,$$

but you can compute

$$[W, \mathbb{CP}^\infty] = H^2(W; \mathbb{Z}) \cong \widehat{\mathbb{Z}}_2/\mathbb{Z}$$

directly using the obvious cell structure. Here $\widehat{\mathbb{Z}}_2$ denotes the 2-adic integers. Alternatively, you can see from the geometry that $\pi_1(W) = \mathbb{Z}[1/2]$, so $H^2(W; \mathbb{Z})$ contains $\mathrm{Ext}^1(\mathbb{Z}[1/2], \mathbb{Z})$, and the cellular cohomology computation is essentially a computation of this Ext group via a free resolution of $\mathbb{Z}[1/2]$.

Although (20) need not by injective, we just gave a geometric description of its kernel: it comprises precisely those maps from the mapping web W which are nullhomotopic when restricted to each vertex. In other words, those maps $g : W \to Z$ which extend over the space W_C obtained by coning off each of the W_v. In symbols, we're talking about those g such that for each v there exists an extension $g_v : CW_v \to Z$ of $g|_{W_v}$. By sliding each cylinder $W_{s(a)} \wedge [0, 1]_+$ down the cones at its ends, then collapsing these cones, we obtain a homotopy equivalence

$$W_C \simeq \bigvee_a \Sigma W_{s(a)},$$

and this extends (20) to an exact sequence (of based sets)

$$\prod_a [\Sigma W_{s(a)}, Z\Big] \xrightarrow{\Phi} [W, Z] \to \lim[W_v, Z] \to *.$$

Now consider exactness at the left-hand end of this sequence. We view elements of $\prod_a[\Sigma W_{s(a)}, Z]$ via $[W_C, Z]$ as maps $g : W \to Z$ equipped with a choice of coning g_v over each vertex. Such an element lies in the kernel of Φ if and only if g extends over CW (i.e. is nullhomotopic), which is equivalent to the existence of a choice g'_v of coning over each vertex such that $g'_{s(a)} \simeq g'_{t(a)} \circ f_a$. Said another way, a map from W_C is a map from W with a choice of coning over each vertex, and it is in $\ker \Phi$ if and only if there exists another choice of coning which is compatible with the gluing maps. We think of the given conings as hanging below the web and the compatible ones as standing above.

We are thus led to consider the space W_Σ obtained from W by suspending each slice, or alternatively from W_C by reconing each slice. Each map $[W_\Sigma, Z]$ carries restrictions r_d and r_u to the downward and upward conings respectively, and the upshot of the previous paragraph is that

$$\ker \Phi = \{r_d(\widehat{g}) : \widehat{g} \text{ is an element of } [W_\Sigma, Z] \text{ such that } r_u(\widehat{g}) \text{ is nullhomotopic}\}$$

Sliding the cylinders down the cones hanging below we obtain a homotopy equivalence

$$W_\Sigma = \underbrace{\bigvee_a \Sigma W_{s(a)}}_{\text{cylinders}} \vee \underbrace{\bigvee_v \Sigma W_v}_{\text{suspensions over vertices}}$$

under which we identify elements $\widehat{g}$ of $[W_\Sigma, Z]$ with tuples

$$\big((\alpha_a), (\beta_v)\big) \in \prod_a [\Sigma W_{s(a)}, Z] \times \prod_v [\Sigma W_v, Z].$$

Using this identification we have

$$r_u(\widehat{g}) = (\alpha_a) \quad \text{and} \quad r_d(\widehat{g}) = (\beta_{s(a)} \cdot \alpha_a \cdot (\beta_{t(a)} \circ f_a)^{-1}),$$

using the group structure on $[\Sigma W_{s(a)}, Z]$, and putting everything together we conclude that

$$\ker \Phi = \operatorname{im} \Big(\prod_v [\Sigma W_v, Z] \xrightarrow{\delta f} \prod_a [\Sigma W_{s(a)}, Z] \Big),$$

where

$$\delta f : (\beta_v) \mapsto (\beta_{s(a)}^{-1} \cdot (\beta_{t(a)} \circ f_a)).$$

We obtain an exact sequence of based sets

$$\prod_v [\Sigma W_v, Z] \xrightarrow{\delta f} \prod_a [\Sigma W_{s(a)}, Z] \xrightarrow{\Phi} [W, Z] \to \lim [W_v, Z] \to *. \tag{21}$$

5.3. **The cohomology long exact sequence.** To obtain information about cohomology, we now just plug an Ω-spectrum into (21) in place of Z. We get an exact sequence of abelian groups (for each i)

$$\prod_v \widetilde{h}^{i-1}(W_v) \xrightarrow{\delta f} \prod_a \widetilde{h}^{i-1}(W_{s(a)}) \to \widetilde{h}^i(W) \to \lim \widetilde{h}^i(W_v) \to 0.$$

But now note that we can express $\lim \widetilde{h}^i(W_v)$ as the kernel of δf (between $\widetilde{h}^i$ groups), so we can rewrite this exact sequence as

$$\prod_v \widetilde{h}^{i-1}(W_v) \xrightarrow{\delta f} \prod_a \widetilde{h}^{i-1}(W_{s(a)}) \to \widetilde{h}^i(W) \to \prod_v \widetilde{h}^i(W_v) \xrightarrow{\delta f} \prod_a \widetilde{h}^i(W_{s(a)}), \tag{22}$$

and these now obviously glue together to give a long exact sequence.

Let's look back at some of the familiar mapping webs from Example 5.2 and see what we get.

Mapping cone. This recovers the long exact sequence of the pair from Section 2.9.

Homotopy pushout. For a triple $X \xleftarrow{f} Z \xrightarrow{g} Y$ with homotopy pushout W we can cancel one copy of $\widetilde{h}^*(Z)$ from each of $\prod_a$ and $\prod_v$ to get the long exact sequence

$$\begin{tikzcd}
\widetilde{h}^*(W) \arrow[rr] & & \widetilde{h}^*(X) \oplus \widetilde{h}^*(Y) \arrow[dl, "(f^*-g^*)"] \\
& \widetilde{h}^*(Z) \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

This is a form of Mayer–Vietoris! To recover the usual version, suppose that W' is a CW complex which is covered by subcomplexes X and Y, and let Z be their intersection, with f and g the inclusion maps. In this case the complex W' is homotopy equivalent to the web W (the key property is that the inclusion of a subcomplex in a CW complex is a *cofibration*), so we get Mayer–Vietoris for CW complexes. To deduce the general case, May [7, Chapter 10, Section 7] shows that any *excisive triad* can be approximated by CW complexes.

Remark 5.5. Once you have Mayer–Vietoris, you also have excision. To see this suppose we have $U \subset A \subset X$ with the closure of U contained in the interior of A, and consider the following four chain complexes: the long exact sequences for the pairs (X, A) and $(X \setminus U, A \setminus U)$, the Mayer–Vietoris sequence for $X = A \cup (X \setminus U)$, and the complex

$$\cdots \to 0 \to \widetilde{h}^i(X, A) \to \widetilde{h}^i(X \setminus U, A \setminus U) \to 0 \to \widetilde{h}^{i+1}(X, A) \to \widetilde{h}^{i+1}(X \setminus U, A \setminus U) \to 0 \to \dots.$$

Call these complexes C_1, C_2, C_3 and C_4 respectively. We know that the first three are acyclic and we want to show that the fourth is. There are restriction chain maps $r_{12} : C_1 \to C_2$ and $r_{43} : C_4 \to C_3$ and their cones are manifestly isomorphic. $\operatorname{Cone}(r_{12})$ is acyclic since C_1 and C_2 are, so $\operatorname{Cone}(r_{34})$ must also be acyclic. From the spectral sequence associated to the obvious filtration of the latter, we conclude that C_4 is acyclic.

Mapping telescope. In this case both the vertices and arrows are indexed by the non-negative integers, and for each i we have a sequence of abelian groups

$$\widetilde{h}^i(W_0) \leftarrow \widetilde{h}^i(W_1) \leftarrow \widetilde{h}^i(W_2) \leftarrow \dots$$

and a map

$$\delta f : \prod_n \widetilde{h}^i(W_n) \to \prod_n \widetilde{h}^i(W_n).$$

The kernel of this map is exactly $\lim_n \widetilde{h}^i(W_n)$, and the cokernel is denoted by $\lim^1_n \widetilde{h}^i(W_n)$, so we can write (22) as a short exact sequence

$$0 \to \lim_n{}^1 \widetilde{h}^{i-1}(W_n) \to \widetilde{h}^i(W) \to \lim_n \widetilde{h}^i(W_n) \to 0. \tag{23}$$

This result was proved by Milnor [9] and is sometimes called the *Milnor sequence*; it's also covered in Hatcher [5, Theorem 3F.8] and May [7, Chapter 19, Section 4]. These references all derive it algebraically by chopping the mapping telescope into the 'odd' and 'even' parts and applying Mayer–Vietoris, but our derivation was purely topological.

The object $\lim^1$ can be defined for any sequence $\ldots \xleftarrow{a_{i-1}} A_i \xleftarrow{a_i} A_{i+1} \leftarrow \ldots$ of abelian groups and by applying the snake lemma to

$$\begin{tikzcd}
0 \arrow[r] & \prod_n A_n \arrow[r] \arrow[d, "\delta a"] & \prod B_n \arrow[r] \arrow[d, "\delta b"] & \prod_n C_n \arrow[r] \arrow[d, "\delta c"] & 0 \\
0 \arrow[r] & \prod_n A_n \arrow[r] & \prod B_n \arrow[r] & \prod_n C_n \arrow[r] & 0
\end{tikzcd}$$

we see that short exact sequences of sequences of groups give rise to long exact sequences in $\lim^*$. With a bit more work one can actually show that $\lim^1$ is the right derived functor of lim. Note however that for us it arose completely naturally (and in a more general form) from the geometry, without any mention of scary-sounding derived functors.

Definition 5.6. (A_i) satisfies the *Mittag-Leffler condition* if for all i there exists $s(i) \geq i$ such that for all $j \geq s(i)$ the image of A_j in A_i is equal to the image of $A_{s(i)}$ in A_i.

Lemma 5.7. *If (A_i) satisfies the Mittag-Leffler condition then* $\lim^1 A_i = 0$.

Proof. Given (y_i) in $\prod A_i$ we need to construct (x_i) such that $\delta a(x_i) = (y_i)$. For an integer n, say an *n-solution* is an element (z_i) of $\prod A_i$ whose image under δa agrees with (y_i) up to (and including) the nth component. Note that for any given n you can construct an n-solution (z_i) by setting $z_i = 0$ for all $i > n$ and then defining $z_i = y_i + a_i(z_{i+1})$ recursively downwards for $i \leq n$.

We claim that if (z_i) is an n-solution with $n \geq s(m) - 1$ then there exists an $(n+1)$-solution (z'_i) which coincides with (z_i) up to (and including) the mth component. Inductively, we can thus build the required solution (x_i).

