Polarizations, torsors and theta groups

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Abstract

Let $\lambda\colon A \to A^\vee$ be a polarization on an abelian variety over a field k. If k is not algebraically closed, there might not exist an ample line bundle on A defined over k that represents λ. To remedy this, Poonen and Stoll have asked the following question: does there exist a line bundle on an A-torsor that represents λ?

We give a criterion for the existence of such a torsor and line bundle which only depends on the kernel of λ. Using this criterion, we show that the answer to the question is yes when the polarization has odd or small even degree. On the other hand, we show that for every $g \geq 7$, there exists a polarized g-dimensional abelian variety for which the answer to the question is no.

1 Introduction

1.1 Context

Let A be an abelian variety over a field k. If L is a line bundle on A, define the homomorphism

$$\phi_L\colon A \to A^\vee,\ a \mapsto t_a^* L \otimes L^{-1},$$

where $t_a\colon A \to A$ denotes translation by a and $A^\vee = \mathbf{Pic}^0_A$ denotes the dual abelian variety. By definition, a polarization on A is a homomorphism $\lambda\colon A \to A^\vee$ such that, over an algebraic closure $\bar{k}$ of k, $\lambda_{\bar{k}} = \phi_L$ for some ample line bundle L on $A_{\bar{k}}$. It is natural to ask: is this base change to $\bar{k}$ necessary? In other words, can we choose L to be defined over k? The following example shows that this is not always possible:

Example. *Let C/k be a (smooth, projective, geometrically connected) curve of genus g with Jacobian variety $J = \mathbf{Pic}^0_C$, a g-dimensional abelian variety. The locus of effective line bundles defines a divisor $\Theta \subset \mathbf{Pic}^{g-1}_C$, the theta divisor. If $x \in \mathbf{Pic}^{g-1}_C(k)$, then the translate of $\mathcal{O}_{\mathbf{Pic}^{g-1}_C}(\Theta)$ by x defines a line bundle L_x on $J_{\bar{k}}$. The morphism $\phi_{L_x}\colon J_{\bar{k}} \to J^\vee_{\bar{k}}$ is independent of the choice of x, hence descends to a morphism $\lambda\colon J \to J^\vee$; this is the canonical principal polarization on a Jacobian. The association $x \mapsto L_x$ induces a bijection between $\mathbf{Pic}^{g-1}_C(k)$ and the set $\{[L] \in \mathrm{Pic}(J)\colon \phi_L = \lambda\}$. In particular, the latter set is empty if $\mathbf{Pic}^{g-1}_C(k) = \varnothing$. If $g = 2$, there are many curves C over $\mathbb{Q}$ with $\mathbf{Pic}^{g-1}_C(\mathbb{Q}) = \varnothing$; see [PS99, Theorem 23].*

In this example, the polarization on J is still constructed by a line bundle that is defined over k, namely the one induced by the theta divisor on the J-torsor $\mathbf{Pic}^{g-1}_C$. In general, given an A-torsor X and a line bundle L on X, the definition of $\phi_L\colon A \to A^\vee$ still makes sense since the translation-by-a map $t_a\colon X \to X$ is well defined and since there is a canonical identification $\mathbf{Pic}^0_X = \mathbf{Pic}^0_A$, see Section 2.4 for details. The following basic question is therefore very natural, and has been explicitly raised by Poonen and Stoll [PS99, §4, p. 1120] around 25 years ago.

Question 1. *If $\lambda\colon A \to A^\vee$ is a polarization on an abelian variety, does there always exist an A-torsor X and a line bundle L on X such that $\lambda = \phi_L$?*

This question also relates to the works of Alexeev and Olsson on compactifying moduli spaces of polarized abelian varieties [Ale02, Ols08]: in these works the moduli space of pairs (A, λ) is replaced by a moduli space of triples (A, X, L), and it is natural to wonder whether the association $(A, X, L) \mapsto (A, \phi_L)$ admits an inverse.

It is known that the answer to Question 1 is yes in all of the following cases:

- (A, λ) is the Jacobian of a curve C with its natural principal polarization; take $X = \mathbf{Pic}^{g-1}_C$ and $L = \mathcal{O}_X(\Theta)$. More generally, (A, λ) is a Prym variety; see [PS99, §4, p. 1120].
- k is a local field; then there even exists a line bundle L on A with $\lambda = \phi_L$ by [PS99, §4, Lemma 1].
- $\lambda = 2\mu$ for some polarization μ; in that case the pullback $L = (1, \mu)^*\mathcal{P}$ of the Poincaré bundle $\mathcal{P}$ on $A \times A^\vee$ satisfies $\phi_L = \lambda$ by [MFK94, Proposition 6.10].
- At least one of the conditions of [PR11, Proposition 3.12] is satisfied (for example, k is finite); in that case there even exists a *symmetric* L on A representing λ.

The definition of ϕ_L even makes sense if we only assume the isomorphism class of L to be Galois-invariant, i.e., if L defines an element of $\mathbf{Pic}_X(k) = \mathrm{Pic}(X_{k^{\mathrm{sep}}})^{\mathrm{Gal}(k^{\mathrm{sep}}|k)}$, where k^{sep} is a separable closure of k. We can therefore ask the (a priori) weaker variant of Question 1:

Question 2. *If $\lambda\colon A \to A^\vee$ is a polarization on an abelian variety over k, does there always exist an A-torsor X and an element $[L] \in \mathrm{Pic}(X_{k^{\mathrm{sep}}})^{\mathrm{Gal}(k^{\mathrm{sep}}|k)}$ such that $\lambda = \phi_L$?*

The purpose of this paper is to show that even though Questions 1 and 2 often have a positive answer, they have a negative answer in general.

1.2 Results

Theorem A. *Let $\lambda\colon A \to A^\vee$ be a polarization whose degree is odd and invertible in k. Then there exists an A-torsor X and a line bundle L on X with $\phi_L = \lambda$.*

To state a more general theorem, recall that if $\lambda\colon A \to A^\vee$ is a polarization whose degree is invertible in k, there exists a unique sequence of positive integers $d_1, \ldots, d_g$, each one dividing the next, with $g = \dim(A)$ and such that the kernel $A[\lambda]$ of λ satisfies

$$A[\lambda](\bar{k}) \simeq (\mathbb{Z}/d_1\mathbb{Z})^2 \times \cdots \times (\mathbb{Z}/d_g\mathbb{Z})^2.$$

The tuple $D = (d_1, \ldots, d_g)$ is called the type of (A, λ). For each i, let 2^{n_i} be the largest power of 2 dividing d_i and write $D_2 = (2^{n_1}, \ldots, 2^{n_g})$.

Theorem B. *Let $\lambda\colon A \to A^\vee$ be a polarization whose degree is invertible in k. Suppose that $D_2 = (1, \ldots, 1)$, $(1, \ldots, 1, 2)$, $(1, \ldots, 1, 2, 2)$ or $(1, \ldots, 1, 2, 2, 2)$. In the last case, additionally suppose that k has characteristic zero. Then there exists an A-torsor X and a line bundle L on X with $\phi_L = \lambda$.*

Using Theorem B and previous results, we show that the answer to Question 1 is always yes when $\dim A \leq 2$, see Proposition 4.13. However, this pattern does not persist: the next theorem shows that there exist polarized abelian varieties of every dimension at least 7 for which the answer to Questions 1 and 2 is no.

Theorem C. *Consider a sequence of the form*

$$D = (\overbrace{1,\dots,1}^{\geq 3 \text{ times}}, \overbrace{2,\dots,2}^{\geq 4 \text{ times}}).$$

Then there exists a field of characteristic zero k and a polarized abelian variety (A,λ) of type D over k with the property that there does not exist an A-torsor X and an element $[L] \in \mathbf{Pic}_X(k)$ such that $\lambda = \phi_L$. In other words, the answer to Question 2 (and hence to Question 1) is no for (A,λ).

The counterexample of Theorem C is generic, in the following sense. For a type $D = (d_1,\dots,d_g)$, let $\mathcal{A}_{g,D} \to \operatorname{Spec}(\mathbb{C})$ be the moduli stack of g-dimensional abelian varieties with a polarization of type D. If $p \geq 3$ is a prime not dividing d_g, let $\mathcal{A}_{g,D}[p] \to \mathcal{A}_{g,D}$ be the cover given by adding full level-p structure, a smooth quasi-projective scheme. Denote the generic fiber of the universal abelian scheme over $\mathcal{A}_{g,D}[p]$ by A; this is an abelian variety over the function field $k = \mathbb{C}(\mathcal{A}_{g,D}[p])$ equipped with a polarization λ of type D. We prove that if D is of the form of Theorem C, then the answer to Question 2 is no for (A,λ). We don't know if such examples exists over arithmetically interesting fields such as $\mathbb{Q}$ or number fields. We also don't know whether the answer to Question 1 is always yes when $\dim A \in \{3,4,5,6\}$; this would involve analyzing types D where some d_i is divisible by 4. See Section 4.4 for a survey of known results towards Question 1 and its variants.

1.3 Methods

Assume that the degree of $\lambda\colon A \to A^\vee$ is invertible in k and let $D = (d_1,\dots,d_g)$ be the type of (A,λ). Then the kernel $A[\lambda]$ is a finite étale group scheme (in other words, a Galois module) of order $(d_1\cdots d_g)^2$ equipped with an alternating nondegenerate pairing $e_\lambda\colon A[\lambda]\times A[\lambda] \to \mathbb{G}_m$, the Weil pairing. The key first step is to prove that the answer to Question 1 only depends on the isomorphism class of the pair $(A[\lambda], e_\lambda)$, as we now explain.

Define a *theta group* for $(A[\lambda], e_\lambda)$ to be a central extension of algebraic k-groups

$$1 \to \mathbb{G}_m \to \mathcal{G} \to A[\lambda] \to 1$$

such that for all $\tilde{x},\tilde{y} \in \mathcal{G}$ lifting $x,y \in A[\lambda]$, the commutator $[\tilde{x},\tilde{y}] \in \mathbb{G}_m$ equals $e_\lambda(x,y)$. If k is algebraically closed, there is a unique isomorphism class of theta groups for $(A[\lambda], e_\lambda)$. In general, both the existence and uniqueness of theta groups can fail. The primordial example of a theta group is due to Mumford [Mum66]: if L is a line bundle on A with $\phi_L = \lambda$, then

$$\mathcal{G}(L) = \{(a,\varphi)\colon a \in A[\lambda],\ \varphi\colon L \xrightarrow{\sim} t_a^*L\}$$

is a theta group for $(A[\lambda], e_\lambda)$. We call a theta group $\mathcal{G}$ for $(A[\lambda], e_\lambda)$ *linear* if it admits an algebraic representation of dimension $d_1\cdots d_g$ on which $\mathbb{G}_m$ acts via scalar multiplication. The formula $(a,\varphi)\cdot s = t_{-a}^*(\varphi(s))$ defines an action of $\mathcal{G}(L)$ on $\mathrm{H}^0(A,L)$ (which has dimension $d_1\cdots d_g$ by Riemann–Roch), showing that Mumford theta groups $\mathcal{G}(L)$ are linear. In this notation, we can state:

Theorem Θ. *Let $\lambda\colon A \to A^\vee$ be a polarization whose degree is invertible in k. Then the following statements are equivalent:*

1. *There exists an A-torsor X and a line bundle L on X such that $\lambda = \phi_L$.*

2. There exists a linear theta group for $(A[\lambda], e_\lambda)$.

The following statements are also equivalent:

1. There exists an A-torsor X and an element $[L] \in \mathbf{Pic}_X(k) = \mathrm{Pic}(X_{k^{\mathrm{sep}}})^{\mathrm{Gal}(k^{\mathrm{sep}}|k)}$ such that $\lambda = \phi_L$.

2. There exists a theta group for $(A[\lambda], e_\lambda)$.

The implications (1) $\Rightarrow$ (2) are straightforward and follow from generalizing the definition of $\mathcal{G}(L)$ to line bundles on A-torsors. The reverse implications are more significant. Once the situation is appropriately categorified, they follow from the fact that a fully faithful morphism between two gerbes is an isomorphism. Theorem Θ can be upgraded to an equivalence of categories and works over an arbitrary base scheme; see Theorems 3.18 and 3.24. We also prove a version of Theorem Θ for symmetric line bundles on "symmetric torsors"; see Theorem 3.31 and Corollary 3.32.