To prove the claim, suppose we have such an n-solution (z_i), with $n \geq s(m) - 1$. We can define an auxiliary $(n+1)$-solution (w_i) by

$$w_i = \begin{cases} 0 & \text{if } i > n+1 \\ y_{n+1} & \text{if } i = n+1 \\ z_i + a^{n+1-i}(y_{n+1} - z_{n+1}) & \text{if } i \leq n. \end{cases}$$

Since $n+1$ is at least $s(m)$, we know that there exists ε in A_{n+2} such that

$$a^{n+2-m}(\varepsilon) = a^{n+1-m}(y_{n+1} - z_{n+1}).$$

Now define (z'_i) by

$$z'_i = \begin{cases} 0 & \text{if } i > n+2 \\ -\varepsilon & \text{if } i = n+2 \\ w_i - a^{n+2-i}(\varepsilon) & \text{if } i \leq n+1. \end{cases}$$

It's easy to check that (z'_i) is an $(n+1)$-solution and that it coincides with (z_i) for $i \leq m$, as required. ☐

6. The Atiyah–Hirzebruch spectral sequence

6.1. **What about (co)chain complexes?** We're used to thinking of (co)homology in terms of (co)chain complexes—I'll drop the tedious (co) from now on—but the homotopy-theoretic picture we have been working with looks very different. It is natural to wonder whether generalised cohomology theories have a chain complex model, and the answer is almost yes. The trick is to try to generalise the *cellular* model of ordinary cohomology, which is just defined in terms of the relative cohomology of skeleta, rather than the *singular* model.

Recall that if X is a CW complex, with n-skeleton X_n, then the cellular cochain complex $C^*_{\text{cell}}(X)$ is defined by

$$C^i_{\text{cell}}(X) = H^i(X_i, X_{i-1}) = \widetilde{H}^i\Big(\bigvee_{i\text{-cells}} S^i\Big),$$

with differential given by

$$\begin{tikzcd} H^i(X_i, X_{i-1}) \arrow[rr] \arrow[dr, "\text{restriction}"'] & & H^{i+1}(X_{i+1}, X_i) \\ & H^i(X_i) \arrow[ur, "\text{LES boundary}"'] & \end{tikzcd}$$

By analysing the long exact sequences of pairs one proves directly that cellular cohomology coincides with ordinary cohomology. However, this crucially uses vanishing results of the form

$$H^i(X_j, X_{j-1}) = \widetilde{H}^i\Big(\bigvee_{j\text{-cells}} S^j\Big) = 0 \text{ for } i \neq j.$$

In other cohomology theories this will not in general be true.

6.2. **The cellular complex is a spectral sequence.** Let's think about the cellular complex a bit more conceptually. We have a filtration of our space by skeleta $X_0 \subset X_1 \subset \ldots$, which gives us a decreasing filtration $F^\bullet C^*(X)$ of the singular cochain complex of X:

$$F^i C^*(X) = \{\phi \in C^*(X) : \phi|_{X_{i-1}} = 0\}.$$

The associated graded complex is

$$\bigoplus_i F^i C^*(X)/F^{i+1}C^*(X) = \bigoplus_i C^*(X_i, X_{i-1}),$$

and its cohomology is precisely the cellular complex $C^*_{\text{cell}}(X)$.

A filtered complex is begging to be turned into a spectral sequence. Later we'll need to modify this construction slightly, so first we'll recap the standard version (following Eisenbud [4, Section A3.13]). Recall that an *exact couple* comprises a diagram

$$\begin{tikzcd} A \arrow[rr, "\alpha"] & & A \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd} \tag{24}$$

of modules such that $\ker \alpha = \operatorname{im} \gamma$ and so on. Given an exact couple we define for $r \geq 1$

$$Z_r = \gamma^{-1}(\operatorname{im} \alpha^{r-1}),\ B_r = \beta(\ker \alpha^{r-1}), \text{ and } E_r = Z_r/B_r.$$

The fact that $\gamma \circ \beta = 0$ ensures that B_r is indeed a submodule of Z_r, and that the map

$$\mathrm{d}_r = \beta \circ \alpha^{-(r-1)} \circ \gamma : E_r \to E_r$$

is well-defined and squares to zero. Moreover, a bit of chasing shows that Z_{r+1} and B_{r+1} are the preimages in Z_r of the kernel and image of d_r respectively, so E_{r+1} is identified with $H(E_r, \mathrm{d}_r)$.

A filtered complex $F^\bullet C^*$ gives a short exact sequence

$$0 \to \bigoplus_i F^i C^* \xrightarrow{\iota} \bigoplus_i F^i C^* \to \operatorname{gr} C^* \to 0,$$

where the map ι is the sum of the inclusions $F^{i+1}C^* \hookrightarrow F^i C^*$, and if we take (24) to be the long exact sequence in homology then the E_r are the pages of the spectral sequence associated to the complex. As usual we'll lay the pages out in a grid, with the $\bullet$-grading labelling the columns and the $(* - \bullet)$-grading labelling the rows. We denote these labels by p and q respectively, so each E_r is bigraded $E_r^{p,q}$. We set

$$Z_\infty = \bigcap_r Z_r,\ B_\infty = \bigcup_r B_r, \text{ and } E_\infty = Z_\infty / B_\infty,$$

and say that the spectral sequence *abuts to* E_∞ if for each fixed p and q the sequence $E_1^{p,q}, E_2^{p,q}, \dots$ eventually stabilises to $E_\infty^{p,q}$. If, in addition, E_∞ is the associated graded of a module M with respect to some filtration, then we say that the spectral sequence *converges to* M.

If we start with the singular cochain complex filtered by skeleta then the long exact sequence giving our exact couple is simply the sum of the long exact sequences of pairs

$$\begin{tikzcd} \bigoplus_i H^*(X_i) \arrow[rr, "f^*"] & & \bigoplus_i H^*(X_i) \arrow[dl, "\delta"] \\ & \bigoplus_i H^*(X_i, X_{i-1}) \arrow[ul] & \end{tikzcd} \tag{25}$$

where f^* is the sum of the pullbacks under the inclusions $f_i : X_i \to X_{i+1}$, and δ is the connecting map. The E_1 page of the spectral sequence is

$$\bigoplus_i H^*(X_i, X_{i-1}) = \bigoplus_{p,q} H^{p+q}(X_p, X_{p-1}) = C^*_{\text{cell}}(X),$$

and the differential d_1 coincides with the cellular differential. The whole page is concentrated in the zeroth row ($q = 0$), so all later differentials vanish and the spectral sequence trivially abuts to E_∞. The fact that the restriction $f_i^* : H^*(X_i) \to H^*(X_{i-1})$ is surjective in degrees $* \le i - 1$ ensures that the sequence converges to $H^*(X)$.

6.3. **Generalising.** The important thing to note about the construction of the spectral sequence of a filtered complex is that the complex only enters through its long exact sequence, so we don't need the full chain-level object to begin with. In particular, the exact couple (25) makes sense for *any* generalised cohomology theory, even if there is no underlying chain complex to filter.

Anyway, let's see what happens. Before, we were considering a CW complex X with skeleta $X_0 \subset X_1 \subset \dots$, but now we'll just assume that

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots$$

is a sequence of spaces and maps. We start with the exact couple

$$\begin{tikzcd} \prod_{p,q} \widetilde{h}^{p+q}(X_p) \arrow[rr, "f^*"] & & \prod_{p,q} \widetilde{h}^{p+q}(X_p) \arrow[dl, "\delta"] \\ & \prod_{p,q} \widetilde{h}^{p+q}(X_p, X_{p-1}) \arrow[ul] & \end{tikzcd}$$

This gives a spectral sequence with rth page $E_r = \prod_{p,q} E_r^{p,q}$, where $E_r^{p,q} = Z_r^{p,q}/B_r^{p,q}$ and

$$Z_r^{p,q} = \{\phi \in \widetilde{h}^{p+q}(X_p, X_{p-1}) : \phi|_{X_p} = (f^*)^r \psi \text{ for some } \psi \in \widetilde{h}^{p+q}(X_{p+r})\}$$
$$B_r^{p,q} = \{\phi \in \widetilde{h}^{p+q}(X_p, X_{p-1}) : \phi = \delta\chi \text{ for some } \chi \in \widetilde{h}^{p+q-1}(X_{p-1}) \text{ with } (f^*)^r \chi = 0\}.$$

In order for this to be useful we need to understand abutment and convergence. So fix an arbitrary p and q and consider each of these issues in turn. Throughout we will write A^i_j for $\widetilde{h}^i(Y_j)$, and let $F^\bullet A^i_j$ denote the decreasing filtration defined by

$$F^k A^i_j = \begin{cases} 0 & \text{if } k > j \\ \ker\left((f^*)^{j-k} : A^i_j \to A^i_{k-1}\right) & \text{otherwise.} \end{cases}$$

Abutment. We want the groups $Z_r^{p,q}$ and $B_r^{p,q}$ to stabilise for large r. The condition $(f^*)^r\chi = 0$ in the definition of $B_r^{p,q}$ is vacuous as long as $r \ge p$, so $B_r^{p,q}$ always stabilises to

$$B_\infty^{p,q} = \operatorname{im}\left(\delta : \widetilde{h}^{p+q-1}(X_{p-1}) \to \widetilde{h}^{p+q}(X_p, X_{p-1})\right).$$

Stabilisation of $Z_r^{p,q}$ is more delicate, but note that it certainly occurs if the sequence $(A^{p+q}_{p+r})_r$ satisfies the Mittag-Leffler condition.

Convergence. We have

$$\begin{aligned} E_\infty^{p,q} &= \frac{\{\phi \in \widetilde{h}^{p+q}(X_p, X_{p-1}) : \text{for all } r \text{ there exists } \psi \in \widetilde{h}^{p+q}(X_{p+r}) \text{ with } \phi|_{X_p} = (f^*)^r\psi\}}{\operatorname{im}\big(\delta : \widetilde{h}^{p+q-1}(X_{p-1}) \to \widetilde{h}^{p+q}(X_p, X_{p-1})\big)} \\ &= \big\{\phi \in \widetilde{h}^{p+q}(X_p) : f^*\phi = 0 \text{ and for all } r \text{ there exists } \psi \in \widetilde{h}^{p+q}(X_{p+r}) \text{ with } \phi = (f^*)^r\psi\big\}. \end{aligned}$$

The map from the first row to the second is restriction to X_p.