To prove Theorem A, it suffices to prove by Theorem Θ that if the degree of λ is odd then there exists a linear theta group for $(A[\lambda], e_\lambda)$. In this case, we can construct such a group "by hand". Since the degree is odd the alternating pairing e_λ admits a square root, i.e., there exists an alternating pairing b on $A[\lambda]$ such that $b^2 = e$. Endowing $\mathcal{G} = \mathbb{G}_m \times M$ with the multiplication $(\lambda, x) \cdot (\lambda', y) = (\lambda\lambda' b(x, y), x + y)$ defines a theta group for $(A[\lambda], e_\lambda)$, and a result of Polishchuk [Pol02] shows that $\mathcal{G}$ is linear. Again, Theorem A works over an arbitrary base. If we insist that the pair (X, L) representing λ is symmetric in a certain sense, then this pair is essentially unique; see Corollary 4.4.

The proof of Theorem B relies on the following surprising consequence of Theorem Θ: if there exists a different polarized abelian variety (B, μ) and an isomorphism $(A[\lambda], e_\lambda) \simeq (B[\mu], e_\mu)$, then the answer to Question 1 is the same for (A, λ) and (B, μ)! To see this in action, consider the simplest nontrivial case where (A, λ) is of type $D = (1, \ldots, 1, 2)$, so that $A[\lambda](\bar{k}) \simeq (\mathbb{Z}/2\mathbb{Z})^2$. The Galois action on $A[\lambda]$ is encoded by a representation $\rho_{A,\lambda} \colon \mathrm{Gal}_k \to \mathrm{Sp}_2(\mathbb{F}_2) = \mathrm{GL}_2(\mathbb{F}_2)$, where $\mathrm{Gal}_k = \mathrm{Gal}(k^{\mathrm{sep}}|k)$. But every such representation is isomorphic to the 2-torsion representation of an elliptic curve: there exists an elliptic curve E/k and an isomorphism of Galois modules $A[\lambda] \simeq E[2]$ intertwining the Weil pairings on both sides. Since the answer to Question 1 is clearly yes for E with the polarization $[2] \colon E \to E$, the same is true for (A, λ). In the cases where $D = (1, \ldots, 1, 2, 2)$ or $(1, \ldots, 1, 2, 2, 2)$, we similarly show that there exists a curve C/k with Jacobian J and an isomorphism $(A[\lambda], e_\lambda) \simeq (J[2], e_2)$. In the first case, we use genus-2 hyperelliptic curves and the exceptional isomorphism $\mathrm{Sp}_4(\mathbb{F}_2) \simeq S_6$. In the second case, we use plane quartic curves with a flex point and the isomorphism $\mathrm{Sp}_6(\mathbb{F}_2) \simeq W(E_7)^+$, where $W(E_7)^+$ is the index-2 subgroup of the Weyl group of E_7. As a by-product of our argument, we answer a question of Chidambaram [Chi24, Question 1.2] affirmatively.

To explain the proof of Theorem C, we relate our problem to Galois cohomology. For simplicity, assume that k is of characteristic zero and contains a primitive n-th root of unity ζ_n for all $n \geq 1$. Define the abelian group $M_D = (\mathbb{Z}/d_1\mathbb{Z})^2 \times \cdots \times (\mathbb{Z}/d_g\mathbb{Z})^2$ and let $e_D \colon M_D \times M_D \to k^\times$ be the direct sums of the standard alternating pairings e_i on $(\mathbb{Z}/d_i\mathbb{Z})^2$ defined by $e_i(v, w) = \zeta_{d_i}^{\det(v\, w)}$. Then $(A[\lambda], e_\lambda)$ is a twist of the split pair (M_D, e_D), in the sense that there exists an isomorphism $A[\lambda](\bar{k}) \simeq M_D$ intertwining e_λ with e_D. A choice of such an isomorphism determines a continuous homomorphism $\rho_{A,\lambda} \colon \mathrm{Gal}_k \to \mathrm{Sp}(M_D)$, where we write $\mathrm{Sp}(M_D) = \mathrm{Aut}(M_D, e_D)$.

We can also explicitly construct a "split" theta group $\mathcal{G}_D$ for the split pair (M_D, e_D); see Section 3.2. Let $\mathrm{Aut}(\mathcal{G}_D)$ be the group of automorphisms $\mathcal{G}_D$ that induce the identity on $\mathbb{G}_m$. This determines an exact sequence of finite groups

$$1 \to M_D \to \mathrm{Aut}(\mathcal{G}_D) \to \mathrm{Sp}(M_D) \to 1, \tag{1.1}$$

where M_D is identified with the set of inner automorphisms of $\mathcal{G}_D$. Since theta groups for $(A[\lambda], e_\lambda)$ are twists of $\mathcal{G}_D$, Theorem Θ has the following concrete consequence (Lemma 5.2): the answer to Question 2 is yes if and only if

$\rho_{A,\lambda}\colon \mathrm{Gal}_k \to \mathrm{Sp}(M_D)$ lifts to a homomorphism $\tilde{\rho}_{A,\lambda}\colon \mathrm{Gal}_k \to \mathrm{Aut}(\mathcal{G}_D)$ under (1.1).

Lifting problems (sometimes called embedding problems) for Galois representations such as this one are well studied, see for example [ILF97]. Let $c_D \in \mathrm{H}^2(\mathrm{Sp}(M_D), M_D)$ be the element in group cohomology classifying the extension (1.1). If $d_1 \dots d_g$ is odd, then (1.1) canonically splits and hence $c_D = 0$; this explains the bare hands construction of a theta group for $(A[\lambda], e_\lambda)$ in the proof of Theorem A. On the other hand, if λ has even degree then (1.1) is typically not split. For example, if $D = (1, \dots, 1, 2, \dots, 2)$, where 2 occurs m times, then $c_D \neq 0$ if $m \geq 3$. Even so, the proof of Theorem B shows that if $m = 3$ then every continuous representation $\rho\colon \mathrm{Gal}_k \to \mathrm{Sp}(M_D)$ lifts to a representation $\tilde{\rho}\colon \mathrm{Gal}_k \to \mathrm{Aut}(\mathcal{G}_D)$! Serre has studied this curious phenomenon and has dubbed it negligible cohomology [Ser02, Chapter III, Appendix 2]. Generally speaking, if K is a field, G is a finite group and M a G-module, a group cohomology class $c \in \mathrm{H}^i(G, M)$ is said to be *negligible over* K if for every field extension L/K and continuous homomorphism $f\colon \mathrm{Gal}_L \to G$, $f^*(c) = 0$ in $\mathrm{H}^i(\mathrm{Gal}_L, M)$. In this terminology, every Galois representation $\mathrm{Gal}_k \to \mathrm{Sp}(M_D)$ lifts to $\mathrm{Aut}(\mathcal{G}_D)$ if and only if $c_D \in \mathrm{H}^2(\mathrm{Sp}(M_D), M_D)$ is negligible over $\mathbb{Q}(\mu_\infty) = \cup_{n\geq 1}\mathbb{Q}(\zeta_n)$.

In a remarkable recent advance, Merkurjev and Scavia have determined the subgroup of negligible classes of $\mathrm{H}^2(G, M)$ over fields containing sufficiently many roots of unity [MS25]. As an application of their result, they show that there exist 3-dimensional Galois representations mod p that do not lift mod p^2 for every odd prime p. Applying their computation to our setting, we show that c_D is *not* negligible if $D = (1, \dots, 1, 2, \dots, 2)$ and 2 occurs at least 4 times. This implies that there exists a field k and Galois representation $\rho\colon \mathrm{Gal}_k \to \mathrm{Sp}(M_D)$ that does not lift to $\mathrm{Aut}(\mathcal{G}_D)$. However, this does not quite show Theorem C yet, since we don't know whether the non-liftable representation $\rho\colon \mathrm{Gal}_k \to \mathrm{Sp}(M_D)$ is of the form $\rho_{A,\lambda}$ for some polarized abelian variety (A, λ) of type D.

To overcome this wrinkle, we use an even more recent result of Totaro who generalized the Merkurjev–Scavia computation to the setting of étale cohomology and twisted Chow groups [Tot25]. Let $\mathcal{A}_{g,D}[p] \to \mathrm{Spec}(\mathbb{C})$ denote the moduli space of g-dimensional polarized abelian varieties of type D with symplectic level-p structure for some prime p not dividing d_g. Then we define a class $\tilde{c}_D \in \mathrm{H}^2(\mathcal{A}_{g,D}[p], M_D)$, which (ignoring the level-p structure) can be thought of as the universal obstruction to lifting the representations $\rho_{A,\lambda}$ to $\mathrm{Aut}(\mathcal{G}_D)$. Using Totaro's results and calculations with Picard groups of covers of $\mathcal{A}_{g,D}[p]$, we show that, if D is of the form of Theorem C, then $\tilde{c}_D$ is nonzero when restricted to the generic point of $\mathcal{A}_{g,D}[p]$. This implies that the answer to Question 2 is no for the generic polarized abelian variety over the function field of $\mathcal{A}_{g,D}[p]$, proving Theorem C. To calculate these Picard groups, we use Borel's determination of the stable cohomology of arithmetic groups [Bor74] and we explicitly calculate abelianizations of certain arithmetic subgroups of $\mathrm{Sp}_{2g}(\mathbb{Q})$.

1.4 Organization

In the preliminary Section 2, we fix our notation and collect some facts from algebraic geometry and abelian schemes over general bases. In Section 3, we systematically analyze theta groups and prove Theorem Θ and its analogue for symmetric line bundles. In Section 4, we prove Theorems A and B. We also collect all known results towards Question 1 and its variants and discuss logical implications between these variants in Section 4.4. In the final Section 5, we discuss moduli spaces of abelian varieties and group cohomology of certain subgroups of twisted symplectic groups $\mathrm{Sp}_{2g}^D(\mathbb{Z})$, proving Theorem C.

Readers who are only interested in the proof of Theorem C (which only uses the easy direction of Theorem Θ) can jump straight to Section 5 after reading Sections 3.1, 3.2 and 3.3.

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2 Preliminaries

We start by setting up notation and recalling facts about torsors, Picard functors, abelian schemes, polarizations, stacks and gerbes. These are mostly standard and can be skipped on a first reading.

2.1 Schemes and sheaves

If $X \to S$ is a morphism of schemes and $T \to S$ a morphism we write X_T for the base change $X \times_S T$, viewed as a T-scheme. If $T = \operatorname{Spec} A$ is an affine scheme we also write X_A for X_T. Write $X(T)$ for the set of sections of $X_T \to T$. Again we write $X(A)$ when $T = \operatorname{Spec} A$. We will often use the Yoneda embedding of the category of S-schemes into the category of sheaves on the (big) étale or fppf site on S. Under this embedding, finite étale morphisms $X \to S$ correspond to finite étale locally constant sheaves [Sta18, Tag 03RV]. If $\mathscr{F}$ is a sheaf of abelian groups on the fppf site on S, write $\mathrm{H}^i(S, \mathscr{F})$ for its associated fppf cohomology groups. If this sheaf arises as the fppf sheaffication of an étale sheaf $\mathscr{G}$, then $\mathrm{H}^i(S, \mathscr{F})$ coincides with étale cohomology of $\mathscr{G}$ [Sta18, Tag 0DDU], which we also denote by $\mathrm{H}^i(S, \mathscr{G})$.

If k is a field, let $k^{\mathrm{sep}} \subset \bar{k}$ be a choice of separable and algebraic closure and let $\mathrm{Gal}_k = \mathrm{Gal}(k^{\mathrm{sep}}/k)$ be its absolute Galois group. We will often use the equivalence of categories $X \mapsto X(k^{\mathrm{sep}})$ between finite étale schemes over k (called finite k-sets) and finite sets with a continuous Gal_k-action. Given a finite étale k-scheme X, denote the cardinality of $X(k^{\mathrm{sep}})$ by $\#X$. Similar remarks apply to the equivalence between finite étale group schemes over k (finite k-groups) and finite groups with a continuous Gal_k-action.

2.2 Torsors

Let G be a fppf group scheme or group sheaf on the fppf site over a scheme S. We write $\mathrm{H}^1(S, G)$ for the set of isomorphism classes of (left) sheaf torsors under G over S. If $S = \operatorname{Spec} R$ we write $\mathrm{H}^1(R, G)$ for the same object. If G is commutative, $\mathrm{H}^1(S, G)$ coincides with fppf cohomology with coefficients in G, so there is no conflict of notation [Sta18, Tag 03AG]. If G is smooth, then every torsor is trivialized étale locally, and if further S is the spectrum of a field k, then $\mathrm{H}^1(k, G)$ coincides with (possibly non-abelian) Galois cohomology $\mathrm{H}^1(\mathrm{Gal}_k, G(k^{\mathrm{sep}}))$ defined using cocycles [Ser02, Chapter III].