Letting $E_{\infty,r}^{p,q}$ be

$$\big\{\phi \in \widetilde{h}^{p+q}(X_p) : f^*\phi = 0 \text{ and there exists } \psi \in \widetilde{h}^{p+q}(X_{p+r}) \text{ with } \phi = (f^*)^r\psi\big\},$$

we see that $E_\infty^{p,q} = \lim_r E_{\infty,r}^{p,q}$, where the limit is taken over the obvious inclusion maps, and that for all r we have a short exact sequence

$$0 \to F^{p+1}A_{p+r}^{p+q} \to F^pA_{p+r}^{p+q} \to E_{\infty,r}^{p,q} \to 0.$$

Taking limits over r we obtain the exact sequence

$$0 \to \lim_r F^{p+1}A_{p+r}^{p+q} \to \lim_r F^pA_{p+r}^{p+q} \to E_\infty^{p,q} \to \lim_r{}^1 F^{p+1}A_{p+r}^{p+q}. \tag{26}$$

Now let X be the mapping telescope of the X_i, and consider the standard filtration $F^\bullet\widetilde{h}^i(X)$ given by

$$F^k\widetilde{h}^i(X) = \ker\big(\widetilde{h}^i(X) \xrightarrow{\text{restriction}} \widetilde{h}^i(X_{k-1})\big).$$

This is the same as $\widetilde{h}^i$ of the mapping telescope of the X_i but with X_k coned off. By the usual Milnor sequence applied to this modified telescope, for each i, j and k we have a short exact sequence

$$0 \to \lim_j{}^1 F^kA_j^{i-1} \to F^k\widetilde{h}^i(X) \to \lim_j F^kA_j^i \to 0.$$

There is an obvious map of sequences from $k+1$ to k

$$\begin{tikzcd} 0 \arrow[r] & \lim_j{}^1 F^{k+1}A_j^{i-1} \arrow[r] \arrow[d] & F^{k+1}\widetilde{h}^i(X) \arrow[r] \arrow[d] & \lim_j F^{k+1}A_j^i \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \lim_j{}^1 F^kA_j^{i-1} \arrow[r] & F^k\widetilde{h}^i(X) \arrow[r] & \lim_j F^kA_j^i \arrow[r] & 0 \end{tikzcd}$$

and the right-hand vertical map is injective so the snake lemma gives a short exact sequence

$$0 \to \frac{\lim_j{}^1 F^kA_j^{i-1}}{\lim_j{}^1 F^{k+1}A_j^{i-1}} \to \frac{F^k\widetilde{h}^i(X)}{F^{k+1}\widetilde{h}^i(X)} \to \frac{\lim_j F^kA_j^i}{\lim_j F^{k+1}A_j^i} \to 0$$

Setting $i = p+q$, $j = p+r$ and $k = p$, we can splice the end of this sequence onto the beginning of (26) to deduce the following exact sequence:

$$0 \to \frac{\lim_r{}^1 F^pA_{p+r}^{p+q-1}}{\lim_r{}^1 F^{p+1}A_{p+r}^{p+q-1}} \to \frac{F^p\widetilde{h}^{p+q}(X)}{F^{p+1}\widetilde{h}^{p+q}(X)} \to E_\infty^{p,q} \to \lim_r{}^1 F^{p+1}A_{p+r}^{p+q}.$$

Therefore if all of the $\lim^1$ terms vanish then E_∞ page is the associated graded of the natural filtration on $\widetilde{h}^*(X)$. This holds if the sequences $(F^pA_{p+r}^{p+q})_r$ satisfy the Mittag-Leffler condition for all values of p and q, and this in turn follows from the same condition with the F^p's removed.

The upshot of all of this discussion is:

Lemma 6.1. *Suppose*

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots$$

is a sequence of spaces with the property that for all i the sequence $(\widetilde{h}^i(X_j))_j$ satisfies the Mittag-Leffler condition, and let X denote the mapping telescope. The spectral sequence constructed above abuts, and converges to $\widetilde{h}^{p+q}(X)$.

6.4. **Identifying the early pages.** The E_1 page has $E_1^{p,q} = \widetilde{h}^{p+q}(X_p, X_{p-1})$. To turn this into something more manageable, we'll mimic the setting of the Serre spectral sequence and assume that $\pi : E \to B$ is a fibration with E and B both CW complexes. For each p let B_p be the p-skeleton of B, define X_p to be $\pi^{-1}(B_p)_+$, and assume that this is a subcomplex of E_+. Note that E_+ is homotopy equivalent to the mapping telescope of the X_p, since they form an exhausting increasing sequence of subcomplexes.

Returning to the E_1 page, the cohomology of the pair $\widetilde{h}^{p+q}(X_p, X_{p-1})$ is really the cohomology of the cone on the inclusion of X_{p-1} in X_p, and since this is an inclusion of subcomplexes the cone is homotopy equivalent to the quotient X_p/X_{p-1}. For $p > 0$, this in turn is equivalent to the space obtained by pulling back π to each p-cell of B and then collapsing everything over the boundaries of these cells to a single point. When $p = 0$ it is simply the disjoint union of the fibres over the 0-cells of B with an extra base point added.

We have seen that for any contractible set $C \subset B$, the space $\pi^{-1}(C)$ is homotopy equivalent to the fibre F of π—for simplicity we'll assume that B is connected so that all fibres are homotopy equivalent. Similarly, the pullback of π to each cell of B is homotopy equivalent to F. This means that for $p > 0$ we have

$$\begin{aligned} X_p/X_{p-1} &\simeq \coprod_{p\text{-cells in } B} D^p \times F \Big/ \coprod_{p\text{-cells in } B} \partial D^p \times F \\ &\simeq \bigvee_{p\text{-cells in } B} (S^p \wedge F_+ = \Sigma^p F_+), \end{aligned}$$

whilst for $p = 0$ we have

$$X_0 = \Big(\coprod_{0\text{-cells in } B} F \Big)_+ = \bigvee_{0\text{-cells in } B} F_+.$$

We conclude that for all p and q we have

$$E_1^{p,q} = \widetilde{h}^{p+q}(X_p, X_{p-1}) = \prod_{p\text{-cells in } B} \widetilde{h}^{p+q}(\Sigma^p F_+) = \prod_{p\text{-cells in } B} h^q(F).$$

In other words, the E_1 page is precisely the cellular complex of B with coefficients in $h^*(F)$. Just as for the Serre spectral sequence, the differential is the cellular differential with local coefficients, and we deduce:

Theorem 6.2 (Atiyah–Hirzebruch spectral sequence)**.** *Suppose that $\pi : E \to B$ is a fibration of CW complexes such that the preimages of the skeleta B_p of B are subcomplexes of E, and B is connected. If for all i the sequence $(h^i(\pi^{-1}(B_j)))_j$ satisfies the Mittag-Leffler condition then there is a spectral sequence*

$$E_2^{p,q} = H^p(B; h^q(F)) \implies h^{p+q}(E),$$

where F is the fibre of π.

Remark 6.3. The Mittag-Leffler condition trivially holds if B is finite-dimensional, since the sequences $(h^i(\pi^{-1}(B_j)))_j$ are all finite.

Remark 6.4. Taking the fibre to be a point, so that π is simply the identity map of B, we get a spectral sequence computing $h(B)$ from $H(B; h(*))$. Alternatively, taking h to be H we recover the usual Serre spectral sequence.

Corollary 6.5. *If X is a finite-dimensional CW complex then there is a spectral sequence*

$$E_2^{p,q} = H^p(X; h^q(*)) \implies h^{p+q}(X).$$

6.5. **Proof of Proposition 4.23.** Recall the statement we wish to prove: if X is a CW complex, $X_0 \subset X_1 \subset \dots$ is an exhausting sequence of finite-dimensional subcomplexes such that $H^*(X_i; \mathbb{Z})$ is concentrated in even degree for all i, and h^* is generalised cohomology theory such that $h^*(*)$ is also concentrated in even degree, then the restriction map

$$h^*(X) \to \lim_i h^*(X_i)$$

is an isomorphism in even degrees.

To prove this, first apply the Atiyah–Hirzebruch spectral sequence to each X_i to see that $h^*(X_i)$ is concentrated in even degree. Now apply the Milnor sequence and use the vanishing of $h^{\mathrm{odd}}(X_i)$ to see that the relevant $\lim^1$ terms vanish.

7. RING STRUCTURE OF K-THEORY AND THE CHERN CHARACTER

7.1. **Cohomology computations.** We now return to the study of K-theory, but before we get into the construction of the product operation we need to make some preliminary cohomology computations. These are all exercises with the Serre spectral sequence.

We start with:

Lemma 7.1. *For the variety* $\mathrm{Fl}(n) = \mathrm{U}(n)/T^n$ *of complete flags (or n-tuples of orthogonal lines) in* $\mathbb{C}^n$ *there is an isomorphism of graded* $\mathbb{Z}$*-modules (not of rings!)*

$$H^*(\mathrm{Fl}(n);\mathbb{Z}) \cong \mathbb{Z}[z_1,\ldots,z_n]/(z_1^n, z_2^{n-1},\ldots,z_n),$$

where the z_i *are the first Chern classes of the n tautological line bundles. This is a free* $\mathbb{Z}$*-module of rank* $n!$*. The symmetric group* S_n *acts on* $\mathrm{Fl}(n)$ *by permuting the lines and the induced action on* $H^*(\mathrm{Fl}(n);\mathbb{C})$ *is the regular representation* $\mathbb{C}[S_n]$*. In particular, the* S_n*-invariant part of* $H^*(\mathrm{Fl}(n);\mathbb{Z})$ *is* H^0*.*

Proof. For each r let F_r denote the space of r-tuples of orthogonal lines in $\mathbb{C}^n$. We have forgetful fibrations

$$\mathbb{CP}^{n-r} \hookrightarrow F_r \to F_{r-1},$$

and the cohomology ring of the fibre is generated by the first Chern class z_r of the rth tautological line bundle over F_r. The Serre spectral sequence shows that $H^*(F_r;\mathbb{Z})$ is a free $H^*(F_{r-1};\mathbb{Z})$-module on the basis $1, z_r, \ldots, z_r^{n-r+1}$ so by induction on r (up to $r = n$, where F_r coincides with $\mathrm{Fl}(n)$) we get the first part of the result.

To prove the representation-theoretic fact (following a suggestion of Oscar Randal-Williams), note that the S_n-action on $\mathrm{Fl}(n)$ is free so the quotient is a manifold and we can lift a cell structure from this quotient to obtain an S_n-equivariant cell structure on the flag variety. The cellular cochain complex over $\mathbb{C}$ then becomes a complex of free $\mathbb{C}[S_n]$-modules.

Let $\mathbb{C}[S_n]$-Mod denote the abelian category of finite-dimensional $\mathbb{C}[S_n]$-modules. By Maschke's theorem, any such module decomposes as a direct sum of simple modules, and is determined up to isomorphism by the multiplicity of each isomorphism class of summand. This means that the Grothendieck group $K(\mathbb{C}[S_n]\text{-}\mathsf{Mod})$ is the free abelian group on the isomorphism classes of simple modules, and the map

$$\{\text{isomorphism classes of } \mathbb{C}[S_n]\text{-module}\} \to K(\mathbb{C}[S_n]\text{-}\mathsf{Mod}) \tag{27}$$

is injective. The Euler characteristic of a chain complex of $\mathbb{C}[S_n]$-modules defines an element in this Grothendieck group, and is equal to the Euler characteristic of the cohomology.

Now, since our cellular chain complex is built from copies of the regular representation, its Euler characteristic is a multiple of the K-class of the regular representation. This means that the Euler characteristic of $H^*(\mathrm{Fl}(n);\mathbb{C})$ is also a multiple of the regular representation, and by counting dimensions the multiplicity is exactly 1. Since the cohomology ring is concentrated in even degree, its Euler characteristic is simply its own class in the Grothendieck group, and by injectivity of (27) we deduce that

$$H^*(\mathrm{Fl}(n);\mathbb{C}) \cong \mathbb{C}[S_n]$$

as $\mathbb{C}[S_n]$-modules.

To prove the final statement observe that any S_n-invariant integral cohomology class defines an invariant complex cohomology class, so it suffices to show that the invariant part of the complex cohomology is H^0. Clearly this part *is* invariant, so it is left to prove that the invariant part is 1-dimensional. This follows from the fact that the invariant part of the regular representation is 1-dimensional. □

Let $\theta : T^n \to \mathrm{U}(n)$ be the inclusion of a maximal torus (e.g. the diagonal matrices). We know that $B\mathrm{U}(1) \simeq \mathbb{CP}^\infty$, so by taking the product of n copies of the universal bundle $S^1 \hookrightarrow S^\infty \to \mathbb{CP}^\infty$ we see that $BT^n \simeq (\mathbb{CP}^\infty)^n$. By Künneth we obtain

$$H^*(BT^n;\mathbb{Z}) \cong \mathbb{Z}[x_1,\ldots,x_n],$$

where each x_i has degree 2.

Lemma 7.2. *The cohomology ring* $H^*(B\mathrm{U}(n);\mathbb{Z})$ *is a polynomial ring on generators* $c_1,\dots,c_n$ *of degree* $2,\dots,2n$.

Proof. The fibration $\mathrm{U}(n-1) \hookrightarrow \mathrm{U}(n) \to S^{2n-1}$ which projects an $n\times n$ matrix to its first column can be delooped (in the sense described below) twice to obtain a fibration sequence homotopy equivalent to

$$S^{2n-1} \to B\mathrm{U}(n-1) \to B\mathrm{U}(n). \tag{28}$$

By induction on n, the Serre spectral sequence gives what we want. □

Remark 7.3. By delooping a G-bundle $E \to B$, we mean considering the fibration

$$E \hookrightarrow E \times_G EG \to BG,$$

where the last map is projection π_2 onto EG/G. The space $E\times_G EG$ is homotopy equivalent to B, by the projection π_1 onto E/G, and the induced map $B \to BG$ is precisely the map classifying the original bundle. To see the last point, note that it suffices to show that π_1^*E is isomorphic to π_2^*EG as G-bundles over BG, and both of these bundles are $E\times EG$.

From the construction of Section 3.7, there is a map $B\theta : BT^n \to B\mathrm{U}(n)$.

Proposition 7.4. *The map*

$$B\theta^* : H^*(B\mathrm{U}(n);\mathbb{Z}) \to H^*(BT^n;\mathbb{Z}) = \mathbb{Z}[x_1,\dots,x_n]$$

identifies $H^*(B\mathrm{U}(n);\mathbb{Z})$ *with the ring of symmetric polynomials, and exhibits* $\mathbb{Z}[x_1,\dots,x_n]$ *as a free module of rank* $n!$ *over this subring. Moreover,* $H^*(B\mathrm{U}(n);\mathbb{Z})$ *is isomorphic to the polynomial ring on the elementary symmetric polynomials.*

Remark 7.5. Depending on what you already know about symmetric polynomials, you might think that various parts of this statement are redundant, but we will prove them all directly by topological methods.

Proof. All cohomology groups are taken with integer coefficients. Recall that $B\theta$ can be viewed as a fibration with fibre given by the homotopy quotient of $\mathrm{U}(n)$ by T^n. (We say 'can be viewed' because the spaces and maps involved are only defined up to homotopy.) Since $\mathrm{U}(n) \to \mathrm{U}(n)/T^n$ is a principal T^n-bundle, this homotopy quotient is a fibration over the ordinary quotient $\mathrm{U}(n)/T^n$ with contractible fibre ET^n, and hence is homotopy equivalent to $\mathrm{U}(n)/T^n = \mathrm{Fl}(n)$. We conclude that there is a fibration sequence whose terms are homotopy equivalent to

$$\mathrm{Fl}(n) \hookrightarrow BT^n \to B\mathrm{U}(n).$$

The spaces $\mathrm{Fl}(n)$ and $B\mathrm{U}(n)$ have cohomology rings concentrated in even degree, so the differentials in the Serre spectral sequence all vanish trivially, proving that $B\theta^*$ is injective. Since $H^*(\mathrm{Fl}(n))$ is a free $\mathbb{Z}$-module, we conclude that $H^*(BT^n) \cong H^*(\mathrm{Fl}(n)) \otimes H^*(B\mathrm{U}(n))$ as $H^*(B\mathrm{U}(n))$-modules via the cup product (this is essentially Leray–Hirsch). We are left to show that $H^*(B\mathrm{U}(n))$ corresponds to the subring of $H^*(BT^n)$ comprising the symmetric polynomials, and that it is generated by the elementary symmetric polynomials.

For the first of these, note that the map θ is homotopic to its composition with any permutation of the factors of T^n. This is because any permutation of rows and columns in $\mathrm{U}(n)$ can be achieved by multiplying by appropriate change-of-basis matrices, and these can be homotoped to the identity within $\mathrm{U}(n)$. This shows that the image of $B\theta^*$ is contained in the symmetric polynomials. Conversely, the symmetric group S_n acts on the whole fibration, preserving the fibres, and any invariant element of $H^*(BT^n)$ must come from an invariant element on the E_2 page. By Lemma 7.1 all such elements lie in $H^*(B\mathrm{U}(n); H^0(\mathrm{Fl}(n)))$, so are in the image of $B\theta^*$.

Finally we show that the generators c_i of $H^*(B\mathrm{U}(n))$ can be chosen to be the elementary symmetric polynomials in the x_i. Return to the fibration (28). We defined c_n as the element of $H^*(B\mathrm{U}(n))$ which is hit by the generator of $H^{2n-1}(S^{2n-1})$ in the Serre spectral sequence; strictly this requires a choice of orientation on S^{2n-1}, but you can do this canonically using the complex orientation on $\mathbb{C}^n \supset S^{2n-1}$ and (say) the outward-pointing normal direction—some particular choice will make the signs turn out correctly! It is therefore annihilated by the pullback $H^{2n}(B\mathrm{U}(n)) \to H^{2n}(B\mathrm{U}(n-1))$. Since the diagram

$$\begin{tikzcd} H^{2n}(B\mathrm{U}(n-1)) \arrow[d] & H^{2n}(B\mathrm{U}(n)) \arrow[l] \arrow[d, "B\theta^*"] \\ H^{2n}(BT^{n-1}) & H^{2n}(BT^n) \arrow[l] \end{tikzcd} \tag{29}$$

commutes, we see that $B\theta^*(c_n)$ is a degree n symmetric polynomial in $x_1, \dots, x_n$ which is annihilated by the map $H^*(BT^n) \to H^*(BT^{n-1})$, which just sets x_n to 0. The only such polynomials are multiples of $x_1 \dots x_n$ and you can see that the multiple must be ± 1 since c_n is indivisible in $H^{2n}(B\mathrm{U}(n))$, $B\theta^*$ is injective, and the quotient $H^{2n}(BT^n)/H^{2n}(B\mathrm{U}(n))$ is free.

So up to sign this shows that c_n corresponds to the nth elementary symmetric polynomial, and the sign can be made $+1$ by choosing the correct orientation on S^{2n-1}. An inductive argument using the diagram (29) shows that the other c_i are also elementary symmetric polynomials. This completes the proof. □

Remark 7.6. A concrete (i.e. not just 'up to homotopy') geometric description of (28) is given by viewing $B\mathrm{U}(n)$ as Gr_n and $B\mathrm{U}(n-1)$ as the S^∞-bundle over Gr_{n-1} whose fibre over an $(n-1)$-dimensional subspace $\Pi \subset \mathbb{C}^\infty$ is the sphere in $\Pi^\perp$. The map $B\mathrm{U}(n-1)$ to $B\mathrm{U}(n)$ then sends a subspace Π and orthogonal vector v to their span.

Let $T = \operatorname{colim} T^n$ be the 'maximal torus' in U, and write H^{**} for the contravariant functor

$$H^{**} : \mathsf{hTop} \to \mathsf{AbGp}$$

defined by

$$H^{**} = \prod_{i=0}^{\infty} H^{2i}(-;\mathbb{Z}).$$

Combining Proposition 7.4 with the Milnor sequence we obtain:

Corollary 7.7. *The ring $H^{**}(BT;\mathbb{Z})$ is the power series ring $\mathbb{Z}[\![x_1, x_2, \dots]\!]$, where each x_i has degree 2. The pullback $H^{**}(B\mathrm{U};\mathbb{Z}) \to H^{**}(BT;\mathbb{Z})$ induced by the inclusion $T \to \mathrm{U}$ identifies $H^{**}(B\mathrm{U};\mathbb{Z})$ with the subring of symmetric power series, which is itself a power series ring on the elementary symmetric power series which we denote by $c_1, c_2, \dots$.*

Remark 7.8. Recall from Section 4.4 that for any CW complex X, a vector bundle E on X can be descrbed by a homotopy class of map $f : X \to B\mathrm{U}$. Pulling back the elements $c_1, \dots, c_n \in H^{**}(B\mathrm{U})$ defines the Chern classes of E in $H^{**}(X)$. Note that this definition factors through $K(\mathrm{Vect}'(X))$ and even through $K^0(X)$. The direct sum map $\oplus : B\mathrm{U} \times B\mathrm{U} \to B\mathrm{U}$ lifts to a map $BT \times BT \to BT$ and using this you can show that the total Chern class $1 + c_1 + c_2 + \dots$ of a bundle $E_1 \oplus E_2$ is the product of the total Chern classes of the E_i. Strictly the total Chern class may have terms of arbitrarily high degree so it lives in

$$\prod_i H^i(-;\mathbb{Z}) \quad \text{rather than} \quad \bigoplus_i H^i(-;\mathbb{Z}).$$

Lifting to BT in this way is a kind of master splitting principle: we're splitting the universal bundle over $B\mathrm{U}$ rather than splitting a specific given bundle over X. For a nice discussion of splitting principles in general, see May's note.

Remark 7.9. An inductive argument using the Serre spectral sequence for the fibration

$$\mathrm{U}(n-1) \hookrightarrow \mathrm{U}(n) \to S^{2n-1}$$

shows that $H^*(\mathrm{U}(n);\mathbb{Z}) = \Lambda(u_1, u_3, \dots, u_{2n-1})$, where u_i has degree i. Now looking at the Serre spectral sequence for the universal bundle $\mathrm{U}(n) \hookrightarrow E\mathrm{U}(n) \to B\mathrm{U}(n)$, we see that the class u_i must survive until the $2i$th page, where the differential sends it to c_i in $\mathbb{Z}[c_1, \dots, c_n]/(c_1, \dots, c_{i-1})$, assuming the sign of u_i is chosen correctly. Now naturality of the Serre spectral sequence tells us that for *any* principal $\mathrm{U}(n)$-bundle $\mathrm{U}(n) \hookrightarrow E \to B$ (which is sufficiently nice that the spectral sequence exists), the class u_i survives to the $2i$th page, where it is sent to the ith Chern class of the bundle in $H^*(B;\mathbb{Z})$. A similar property holds for U-bundles.