Every G-torsor is represented by an algebraic space, but not necessarily by a scheme, even if $G \to S$ is an abelian scheme [Ray70, Section XIII 3.2, page 200]. However, if either $G \to S$ is affine, or $G \to S$ is separated of finite presentation and S is locally noetherian and $\dim S \leq 1$, then every G-torsor is representable by a scheme, see [Poo17, Theorem 6.5.10].

2.3 Picard schemes

If X is a scheme, we denote the Picard group of isomorphism classes of line bundles (equivalently, invertible sheaves) on X by $\mathrm{Pic}(X)$. Let $X \to S$ be a smooth proper morphism of schemes with geometrically connected fibers. The Picard functor $\mathbf{Pic}_{X/S}$ (or simply $\mathbf{Pic}_X$ whenever S is clear from the context) is the étale sheaffication on the category of S-schemes of the presheaf $T \mapsto \mathrm{Pic}(X_T)/\mathrm{Pic}(T)$, see [BLR90, Section 8.1]. (This presheaf is already a sheaf in the Zariski topology. Moreover, the sheaffication step is unnecessary when $X \to S$ has a section.) The Picard functor is represented by an algebraic space over S by results of Raynaud [BLR90, Section 8.3], and we denote this algebraic space again by $\mathbf{Pic}_{X/S}$ or $\mathbf{Pic}_X$. If $X \to S$ is projective or if S is the spectrum of a field, then $\mathbf{Pic}_X$ is represented by a scheme [BLR90, Section 8.2]. Let $\mathbf{Pic}^0_{X/S} \subset \mathbf{Pic}_{X/S}$ (or simply $\mathbf{Pic}^0_X$) be the subsheaf consisting of those elements whose restriction to every closed point $s\colon \mathrm{Spec}(k) \to S$ lies in the identity component of the group scheme $\mathbf{Pic}_{X_s/k}$. This is an open subfunctor of $\mathbf{Pic}_X$ [BLR90, Section 8.4, p. 233], so is representable whenever $\mathbf{Pic}_X$ is.

There exists an exact sequence [BLR90, Section 8.1, Proposition 4]

$$1 \to \mathrm{Pic}(S) \to \mathrm{Pic}(X) \to \mathbf{Pic}_{X/S}(S) \xrightarrow{\mathrm{ob}} \mathrm{H}^2(S, \mathbb{G}_m) \to \mathrm{H}^2(X, \mathbb{G}_m) \tag{2.1}$$

and ob measures the obstruction for a class $\ell \in \mathbf{Pic}_{X/S}(S)$ to be represented by a line bundle on X. Given a line bundle L on X, we denote the image of its isomorphism class under the map $\mathrm{Pic}(X) \to \mathbf{Pic}_X(S)$ by $[L] \in \mathbf{Pic}_X(S)$.

Suppose that $X \to S$ admits a section $e\colon S \to X$. Then the obstruction map $\mathrm{ob}\colon \mathbf{Pic}_X(S) \to \mathrm{H}^2(S, \mathbb{G}_m)$ of (2.1) vanishes. Moreover $\mathbf{Pic}_X$ is isomorphic to the functor sending an S-scheme T to isomorphism classes rigidified line bundles on X_T, i.e., the data of a line bundle L on X_T together with an isomorphism between e^*L and the trivial line bundle on T. If $S = \mathrm{Spec}(k)$ where k is a field, then $\mathbf{Pic}_X(k) = \mathrm{Pic}(X_{k^{\mathrm{sep}}})^{\mathrm{Gal}_k}$ and the inclusion $\mathrm{Pic}(X) \subset \mathbf{Pic}_{X/k}(k)$ is an equality when $X(k) \neq \varnothing$, but not in general.

Define the Néron–Severi group sheaf as the sheaf quotient $\mathbf{NS}_X = \mathbf{NS}_{X/S} = \mathbf{Pic}_{X/S}/\mathbf{Pic}^0_{X/S}$ and define the Néron–Severi group as its group of S-points $\mathbf{NS}_{X/S}(S)$. The exact sequence of sheaves $1 \to \mathbf{Pic}^0_X \to \mathbf{Pic}_X \to \mathbf{NS}_X \to 1$ induces an exact sequence

$$1 \to \mathbf{Pic}^0_X(S) \to \mathbf{Pic}_X(S) \to \mathbf{NS}_X(S) \to \mathrm{H}^1(S, \mathbf{Pic}^0_X). \tag{2.2}$$

2.4 Polarizations on abelian varieties

For general references concerning abelian schemes, see [MFK94, Chapter 6] and [FC90, Chapter 1, Section 1], or [ACM24, Section 1] for a nice summary. Let $A \to S$ be an abelian scheme, i.e., a smooth proper group scheme with geometrically connected fibers. Then $A^\vee \coloneqq \mathbf{Pic}^0_{A/S}$ is representable by an abelian scheme over S (see [FC90, Theorem 1.9]) and is called the *dual abelian scheme* of A. If L is a line bundle on A, then $[L] \in A^\vee(S)$ if and only if for every geometric point $s = \mathrm{Spec}(k) \to S$ and every $x \in A_s(k)$, L_s is isomorphic to $t_x^* L_s$, where $t_x\colon A_s \to A_s$ denotes translation by x; see [Ols08, Section 2.1].

Let $X \to S$ be a (left) A-torsor. If $T \to S$ is a surjective étale morphism and $\alpha\colon X_T \xrightarrow{\sim} A_T$ is a trivialization, then α induces isomorphisms $(\mathbf{Pic}^0_X)_T \xrightarrow{\sim} (\mathbf{Pic}^0_A)_T$ and $(\mathbf{NS}_X)_T \xrightarrow{\sim} (\mathbf{NS}_A)_T$. Since translation by a point $t_x\colon A \to A$ induces the identity on $\mathbf{Pic}^0_A$ and $\mathbf{NS}_A$, these isomorphisms do not depend on the choice of trivialization. Therefore, they descend to canonical isomorphisms $\mathbf{Pic}^0_X \simeq \mathbf{Pic}^0_A$ and $\mathbf{NS}_X \simeq \mathbf{NS}_A$; we will make these identifications without further mention (see also [Ale02, Theorem 3.0.3]). We thus obtain the fundamental exact sequence