The final fact we will need is:

Lemma 7.10. *The cohomology ring $H^*(\mathrm{Gr}(k,N);\mathbb{Z})$ of the Grassmannian of k-planes in $\mathbb{C}^N$ is the ring of polynomials in $x_1,\dots,x_N$ (with each x_i of degree 2) which are symmetric under the action of $S_k \times S_{N-k} \subset S_N$, modulo the ideal generated by the totally symmetric polynomials of positive degree.*

Proof. We have $\mathrm{Gr}(k,N) \cong \mathrm{U}(N)/(\mathrm{U}(k)\times\mathrm{U}(N-k))$ so by delooping we obtain a prinicipal $\mathrm{U}(N)$-bundle

$$\mathrm{Gr}(k,N) \simeq \mathrm{U}(N) \times_{\mathrm{U}(k)\times\mathrm{U}(N-k)} E(\mathrm{U}(k)\times\mathrm{U}(N-k)) \to B\mathrm{U}(k)\times B\mathrm{U}(N-k),$$

which is simply the direct sum of the tautological $\mathrm{U}(k)$ and $\mathrm{U}(N-k)$-bundles.

By Remark 7.9 we see that the E_2 page of the Serre spectral sequence looks like

$$\Lambda(u_1,u_3,\dots,u_{2N-1})\otimes\mathbb{Z}[c_1,\dots,c_k,c'_1,\dots,c'_{N-k}],$$

and u_i survives to the $2i$th page where it is sent to the degree $2i$ term in

$$(1+c_1+\dots+c_k)(1+c'_1+\dots+c'_{N-k})$$

(this is the total Chern class of the direct sum of the two tautological bundles). Another way to express this is that after pulling everything back to $H^*(BT^N;\mathbb{Z}) = \mathbb{Z}[x_1,\dots,x_N]$, so that c_i represents the ith elementary symmetric polynomial in $x_1,\dots,x_k$ and c'_i represents the ith elementary symmetric polynomial in $x_{k+1},\dots,x_N$, the class u_i is sent to the ith elementary symmetric polynomial in $x_1,\dots,x_N$. We deduce that all u_i are killed by the spectral sequence, and all that remains is the claimed quotient lying in the zeroth row. ☐

7.2. **Constructing the multiplication.** Given vector bundles E and F on spaces X and Y, their *external tensor product* $E\boxtimes F$ is defined to be the vector bundle

$$\pi_1^*E\otimes\pi_2^*F$$

on $X\times Y$, where π_1 and π_2 are the projections from $X\times Y$ to X and Y respectively. Since this is bilinear in E and F it induces a product on Grothendieck groups

$$\boxtimes : K(\mathrm{Vect}'(X))\otimes K(\mathrm{Vect}'(Y)) \to K(\mathrm{Vect}'(X\times Y)),$$

and our goal is to extend this to a product

$$\boxtimes : K^0(X)\times K^0(Y)\to K^0(X\times Y). \tag{30}$$

In fact we will define a product

$$\boxtimes : \widetilde{K}^0(X)\times\widetilde{K}^0(Y)\to\widetilde{K}^0(X\wedge Y), \tag{31}$$

which induces (30) via the identification $K^0 = \widetilde{K}^0\oplus\mathbb{Z}$ (with the $\mathbb{Z}$ recording copies of the trivial bundle) and the pullback $K^*(X\wedge Y)\to K^*(X\times Y)$.

Taking $X=Y$ and pulling back by the diagonal inclusion $\Delta : X\to X\times X$ we obtain a product $\otimes$ on $\widetilde{K}^0(X)$ (or $K^0(X)$) making it into a commutative ring. This can be extended to a graded-commutative ring structure on the full $\widetilde{K}^*(X)$ (respectively $K^*(X)$) but this will not concern us. Clearly this parallels the construction of the cup product in ordinary cohomology via the cross product.

The desired operation (31) is defined to be the composition of

$$[X,\mathbb{Z}\times B\mathrm{U}]\times[Y,\mathbb{Z}\times B\mathrm{U}]\xrightarrow{\wedge}[X\wedge Y,(\mathbb{Z}\times B\mathrm{U})\wedge(\mathbb{Z}\times B\mathrm{U})]$$

with the pushforward by a map $(\mathbb{Z}\times B\mathrm{U})\wedge(\mathbb{Z}\times B\mathrm{U})\to\mathbb{Z}\times B\mathrm{U}$. Our aim is now to construct this map—or really its homotopy class M—and show that the induced product operation has all of the required properties: that it is associative, that it distributes over addition, that the trivial line bundle acts as a multiplicative identity (this class only exists in the unreduced K^0), and that it is intertwined with the external tensor tensor product on $K(\mathrm{Vect}'(-))$. Note that it will automatically commute with pullbacks.

Before getting into the construction, we have:

Lemma 7.11. *Suppose X is a CW complex with countably many cells whose cohomology is concentrated in even degree.*

(i) $X\vee X$, $X\times X$ and $X\wedge X$ also have these properties.
(ii) If, in addition, X is finite dimensional then

$$\widetilde{K}^0(X\wedge X) = \ker(i^* : K^0(X\times X)\to K^0(X\vee X)),$$

where i is the inclusion $X\vee X\to X\times X$.

Proof. The only non-trivial part in showing that $X \vee X$, $X \times X$ and $X \wedge X$ are countable CW complexes is to check that the product topology on $X \times X$ coincides with the CW topology, and this is done in Hatcher [5, Theorem A.6]. By Mayer–Vietoris we have

$$\widetilde{H}^*(X \vee X) = \widetilde{H}^*(X) \times \widetilde{H}^*(X),$$

so $X \vee X$ has cohomology concentrated in even degree, and the same is true of $X \times X$ from the Serre spectral sequence. Moreover the map $i^* : H^*(X \times X) \to H^*(X \wedge X)$ induces a surjection on cohomology, since for any α and β in $\widetilde{H}^*(X)$ we have

$$i^*(\alpha \times 1 + 1 \times \beta) = (\alpha, \beta).$$

From the long exact sequence associated to

$$X \vee X \xrightarrow{i} X \times X \to Ci,$$

we conclude that the cone Ci also has cohomology concentrated in even degree, and since $X \vee X \to X \times X$ is an inclusion of a subcomplex into a CW complex this cone is homotopy equivalent to the quotient $X \wedge X$.

If X is finite-dimensional then Atiyah–Hirzebruch ensures that $X \vee X$, $X \times X$ and $X \wedge X$ also have K-theory concentrated in even degree, and the long exact sequence in K-theory associated to this triple then tells us that

$$\widetilde{K}^0(X \wedge X) = \ker(i^* : K^0(X \times X) \to K^0(X \vee X)). \qquad \square$$

Now, returning to the construction of the K-theory product, M is supposed to be a class in

$$[(\mathbb{Z} \times B\mathrm{U}) \wedge (\mathbb{Z} \times B\mathrm{U}), \mathbb{Z} \times B\mathrm{U}] = \widetilde{K}^0((\mathbb{Z} \times B\mathrm{U}) \wedge (\mathbb{Z} \times B\mathrm{U})),$$

so we need to understand something about this K-group. The space $\mathbb{Z} \times B\mathrm{U}$ can be viewed as $\operatorname{colim} G_n$, where

$$G_n \coloneqq \{-n, -n+1, \ldots, n\} \times \mathrm{Gr}(n, 2n)$$

and $\mathrm{Gr}(n, 2n)$ is included as a subcomplex in $\mathrm{Gr}(n+1, 2(n+1))$ via

$$(\Pi \subset \mathbb{C}^{2n}) \mapsto (\mathbb{C} \oplus \Pi \oplus 0 \subset \mathbb{C} \oplus \mathbb{C}^{2n} \oplus \mathbb{C}),$$

so $(\mathbb{Z} \times B\mathrm{U}) \wedge (\mathbb{Z} \times B\mathrm{U})$ is a CW complex exhausted by the subcomplexes $G_n \wedge G_n$. By Lemma 7.10 and Lemma 7.11(i) each of these subcomplexes has cohomology concentrated in even degree, so Proposition 4.23 tells us that the restriction map

$$\widetilde{K}^0((\mathbb{Z} \times B\mathrm{U}) \wedge (\mathbb{Z} \times B\mathrm{U})) \to \lim_n \widetilde{K}^0(G_n \wedge G_n)$$

is an isomorphism.

More generally, consider triples (d, k, n) of positive integers with $k \leq n$, partially ordered by setting $(d, k, n) \leq (d', k', n')$ if and only if $d \leq d'$, $k \leq k'$, and $n - k \leq n' - k'$. Let

$$G_{d,k,n} = \{-d, -d+1, \ldots, d\} \times \mathrm{Gr}(k, n)$$

and for $(d, k, n) \leq (d', k', n')$ consider the stabilisation map $G_{d,k,n} \to G_{d',k',n'}$ given by

$$\{a\} \times (\Pi \subset \mathbb{C}^n) \mapsto \{a\} \times (\mathbb{C}^{k'-k} \oplus \Pi \oplus 0 \subset \mathbb{C}^{k'-k} \oplus \mathbb{C}^n \oplus \mathbb{C}^{(n'-k')-(n-k)}).$$

The $G_{d,k,n}$ form a directed system of subcomplexes of $\mathbb{Z} \times B\mathrm{U}$, and the $G_n = G_{n,n,2n}$ form a subsystem. Since for all (d, k, n) there exists n' with $(d, k, n) \leq (n', n', 2n')$, the two limits

$$\lim_{(d,k,n)} \widetilde{K}^0(G_{d,k,n} \wedge G_{d,k,n}) \quad \text{and} \quad \lim_n \widetilde{K}^0(G_n \wedge G_n)$$

coincide, and hence the restriction map

$$\widetilde{K}^0((\mathbb{Z} \times B\mathrm{U}) \wedge (\mathbb{Z} \times B\mathrm{U})) \to \lim_{(d,k,n)} \widetilde{K}^0(G_{d,k,n} \wedge G_{d,k,n})$$

is an isomorphism.

We conclude that we can define the required map M by a consistent sequence of classes $M_{d,k,n}$ in $\widetilde{K}^0(G_{d,k,n} \wedge G_{d,k,n})$, and by Lemma 7.11(ii) such a class is an element of

$$\begin{aligned} &\ker(i^* : K^0(G_{d,k,n} \times G_{d,k,n}) \to K^0(G_{d,k,n} \vee G_{d,k,n})) \\ &\qquad\qquad = \ker(i^* : K(\mathrm{Vect}(G_{d,k,n} \times G_{d,k,n})) \to K(\mathrm{Vect}(G_{d,k,n} \vee G_{d,k,n}))), \end{aligned}$$

i.e. a virtual vector bundle on $G_{d,k,n} \times G_{d,k,n}$ whose restrictions to $G_{d,k,n} \times \{*\}$ and $\{*\} \times G_{d,k,n}$ are trivial.