$$1 \to A^\vee \to \mathbf{Pic}_X \to \mathbf{NS}_A \to 1. \tag{2.3}$$

Every line bundle L on A induces a morphism $\phi_L \colon A \to A^\vee$ that on T-points is given by $a \mapsto [t_a^* L \otimes L^{-1}]$, where $t_a \colon A_T \to A_T$ denotes translation by $a \in A(T)$. By the theorem of the square, ϕ_L is a homomorphism. It only depends on the image of L in $\mathbf{NS}_{A/S}(S)$. It is symmetric, in the sense that $\phi_L^\vee = \phi_L$ using the double duality $A = A^{\vee\vee}$. More generally, if X is an A-torsor and L is a line bundle on X, the same formula defines a homomorphism $\phi_L \colon A \to A^\vee$, using the identification $\mathbf{Pic}^0_X = A^\vee$ (see also [Ale02, Lemma 3.0.1]). Even more generally, if $\ell \in \mathbf{Pic}_X(S)$ then by descent there exists a unique morphism $\phi_\ell \colon A \to A^\vee$ such that $(\phi_\ell)_T = \phi_L$ whenever $T \to S$ is surjective étale and L is a line bundle on X_T with $[L] = \ell_T$ under (2.1).

By definition ([FC90, Definition 1.6] or [MFK94, Chapter 6, Section 2]), a homomorphism of abelian S-schemes $\lambda \colon A \to A^\vee$ is called a *polarization* if there exists a surjective étale morphism $T \to S$ such that $\lambda_T = \phi_L$ for some relatively ample line bundle L on $A_T \to T$. If L and L' are line bundles on A, then $\phi_L = \phi_{L'}$ if and only if L and L' have the same image in $\mathbf{NS}_{A/S}(S)$. Therefore the assignment $L \mapsto \phi_L$ induces a bijection between the set of elements of $\mathbf{NS}_{A/S}(S)$ that are étale locally the image of an ample line bundle on A and the set of polarizations on A. Consequently, we may and often do identify a polarization λ with the corresponding element of $\mathbf{NS}_{A/S}(S)$. If X is an A-torsor and $\ell \in \mathbf{Pic}_X(S)$ satisfies $\phi_\ell = \lambda$, then we say that ℓ *represents* λ. Given any class $\lambda \in \mathbf{NS}_{A/S}(S)$, write $\mathbf{Pic}^\lambda_A$ for the preimage of λ under the projection $\mathbf{Pic}_A \to \mathbf{NS}_A$, and similarly define $\mathbf{Pic}^\lambda_X$ for any A-torsor X using (2.3). The class $\mathbf{Pic}^\lambda_X$ is a torsor under $\mathbf{Pic}^0_X = \mathbf{Pic}^0_A = A^\vee$. By definition, the class of $\mathbf{Pic}^\lambda_X$ in $\mathrm{H}^1(S, \mathbf{Pic}^0_X)$ equals the image of $\lambda \in \mathrm{NS}_X(S)$ under the connecting map of (2.2).

Given a polarization λ on an abelian scheme A over a field k, the order of the kernel $A[\lambda]$ (as a group scheme) is called the degree of λ, denoted by $\deg(\lambda)$. If $\deg(\lambda)$ is invertible in k, there further exist unique positive integers $d_1 \mid d_2 \mid \cdots \mid d_g$ with $g = \dim A$ such that $A[\lambda](k^{\mathrm{sep}}) \simeq (\mathbb{Z}/d_1\mathbb{Z})^2 \times \cdots \times (\mathbb{Z}/d_g\mathbb{Z})^2$; we call $D = (d_1, \ldots, d_g)$ the type of λ. If (A, λ) is a polarized abelian scheme over a general scheme S, then the function which associates to a point $s \in S$ the degree of $(A, \lambda)_s$ is locally constant; if this degree is invertible for every s, we say the degree of (A, λ) is invertible on S. If the degree of (A, λ) is invertible on S, then the type of $(A, \lambda)_s$ is also locally constant. In this paper, we will only consider polarizations whose degree is invertible on the base scheme.

2.5 Stacks and gerbes

We will make use of basic properties of stacks and gerbes; we briefly recall these notions here. Let $\mathcal{C}$ be a site, for example the (big) étale or fppf site of a scheme S. Given an object U of $\mathcal{C}$, let $\mathcal{C}/U$ be the site whose underlying category is the slice category over U and whose coverings are restrictions of coverings in $\mathcal{C}$. Given a category fibered in groupoids $\mathscr{X} \to \mathcal{C}$ and two objects x, y of $\mathscr{X}(U)$, let $\mathbf{Isom}(x, y)$ denote the presheaf on $\mathcal{C}/U$ with the property that $\mathbf{Isom}(x, y)(T) = \mathrm{Hom}_{\mathscr{X}(T)}(f^*x, f^*y)$ for all objects $f \colon T \to U$ in $\mathcal{C}/U$. Similarly write $\mathbf{Aut}(x) = \mathbf{Isom}(x, x)$.

Recall from [Vis05, Section 4] or [Sta18, Tag 02ZH] that a *stack* is a category fibered in groupoids $\mathscr{X} \to \mathcal{C}$ such that for every object U of $\mathcal{C}$, the presheaf $\mathbf{Isom}(x, y)$ is a sheaf for all objects $x, y \in \mathscr{X}(U)$ and such that the pullback functor $\mathscr{X}(U) \to \mathscr{X}(\{U_i \to U\})$ to the groupoid $\mathscr{X}(\{U_i \to U\})$ of descent data in the sense of [Vis05, Section 4.1.2] is an equivalence of categories for every covering $\{U_i \to U\}$.

As an example, let $\mathscr{G}$ be a sheaf of groups on $\mathcal{C}$. The *classifying stack* for $\mathscr{G}$, denoted $\mathrm{B}\mathscr{G}$, is the stack on $\mathcal{C}$ such that for each U of $\mathcal{C}$, the objects of the groupoid $(\mathrm{B}\mathscr{G})(U)$ are $\mathscr{G}_U$-torsors on $\mathcal{C}/U$, and morphisms given by isomorphisms of $\mathscr{G}_U$-torsors. This is indeed a stack by [Sta18, Tag 04TQ].

A stack $\mathscr{X}$ on $\mathcal{C}$ is a *gerbe* [Sta18, Tag 06NY] if it satisfies the following two properties: for every object U of $\mathcal{C}$, there exists a cover $\{U_i \to U\}$ such that the groupoid $\mathscr{X}(U_i)$ is non-empty for all i; for every object U of $\mathcal{C}$ and objects x, y in $\mathscr{X}(U)$, there exists a covering $\{U_i \to U\}$ such that $x|_{U_i} \simeq y|_{U_i}$ for all i. For example, classifying stacks of sheaves of groups are gerbes.

We will encounter a few gerbes in this paper in Section 3. We will often use the following formal but very useful lemma:

Lemma 2.1. *Let $F\colon \mathscr{X} \to \mathscr{Y}$ be a morphism between gerbes such that for every object U of $\mathcal{C}$ and x of $\mathscr{X}(U)$, the induced map $\mathrm{Aut}_{\mathscr{X}(U)}(x) \to \mathrm{Aut}_{\mathscr{Y}(U)}(F(x))$ is an isomorphism. Then F is an isomorphism. In particular, F_U is an equivalence of groupoids for all U.*

Proof. This must be well known; we give a quick proof. It suffices to prove that F is fully faithful and essentially surjective. To prove F is fully faithful, we need to show that for every U of $\mathcal{C}$ and x, y of $\mathscr{X}(U)$, the map of sheaves $\mathbf{Isom}(x, y) \to \mathbf{Isom}(F(x), F(y))$ is an isomorphism. Since this can be checked locally and since $\mathscr{X}$ is a gerbe, we may assume $x = y$, in which case it follows from our assumptions. We now prove that for every U and y in $\mathscr{Y}(U)$, y lies in the essential image of F_U. Since objects in $\mathscr{X}$ exists locally and since all objects in $\mathscr{Y}$ are locally isomorphic, there exists a covering $\{U_i \to U\}$, objects x_i in $\mathscr{X}(U_i)$ and isomorphisms $F_{U_i}(x_i) \simeq y|_{U_i}$. Since F_V is fully faithful for each V, there exists a unique descent datum $D = (\{x_i\}, \{\alpha_{ij}\})$ with respect to $\{U_i \to U\}$ that maps to the descent datum of y under F. Since $\mathscr{X}$ is a stack, there exists an object x in $\mathscr{X}(U)$ corresponding to this descent datum, and this object satisfies $F_U(x) \simeq y$. ☐

Gerbes are twisted versions of classifying stacks, in the following sense.

Lemma 2.2. *Let $\mathscr{X}$ be a gerbe, suppose that S is a terminal object of $\mathcal{C}$ and suppose that there exists an object x in $\mathscr{X}(S)$. Then the assignment $y \mapsto \mathbf{Isom}(y, x)$ induces an isomorphism $\mathscr{X} \simeq \mathrm{B}\mathbf{Aut}(x)$.*

Proof. Since $\mathscr{X}$ is a gerbe, x and y are locally isomorphic, so $\mathbf{Isom}(y, x)$ with its natural left action is a torsor under $\mathbf{Aut}(x)$. Moreover if y, y' are objects of $\mathscr{X}(T)$, then the map $\mathrm{Isom}_{\mathscr{X}(T)}(y, y') \to \mathrm{Isom}_{B\mathscr{G}(T)}(\mathbf{Isom}(y, x), \mathbf{Isom}(y', x))$ is a bijection. Now apply Lemma 2.1. ☐

Whenever we are in the situation of Lemma 2.2, we say the gerbe is *neutral*.

When all automorphism groups are commutative, we can say more. Suppose that $\mathscr{X}$ is a gerbe such that for every object U of $\mathcal{C}$ and object x of $\mathscr{X}(U)$, the group sheaf $\mathbf{Aut}(x)$ on the slice category $\mathcal{C}/U$ is commutative. Then by [Sta18, Tag 06NY] there exists a sheaf of commutative groups $\mathscr{G}$ on $\mathcal{C}$ such that $\mathscr{G}|_U \simeq \mathbf{Aut}(x)$ for every object x of $\mathscr{X}(U)$ and every U of $\mathcal{C}$. In that case, we say the gerbe $\mathscr{X}$ is *banded by* $\mathscr{G}$.

Lemma 2.3. *Let $\mathcal{C}$ be the big etale or fppf site of a scheme S and let $\mathscr{G}$ be a sheaf of commutative groups on $\mathcal{C}$. Then there exists a canonical bijection $\mathscr{X} \mapsto [\mathscr{X}]$, compatible with base change on S, between the set of isomorphism classes of gerbes banded by $\mathscr{G}$ and $\mathrm{H}^2(S, \mathscr{G})$, sending a neutral gerbe to the trivial class.*

Proof. See [Gir71, Théorème IV.3.4.2]. ☐

3 Polarizations and theta groups

We start by discussing formal properties of commutative finite étale group schemes with a nondegenerate alternating pairing such as $(A[\lambda], e_\lambda)$, which we call symplectic modules. We then discuss abstract theta groups, Mumford theta groups and linear theta groups. Theorem Θ follows from combining Corollaries 3.20 and 3.25. We end by proving an analogue of Theorem Θ for symmetric line bundles on symmetric torsors.

3.1 Symplectic modules

Let S be a scheme.

Definition 3.1. *A* symplectic module *over S is a pair (M, e), where $M \to S$ is a commutative finite étale group scheme and $e\colon M \times M \to \mathbb{G}_m$ is a nondegenerate alternating pairing. (Nondegenerate here means that e induces an isomorphism $M \xrightarrow{\sim} M^\vee := \mathbf{Hom}(M, \mathbb{G}_m)$.)*

If (M, e) is a symplectic module and $T \to S$ a morphism, the base change $(M, e)_T = (M_T, e_T)$ is a symplectic module over T. An isomorphism of symplectic modules $(M, e) \xrightarrow{\sim} (M', e')$ is simply an isomorphism of S-group group schemes $M \to M'$ intertwining e and e'; denote the set of such isomorphisms by $\mathrm{Isom}((M, e), (M', e'))$. We recall the classification of symplectic modules over a separably closed field.