Let $E_{k,n}$ denote the tautological bundle on $\mathrm{Gr}(k,n)$, and consider the virtual vector bundle on $G_{d,k,n} \times G_{d,k,n}$ defined by

$$E_{k,n} \boxtimes E_{k,n} + (a-k)\underline{\mathbb{C}} \boxtimes E_{k,n} + (b-k)E_{k,n} \boxtimes \underline{\mathbb{C}} + (a-k)(b-k)\underline{\mathbb{C}} \boxtimes \underline{\mathbb{C}} \tag{32}$$

on $(\{a\} \times \mathrm{Gr}(k,n)) \times (\{b\} \times \mathrm{Gr}(k,n))$, where $\boxtimes$ is the usual external tensor product of bundles. Since the base point $*$ in $G_{d,k,n}$ lies in the $\{0\} \times B\mathrm{U}$ piece, the restriction of (32) to $G_{d,k,n} \times \{*\}$ is

$$kE_{k,n} \boxtimes \underline{\mathbb{C}} + (a-k)k\underline{\mathbb{C}} \boxtimes \underline{\mathbb{C}} - kE_{k,n} \boxtimes \underline{\mathbb{C}} - (a-k)k\underline{\mathbb{C}} \boxtimes \underline{\mathbb{C}} = 0.$$

Similarly, the restriction of (32) to $G_{d,k,n} \times \{*\}$ is trivial. This means that (32) defines a class in $\widetilde{K}^0(G_{d,k,n} \wedge G_{d,k,n})$: this will be our $M_{d,k,n}$.

For $(d',k',n') \geq (d,k,n)$, the restriction of $M_{d',k',n'}$ to $G_{d,k,n} \wedge G_{d,k,n}$ under the stabilisation map corresponds to the virtual bundle

$$\begin{aligned}(\underline{\mathbb{C}}^{k'-k} \oplus E_{k,n}) \boxtimes (\underline{\mathbb{C}}^{k'-k} \oplus E_{k,n}) + (a-k')\underline{\mathbb{C}} \boxtimes (\underline{\mathbb{C}}^{k'-k} \oplus E_{k,n})& \\ + (b-k')(\underline{\mathbb{C}}^{k'-k} \oplus E_{k,n}) \boxtimes \underline{\mathbb{C}} + (a-k')(b-k')\underline{\mathbb{C}} \boxtimes \underline{\mathbb{C}}& \\ = E_{k,n} \boxtimes E_{k,n} + (a-k)\underline{\mathbb{C}} \boxtimes E_{k,n} + (b-k)E_{k,n} \boxtimes \underline{\mathbb{C}} + (a-k)(b-k)\underline{\mathbb{C}} \boxtimes \underline{\mathbb{C}}&\end{aligned}$$

on $G_{d,k,n} \times G_{d,k,n}$, so is isomorphic to $M_{d,k,n}$. Therefore the classes $M_{d,k,n}$ fit together to define a unique class in

$$\lim_n \widetilde{K}^0(G_{d,k,n} \wedge G_{d,k,n}) = \widetilde{K}^0((\mathbb{Z} \times B\mathrm{U}) \wedge (\mathbb{Z} \times B\mathrm{U})).$$

This is M.

Remark 7.12. Where did this definition come from? Well, recall from Section 4.4 that virtual vector bundles correspond to homotopy classes of maps to $\mathbb{Z} \times B\mathrm{U}$ by recording the rank and the stable isomorphism class, so a map to $\{a\} \times \mathrm{Gr}(k,n) \subset \mathbb{Z} \times B\mathrm{U}$ represents the virtual vector bundle of rank a whose stable isomorphism class coincides with that of the pullback of the tautological bundle on $\mathrm{Gr}(k,n)$. In other words, we can think of it as the pullback of the tautological bundle plus $a-k$ copies of the trivial line bundle to arrive at the correct rank. The formula (32) then just expresses the standard external tensor product.

7.3. **Properties of multiplication.**

Now we check the basic properties of the product.

Lemma 7.13. *The external K-theory product*

$$\boxtimes : K^0(X) \times K^0(Y) \to K^0(X \times Y)$$

is associative.

Remark 7.14. It is awkward to discuss associativity of the reduced product to $\widetilde{K}^0(X \wedge Y)$ since $\wedge$ itself is not in general associative. For CW complexes $\wedge$ *is* associative and the proof below shows that the reduced external product is too.

Proof. We need to show that the maps $M \circ (M \wedge \mathrm{id})$ and $M \circ (\mathrm{id} \wedge M)$ from $\wedge^3(\mathbb{Z} \times B\mathrm{U})$ to $\mathbb{Z} \times B\mathrm{U}$ are homotopic. Arguing as in Section 7.2, we have that the restriction map

$$\widetilde{K}^0(\wedge^3(\mathbb{Z} \times B\mathrm{U})) \to \lim_{d,k,n} \widetilde{K}^0(\wedge^3 G_{d,k,n})$$

is injective, and then the required homotopy follows from associativity of the construction (32).

More precisely, for $a, b \geq k$ the virtual bundle (32) is represented up to stable isomorphism by the (actual, not just virtual) vector bundle

$$(E_{k,n} \boxtimes E_{k,n}) \oplus (\underline{\mathbb{C}}^{a-k} \boxtimes E_{k,n}) \oplus (E_{k,n} \boxtimes \underline{\mathbb{C}}^{b-k}),$$

of rank $k(a+b-k)$ inside the trivial bundle of rank $n(n+a+b-2k)$. Meanwhile, for $a \geq k > b$ it is represented by

$$(E_{k,n} \boxtimes E_{k,n}) \oplus (\underline{\mathbb{C}}^{a-k} \boxtimes E_{k,n}) \oplus (E_{k,n}^{\perp} \boxtimes \underline{\mathbb{C}}^{k-b}), \tag{33}$$

of rank $k(a+b-k) + n(k-b)$ inside the trivial bundle of rank $n(n+a-b)$. One can make similar constructions for $b \geq k > a$ and for $k > a, b$, and we conclude that in all cases (32) can be represented by a bundle of rank at most $k^2 + 2(d+k)n$ inside a trivial bundle of rank at most $n^2 + 2(d+k)n$. The actual bounds here are irrelevant; all that matters is that there exist positive integers $\kappa \leq \nu$ depending on d, k and n such that the homotopy class $M_{d,k,n}$ of map

$$G_{d,k,n} \wedge G_{d,k,n} \to \mathbb{Z} \times B\mathrm{U}$$

factors through

$$G_{d^2,\kappa,\nu} \subset \mathbb{Z} \times B\mathrm{U}.$$

(To be concrete, we can take $\kappa = k^2 + 2(d+k)n$ and $\nu = n^2 + k^2 + 4(d+k)n$.)

We can thus express the restriction of $M \circ (M \wedge \mathrm{id})$ to $\wedge^3 G_{d,k,n}$ as

$$\wedge^3 G_{k,d,n} \xrightarrow{M_{d,k,n} \wedge \text{stabilise}} G_{d^2,\kappa,\nu} \wedge G_{d^2,\kappa,\nu} \xrightarrow{M_{d^2,\kappa,\nu}} \mathbb{Z} \times B\mathrm{U},$$

and see that the tautological stable isomorphism class of vector bundle on $\mathbb{Z} \times B\mathrm{U}$ pulls back to the class of

$$(\underline{\mathbb{C}}^{a-k} \oplus E_{k,n}) \boxtimes (\underline{\mathbb{C}}^{b-k} \oplus E_{k,n}) \boxtimes (\underline{\mathbb{C}}^{c-k} \oplus E_{k,n})$$

on

$$(\{a\} \times \mathrm{Gr}(k,n)) \wedge (\{b\} \times \mathrm{Gr}(k,n)) \wedge (\{c\} \times \mathrm{Gr}(k,n))$$

when $a, b, c \geq k$. There are similar expressions along the lines of (33) when a, b or c is less than k. Associativity of the external tensor product of bundles tells us that we get the same answer if we pull back under $M \circ (\mathrm{id} \wedge M)$ instead, and the conclusion is that the K-theory product is associative. ☐

Remark 7.15. This proof looks much worse than it is. Really we're just factoring the first application of M, in the form $M_{d,k,n}$, through $G_{d^2,\kappa,\nu}$ so that we can then use the expression $M_{d^2,\kappa,\nu}$ for the second application of M.

Lemma 7.16. *The K-theory product distributes over addition.*

Proof. Factor the maps through appropriate $M_{d',k',n'}$ as above and use that fact that external tensor product of bundles distributes over direct sum, along with the fact that addition on K-groups corresponds to direct sum of bundles. ☐

Lemma 7.17. *The class $\underline{\mathbb{C}}$ in $K^0(X)$ is a multiplicative identity element.*

Proof. It's the identity for ordinary tensor product of bundles. ☐

We will therefore write $\underline{\mathbb{C}}$ as 1.

Lemma 7.18. *The external K-theory product is intertwined with external tensor product of bundles by the maps $K(\mathrm{Vect}'(-)) \to K^0$.*

Proof. Vector bundles of rank k_X and k_Y over CW complexes X and Y are represented by maps $X \to B\mathrm{U}(k_X)$ and $Y \to B\mathrm{U}(k_Y)$ respectively. Pick $k \geq k_X, k_Y$, and by stabilising view the targets of both maps as $B\mathrm{U}(k)$. By arguments similar to those use for $\mathbb{Z} \times B\mathrm{U}$ we have that the restriction map

$$\widetilde{K}^0(B\mathrm{U}(k) \wedge B\mathrm{U}(k)) \to \lim_n \widetilde{K}^0((\{k_X\} \times \mathrm{Gr}(k,n)) \times (\{k_Y\} \times \mathrm{Gr}(k,n)) \subset G_{k,k,n} \wedge G_{k,k,n})$$

is an isomorphism. And on

$$(\{k_X\} \times \mathrm{Gr}(k,n)) \times (\{k_Y\} \times \mathrm{Gr}(k,n))$$

the map $M_{k,k,n}$ is exactly the external tensor product of bundles of rank k_X and k_Y. ☐

The upshot of all of this is that for any space X the abelian group $K^0(X)$ is naturally a unital commutative ring, and if X is a CW complex then the product operation can be interpreted in terms of the tensor product of bundles. The subgroup $\widetilde{K}^0(X)$ is an ideal.

7.4. **Spheres and the fundamental product theorem.** The first interesting example is the 2-sphere:

Example 7.19. Let H in $K^0(S^2)$ denote the class of $\mathcal{O}_{\mathbb{CP}^1}(1)$. Letting z_0 and z_1 be homogeneous coordinates on $\mathbb{CP}^1$ we have a short exact sequence

$$\begin{gathered} 0 \to \mathcal{O} \to \mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathcal{O}(2) \to 0 \\ 1 \mapsto (z_0, z_1) \end{gathered}$$

and this shows that $\mathcal{O} + \mathcal{O}(2) \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$ as topological vector bundles. Passing to K-theory we get $1 + H^2 = 2H$, or $(H-1)^2 = 0$.

Proposition 7.20. *We have*

$$K^0(S^2) = \mathbb{Z}[H]/(H-1)^2 \quad \textit{and} \quad \widetilde{K}^0(S^2) = \mathbb{Z}[H-1]/(H-1)^2.$$

Proof. We identify $\widetilde{K}^0(S^2) = [S^2, \mathbb{Z} \times B\mathrm{U}]$ with $[S^1, \mathrm{U}] = \pi_1(\mathrm{U})$ by the clutching construction. For each n we have a fibration

$$\mathrm{U}(n) \xrightarrow{\text{first column}} S^{2n-1},$$

with fibre $\mathrm{U}(n-1)$, and an inductive argument using the long exact sequence shows that the inclusion $\mathrm{U}(1) \to \mathrm{U}$ given by

$$e^{i\theta} \mapsto \begin{pmatrix} e^{i\theta} & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}$$

induces an isomorphism on π_1. The image of the standard generator in $\pi_1(\mathrm{U}(1))$, defines the virtual bundle $\mathcal{O}(1) - 1$ by clutching, so translating to K-theory we see that $H - 1$ spans $\widetilde{K}^0(S^2)$. The result now follows from Example 7.19. □

Using the exterior product we can give a more explicit description of the Bott periodicity map:

Theorem 7.21 (Fundamental product theorem)**.** *For any CW complex X the map*

$$\begin{gathered} [X, \mathbb{Z} \times B\mathrm{U}] = \widetilde{K}^0(X) \cong \widetilde{K}^0(S^2) \otimes \widetilde{K}^0(X) \xrightarrow{\boxtimes} \widetilde{K}^0(S^2 \wedge X = \Sigma^2 X) = \widetilde{K}^{-2}(X) = [X, \Omega^2(\mathbb{Z} \times B\mathrm{U})] \\ E \mapsto (\mathcal{O}(1) - 1) \otimes E \end{gathered}$$

is the map induced by the Bott homotopy equivalence $\mathbb{Z} \times B\mathrm{U} \simeq \Omega^2(\mathbb{Z} \times B\mathrm{U})$. In particular, it's an isomorphism.

We will not prove this here. In fact, Bott periodicity follows from the fact that the map is an isomorphism in the case $X = \mathbb{Z} \times B\mathrm{U}$, so the result is genuinely deep.

Corollary 7.22. *For all n the group $\widetilde{K}^0(S^{2n})$ is spanned by $(H-1)^{\boxtimes n}$.*

7.5. **K-theory of $\mathbb{CP}^n$.** Let $\mathcal{L}$ be the class in $K^0(\mathbb{CP}^n)$ corresponding to the line bundle $\mathcal{O}_{\mathbb{CP}^n}(1)$.

Proposition 7.23. *As a $\mathbb{Z}/2$-graded ring, we have*

$$K^*(\mathbb{CP}^n) = \mathbb{Z}[\mathcal{L}]/(\mathcal{L} - 1)^{n+1}.$$

(Since K-theory is 2-periodic in its $\mathbb{Z}$-grading, we may as well collapse it to $\mathbb{Z}/2$.)

Proof. We'll go by induction on n. The long exact sequence associated to the homotopy cofibre sequence $\mathbb{CP}^{n-1} \xrightarrow{i} \mathbb{CP}^n \to Ci \simeq S^{2n}$ is

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, equal] & K^0(S^{2n}) \arrow[r] & K^0(\mathbb{CP}^n) \arrow[r] & K^0(\mathbb{CP}^{n-1}) \arrow[r, equal] \arrow[d] & \mathbb{Z}[\mathcal{L}]/(\mathcal{L}-1)^n \\
0 \arrow[r, equal] & K^1(\mathbb{CP}^{n-1}) \arrow[u] & K^1(\mathbb{CP}^n) \arrow[l] & K^1(S^{2n}) \arrow[l] \arrow[r, equal] & 0,
\end{tikzcd}$$

which shows that $K^1(\mathbb{CP}^n) = 0$ and that additively $K^0(\mathbb{CP}^n)$ is free of rank $n+1$.

Now note that the homogeneous coordinate functions $z_0, \ldots, z_n$ define a nowhere-zero section of $E := \mathcal{O}(1)^{\oplus(n+1)}$, with associated Koszul complex

$$0 \to \Lambda^0 E = \mathcal{O} \to E \to \Lambda^2 E \to \cdots \to \Lambda^{n+1} E \to 0.$$

The term $\Lambda^r E$ is exactly $\mathcal{O}(r)^{\oplus\binom{n+1}{r}}$, so in K-theory this gives the relation

$$\sum_{r=0}^{n+1}(-1)^r\binom{n+1}{r}\mathcal{L}^r = 0,$$

or in other words $(\mathcal{L}-1)^{n+1} = 0$. This demonstrates that $1, \dots, \mathcal{L}^n$ span all powers of $\mathcal{L}$.

It's now just left to show that the powers of $\mathcal{L}$ span $K^0(\mathbb{CP}^n)$, or in other words that any vector bundle on $\mathbb{CP}^n$ can be written as a formal difference of sums of the $\mathcal{O}(r)$. Suppose then that we're given an arbitrary vector bundle F. By approximating the transition functions by polynomials, we may assume that F is actually holomorphic. Beilinson's 'resolution of the diagonal' argument (see [12, Section 2.1]) then gives an explicit resolution for such an F in terms of $\mathcal{O}, \dots, \mathcal{O}(1-n)$, and translating to K-theory we get exactly what we want. ☐

Remark 7.24. This can be proved without appealing directly to holomorphic approximation or to Beilinson, using the fundamental product theorem and Chern character.

From the Milnor sequence we obtain:

Corollary 7.25.

$$K^*(\mathbb{CP}^\infty) = \mathbb{Z}[\![\mathcal{L}-1]\!]$$

Remark 7.26. This is *not* the same as $\mathbb{Z}[\![\mathcal{L}]\!]$: the latter does not contain

$$1-(\mathcal{L}-1)+(\mathcal{L}-1)^2-(\mathcal{L}-1)^3+\dots$$

for example.

7.6. **The K-theory product in cohomology.** We saw in Remark 7.8 that the K-theory addition operation $\oplus : B\mathrm{U}\times B\mathrm{U}\to B\mathrm{U}$ induces the cup product on total Chern classes. In other words, as a map

$$\oplus^* : H^{**}(B\mathrm{U};\mathbb{Z}) = \mathbb{Z}[\![c_1, c_2, \dots]\!] \to H^{**}(B\mathrm{U}\times B\mathrm{U};\mathbb{Z}) = \mathbb{Z}[\![c_1, c_2, \dots, c_1', c_2', \dots]\!]$$

we have

$$\oplus^*(1+c_1+c_2+\dots) = (1+c_1+c_2+\dots)(1+c_1'+c_2'+\dots).$$

By looking at each graded piece in turn, this completely determines $\oplus^*$. We now wish to describe the map

$$\boxtimes^* : H^{**}(\mathbb{Z}\times B\mathrm{U}) = \prod_{a=-\infty}^{\infty}\mathbb{Z}[\![c_i]\!]_{i=1}^{\infty} \to H^{**}((\mathbb{Z}\times B\mathrm{U})\times(\mathbb{Z}\times B\mathrm{U})) = \prod_{a,b=-\infty}^{\infty}\mathbb{Z}[\![c_i, c_i']\!]_{i=1}^{\infty},$$

where c_i in the ath factor on the left-hand side (respectively c_i and c_i' in the (a,b)th factor on the right-hand side) is the ith Chern classes on $\{a\}\times B\mathrm{U}$ (respectively $\{(a,b)\}\times B\mathrm{U}\times\{*\}$ and $\{(a,b)\}\times\{*\}\times B\mathrm{U}$).

Well, the tuple $(1+c_1+c_2+\dots)_a$ of total Chern classes is pulled back by $\boxtimes^*$ to some doubly-indexed tuple $(f_{a,b})_{a,b}$ of power series in the c_i and c_i', and again by looking at the graded pieces these $f_{a,b}$ completely determine the map $\boxtimes^*$. By construction of $\boxtimes$, we have that $f_{a,b}$ is the limit of the total Chern classes of the virtual bundles

$$E_k\boxtimes E_k + (a-k)\underline{\mathbb{C}}\boxtimes E_k + (b-k)E_k\boxtimes\underline{\mathbb{C}} + (a-k)(b-k)\underline{\mathbb{C}}\boxtimes\underline{C}$$

on $B\mathrm{U}(k)\times B\mathrm{U}(k)$, where E_k is the tautological bundle. We can compute these Chern classes by pulling back from $B\mathrm{U}(k)$ to BT^k—where c_i and c_i' become the ith elementary symmetric polynomials in variables x_l and x_l', with l ranging from 1 to k—and splitting the tautological bundles into sums of lines. We get

$$f_{a,b} = \lim_k\left(\prod_{i,j=1}^{k}(1+x_i+x_j')\right)\left(\prod_{j=1}^{k}(1+x_j')\right)^{a-k}\left(\prod_{i=1}^{k}(1+x_i)\right)^{b-k}. \tag{34}$$

Remark 7.27. In the right-hand side of (34) the three factors do not each have well-defined limits. For example, the coefficient of each x_l in the first factor tends to infinity with k. However, we can rewrite it as

$$\lim_k\left(\prod_{i,j=1}^{k}\frac{1+x_i+x_j'}{(1+x_i)(1+x_j')}\right)\left(\prod_{j=1}^{k}(1+x_j')\right)^{a}\left(\prod_{i=1}^{k}(1+x_i)\right)^{b},$$

and now each of the three factors *does* have a well-defined limit.

7.7. **The Chern character.** Recall the functor

$$H^{**} = \prod_{i=0}^{\infty} H^{2i}(-;\mathbb{Z})$$

from Section 7.1. We have already constructed (although not quite in this form) the total Chern class $1 + c_1 + c_2 + \dots$, which defines a natural transformation $[-, B\mathrm{U}] \to H^{**}$ that intertwines $\oplus$ on $B\mathrm{U}$ with cup product on H^{**} (in our setting, naturality just means that it commutes with pullbacks). In particular, it turns stable isomorphism classes of vector bundles into cohomology classes in a way which respects pullbacks and turns direct sums of bundles into cup products. This is a powerful invariant, but it has the inconvenience of interacting poorly with tensor products.

To remedy this we will now construct the *Chern character*, which is a natural *ring* homomorphism $K^0 \to H^{**}_{\mathbb{Q}}$, where

$$H^{**}_{\mathbb{Q}} = \prod_{i=0}^{\infty} H^{2i}(-;\mathbb{Q})$$

(the necessity of denominators will become clear later). Here 'natural ring homomorphism' means natural transformation of functors $\mathsf{hTop}_* \to \mathsf{Ring}$. The construction relies on some abstract nonsense which showcases the power of the spectrum approach to K-theory.

The key gadget is:

Lemma 7.28 (The Yoneda lemma). *Let $\mathscr{C}$ be a category, A an object of $\mathscr{C}$, and $F : \mathscr{C} \to \mathsf{Set}$ a contravariant functor. Viewing $\mathscr{C}(-, A)$ as another contravariant functor $\mathscr{C} \to \mathsf{Set}$, the map*

$$\{\textit{natural transformations } \mathscr{C}(-, A) \to F\} \to F(A)$$

given by $\eta \mapsto \eta(\mathrm{id}_A)$ is a bijection (and is natural in every conceivable way).

This means that giving a natural transformation $\eta : \widetilde{K}^0 \to H^{**}_{\mathbb{Q}}$ of functors $\mathsf{hTop}_* \to \mathsf{Set}$ is equivalent to specifying an element Y_η of $H^{**}_{\mathbb{Q}}(\mathbb{Z} \times B\mathrm{U})$, which is precisely a tuple f_a of power series in the c_i with rational coefficients.

In general, a natural transformation constructed by Yoneda will not respect the addition or multiplication, since the lemma is about Set-valued functors. Respect for addition is encoded by commutativity of the following diagram

$$\begin{tikzcd}
{[-, (\mathbb{Z} \times B\mathrm{U}) \times (\mathbb{Z} \times B\mathrm{U}')]} \arrow[r, "\oplus\circ"] \arrow[d, equal] & {[-, \mathbb{Z} \times B\mathrm{U}]} \arrow[dd, "\eta"] \\
{[-, \mathbb{Z} \times B\mathrm{U}] \times [-, \mathbb{Z} \times B\mathrm{U}]} \arrow[d, "\eta\times\eta"] & \\
H^{**}_{\mathbb{Q}} \times H^{**}_{\mathbb{Q}} \arrow[r, "+"] & H^{**}_{\mathbb{Q}}
\end{tikzcd}$$

where $\oplus$ is the addition operation on $\mathbb{Z} \times B\mathrm{U}$. The two paths from top left to bottom right are natural transformations $[-, (\mathbb{Z} \times B\mathrm{U}) \times (\mathbb{Z} \times B\mathrm{U})] \to H^{**}_{\mathbb{Q}}$ so are described by elements of

$$H^{**}_{\mathbb{Q}}((\mathbb{Z} \times B\mathrm{U}) \times (\mathbb{Z} \times B\mathrm{U})),$$

i.e. doubly-indexed tuples of rational power series in $c_1, c_2, \dots$ and $c_1', c_2', \dots$. Call these elements l and r for the lower left and upper right paths respectively; explicitly they are the images of $\mathrm{id}_{(\mathbb{Z}\times B\mathrm{U})^2}$ under the two paths.

We can now simply read off that

$$l = \mathrm{pr}_1^* Y_\eta + \mathrm{pr}_2^* Y_\eta,$$

where pr_i is projection to the ith factor of $(\mathbb{Z} \times B\mathrm{U}) \times (\mathbb{Z} \times B\mathrm{U})$, whilst

$$r = \oplus^* Y_\eta.$$

In other words

$$l_{a,b} = f_a(c_1, c_2, \dots) + f_b(c_1', c_2', \dots)$$

whilst

$$r_{a,b} = f_{a+b}(c_1 + c_1', c_2 + c_1 c_1' + c_2', \dots).$$

Viewing c_i and c'_i as the ith elementary symmetric polynomials in x_j andx'_j respectively, we deduce that η respects addition if and only if for all a and b we have

$$f_a(x_1, x_2, \dots) + f_b(x'_1, x'_2, \dots) = f_{a+b}(x_1, x'_1, x_2, x'_2, \dots). \tag{35}$$

Note that since $f_{a,b}$ is symmetric the ordering of the variables on the right-hand side is irrelevant.

For multiplication we can do exactly the same, but the expression (34) is more complicated than the pullback formula for addition. If the transformation respects addition, however, then we can simplify the condition for respect for multiplication to

$$\begin{aligned}\lim_k f_a(x_1, \dots, x_k, 0, \dots) f_b(x'_1, \dots, x'_k, 0, \dots) \\ = \lim_k f_{ab}(x_1 + x'_1, x_1 + x'_2, x_2 + x'_1, \dots, x_k + x'_k, 0, \dots) \\ + (a-k) f_0(x'_1, \dots, x'_k, 0, \dots) + (b-k) f_0(x_1, \dots, x_k, 0, \dots)\end{aligned} \tag{36}$$

for all a and b.

We obtain:

Proposition 7.29. *A natural transformation* $\eta : \widetilde{K}^0 \to H^{**}_{\mathbb{Q}}$ *of* Set*-valued functors, represented by a tuple of symmetric power series* (f_a) *in the variables* $x_1, x_2, \dots$ *with rational coefficients, respects addition and multiplication if and only if* (35) *and* (36) *hold.*

We now solve these constraints.

Lemma 7.30. *The transformation* η *respects addition if and only if there exist* λ *in* $\mathbb{Q}$ *and a single-variable rational power series* g *with vanishing constant term such that each* f_a *is of the form*

$$f_a(x_1, x_2, \dots) = \lambda a + \sum_i g(x_i)$$

Proof. The if direction is clear, so suppose conversely that η respects addition. Let $\lambda = f_1(0, 0, \dots)$ and let $g(x) = f_0(x, 0, 0, \dots)$. By repeatedly applying (35) we get

$$f_a(x_1, x_2, \dots) = \lambda a + f_0(x_1, x_2, \dots)$$

for all a. Note also that the $a = b = 0$ case of (35) tells us that f_0 and g have no constant terms. We claim that $f_0(x_1, x_2, \dots) = \sum_i g(x_i)$, which implies the lemma.

To prove the claim, observe that (35) implies that

$$f_0(x_1, x_2, \dots) = f_0(x_2, x_3, \dots) + g(x_1)$$

so the terms involving x_1 on the left-hand side are precisely the terms in $g(x_1)$. In particular there are no cross terms involving both x_1 and some other x_i. By symmetry of f_0 we conclude that $f_0(x_1, x_2, \dots) = \sum_i g(x_i)$, completing the proof. □

Proposition 7.31. *The natural transformation* η *respects both addition and multiplication if and only if there exists* μ *in* $\mathbb{Q}$ *such that*

$$f_a(x_1, x_2, \dots) = a + \sum_i (e^{\mu x_i} - 1)$$

for all a.

Proof. By (36) and Lemma 7.30 we know that η respects addition and multiplication if and only if the f_a are of the form

$$f_a(x_1, x_2, \dots) = \lambda a + \sum_i g(x_i)$$

and

$$\begin{aligned}\lim_k \lambda^2 ab + \lambda a \sum_{j=1}^k g(x'_j) + \lambda b \sum_{i=1}^k g(x_i) + \sum_{i,j=1}^k g(x_i) g(x'_j) \\ = \lim_k \lambda ab + \sum_{i,j=1}^k g(x_i + x'_j) + (a-k) \sum_{j=1}^k g(x'_j) + (b-k) \sum_{i=1}^k g(x_i)\end{aligned} \tag{37}$$

for all a and b.

Setting $a = b = x_i = x'_j = 0$ for $i, j \geq 2$, and letting $h = g + 1$, the latter condition becomes

$$h(x_1)h(x'_1) = h(x_1 + x'_1).$$

Recalling that g has vanishing constant term, we see that $h(x)$ must begin $1 + \mu x + \dots$ for some rational μ, and the coefficients can then be inductively computed to give $h(x) = e^{\mu x}$. Plugging back into the general version of (37) we see that we have a solution if and only if $\lambda = 1$. □

Definition 7.32. The *Chern character* ch is the natural ring homomorphism $K^0 \to H^{**}_{\mathbb{Q}}$ defined by

$$f_a(x_1, x_2, \dots) = a + \sum_i (e^{x_i} - 1).$$

Concretely, it sends a sum of line bundles $\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$ to

$$\sum_{i=1}^{n} e^{c_1(\mathcal{L}_i)}.$$

Recalling that the Chern classes are the elementary symmetric polynomials in the x_i, for a general rank n bundle E we have

$$\operatorname{ch}(E) = n + c_1(E) + \frac{c_1(E)^2 - 2c_2(E)}{2!} + \frac{c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)}{3!} + \dots. \tag{38}$$

Remark 7.33. The later terms in (38) become increasingly cumbersome, but can be worked out using Newton's identities. One useful point to note is that the coefficient of c_n in the degree n part is $(-1)^{n-1}/(n-1)!$. To prove this it suffices to show that when $\sum_i x_i^n$ is expressed as a polynomial in the elementary symmetric polynomials $c_1, c_2, \dots, c_n$ the coefficient of c_n is $(-1)^{n-1}n$. And to do this simply substitute $x_i = \zeta^i$ for $i = 1, \dots, n$, where ζ is a primitive nth root of unity, and $x_i = 0$ for $i > n$. The expressions $c_1, \dots, c_{n-1}$ all vanish, c_n becomes $(-1)^{n-1}$, and $\sum_i x_i^n$ becomes n.

If E and F are K-theory classes on spaces X and Y respectively then since ch is a ring map we have that

$$\operatorname{ch}(\pi_1^* E \otimes \pi_2^* F) = \operatorname{ch}(\pi_1^* E) \operatorname{ch}(\pi_2^* F),$$

and hence

$$\operatorname{ch}(E \boxtimes F) = \operatorname{ch}(E) \times \operatorname{ch}(F).$$

In other words, the Chern character respects the external product as well as the internal one.

7.8. **Almost complex structures on spheres.** We end with a fun application:

Theorem 7.34. *If S^{2n} admits an almost complex structure then n is 1, 2 or 3.*

Remark 7.35. Using 4-manifold theory the $n = 2$ case can be ruled out. S^2 has an integrable complex structure coming from its identification with $\mathbb{CP}^1$ whilst S^6 has a non-integrable almost complex structure coming from octonion multiplication. It is unknown whether S^6 admits an integrable complex structure.

Proof. Recall that by the fundamental product theorem $K^0(S^{2n})$ is spanned by 1 and by $(H-1)^{\boxtimes n}$. These are sent to 1 and $[S^2]^{\times n} \in H^{2n}(S^{2n} = \wedge^n S^2; \mathbb{Z}) \subset H^{2n}(\times^n S^2; \mathbb{Z})$ respectively by ch, and since $[S^2]^{\times n}$ is exactly the fundamental class $[S^{2n}]$ we see that ch in fact defines an isomorphism

$$K^0(S^{2n}) \to H^*(S^{2n}; \mathbb{Z})$$

(note the $\mathbb{Z}$ on the right-hand side, not $\mathbb{Q}$).

Now suppose that E is a complex vector bundle over S^{2n} and consider $\operatorname{ch}(E)$. The degree n part is exactly $(-1)^{n-1}c_n(E)/(n-1)!$, by Remark 7.33, and since ch lands in the *integral* cohomology of S^{2n} we deduce that the top Chern number of E is divisible by $(n-1)!$.

Finally suppose that S^{2n} admits an almost complex structure, and take E to be its tangent bundle. The top Chern number is then the Euler characteristic of S^{2n}, i.e. 2, so we see that $(n-1)!$ divides 2. This is only possible if $n \leq 3$. □

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