Definition 3.2. *Let $g \geq 1$ be an integer. A* type of length g *is a sequence of positive integers $D = (d_1, \ldots, d_g)$ such that d_i divides d_{i+1} for all $i = 1, \ldots, g-1$. Let $\#D = d_1 \cdots d_g$.*

Given a type D, we define the *standard symplectic module* (M_D, e_D) *of type* D over $\mathbb{Z}[1/\#D]$ as follows. Consider the constant group scheme $K_D = \mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_g\mathbb{Z}$ and let $K_D^\vee = \mathbf{Hom}(K_D, \mathbb{G}_m) = \mu_{d_1} \times \cdots \times \mu_{d_g}$ be its Cartier dual. Let $M_D = K_D \times K_D^\vee$, and define the pairing e_D via $e_D((x, \chi), (x', \chi')) = \chi'(x)\chi(x')^{-1}$. Since e_D is isotropic on K_D and $K_D^\vee$ and puts these two groups in perfect duality, this pairing is nondegenerate. If $\#D$ is invertible on S, we obtain by base change the standard symplectic module of type D over S which we again denote by (M_D, e_D).

Lemma 3.3. *Let (M, e) be a symplectic module over S and assume S connected. Then there exists a unique type $D = (d_1, \ldots, d_g)$ with the property that $d_1 \geq 2$, $\#D$ invertible on S and such that $(M, e)_T \simeq (M_D, e_D)$ for some surjective étale $T \to S$.*

Proof. There exists a surjective étale morphism $T \to S$ such that M_T is constant, corresponding to a finite abelian group A. Since $M^\vee \simeq M$ is also étale, the order of A is invertible on S. After a further base change we may assume $M_T^\vee$ is constant and corresponds to $\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$. Then e corresponds to a nondegenerate bilinear alternating form $A \times A \to \mathbb{Q}/\mathbb{Z}$. In this case, it is folklore and elementary that there exists a Lagrangian decomposition $A \simeq L \times L'$ that puts L' and L in duality; see for example [PSV10, Corollary 5.7]. The lemma follows by taking D to be the sequence of elementary divisors of L. $\square$

We say a symplectic module (M, e) *has type* D if for every geometric point $\bar{s}\colon \mathrm{Spec}(k) \to S$ with k a separably closed field, the pullback $(M, e)_{\bar{s}}$ is isomorphic to (M_D, e_D) over k. In that case, an argument similar to the proof of Lemma 3.3 shows that $(M, e)_T \simeq (M_D, e_D)$ for some étale surjective base change $T \to S$.

Definition 3.4. *Let (M, e) and (M', e') be symplectic modules over S. Let $\mathbf{Isom}((M, e), (M', e'))$ be the étale sheaf on S with $\mathbf{Isom}((M, e), (M', e'))(T) = \mathrm{Isom}((M, e)_T, (M', e')_T)$ for all morphisms $T \to S$. We write $\mathbf{Sp}(M) = \mathbf{Isom}((M, e), (M, e))$ if the pairing e is clear from the context. Similarly we write $\mathrm{Sp}(M) = \mathrm{Isom}((M, e), (M, e))$.*

Since $\mathbf{Isom}((M, e), (M', e'))$ is finite and locally constant by Lemma 3.3, it is represented by a finite étale scheme over S. Hence $\mathbf{Sp}(M) \to S$ is a finite étale group scheme too. Given a symplectic module $(M, e)/S$ of type D, $\mathbf{Isom}((M, e), (M_D, e_D))$ receives a left action of $\mathbf{Sp}(M_D)$.

Lemma 3.5. *The assignment $(M, e) \mapsto \mathbf{Isom}((M, e), (M_D, e_D))$ induces a bijection between isomorphism classes of the following objects:*

- *Symplectic modules over S of type D;*

- $\mathbf{Sp}(M_D)$*-torsors over* S.

Proof. This follows from étale descent (sometimes called the twisting principle) and Lemma 3.3. (More formally, the category of symplectic modules of type D is a stack in the étale topology on S and is a gerbe by Lemma 3.3. Now apply Lemma 2.2.) ☐

Example 3.6. *If* $D = (2, 2, \ldots, 2)$ *has length* n*, then* $\mathbf{Sp}(M_D)$ *is the constant group scheme* $\mathrm{Sp}_{2n}(\mathbb{F}_2)$*. If* S *is the spectrum of a field* k*, then isomorphism classes symplectic modules of type* D *are in bijection with* $\mathrm{H}^1(k, \mathrm{Sp}_{2n}(\mathbb{F}_2))$*, which is itself in bijection with conjugacy classes of continuous representations* $\mathrm{Gal}_k \to \mathrm{Sp}_{2n}(\mathbb{F}_2)$*.*

3.2 Abstract theta groups

Let S be a scheme and $M \to S$ a finite étale commutative group scheme. Let $1 \to \mathbb{G}_m \to \mathcal{G} \to M \to 1$ be a central extension of group schemes. If $T \to S$ is a morphism and $x, y \in M(T)$ lift to elements $\tilde{x}, \tilde{y} \in \mathcal{G}(T)$, then the commutator $[\tilde{x}, \tilde{y}] = \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}$ lies in the subgroup $\mathbb{G}_m(T)$ and is independent of the choice of lift of x and y. By descent, it follows that the assignment $(x, y) \mapsto [\tilde{x}, \tilde{y}]$ defines a morphism $M \times M \to \mathbb{G}_m$. This morphism is easily checked to be bilinear and alternating; it is called the *commutator pairing* associated to the central extension.

Definition 3.7. *Let* (M, e) *be a symplectic module over* S*. A* theta group for (M, e) *is a central extension of group schemes*

$$1 \to \mathbb{G}_m \to \mathcal{G} \to M \to 1 \tag{3.1}$$

whose commutator pairing equals e*.*

The data of a theta group includes not just the group scheme $\mathcal{G}$ but also the central extension structure; we will usually suppress this additional data in the notation. If $\mathcal{G}$ is a theta group for (M, e) and $T \to S$ is a morphism, then $\mathcal{G}_T$ is a theta group for $(M, e)_T$.

Let $\mathcal{G}$ and $\mathcal{G}'$ be theta groups for the symplectic modules (M, e) and (M', e') respectively. Define $\mathrm{Isom}(\mathcal{G}, \mathcal{G}')$ to be the set of isomorphisms of group schemes $\alpha\colon \mathcal{G} \to \mathcal{G}'$ such that $\alpha|_{\mathbb{G}_m}\colon \mathbb{G}_m \to \mathbb{G}_m$ is the identity. If such an α exists, we say $\mathcal{G}$ and $\mathcal{G}'$ are isomorphic. Define $\mathbf{Isom}(\mathcal{G}, \mathcal{G}')$ to be étale sheaf on S with $\mathbf{Isom}(\mathcal{G}, \mathcal{G}')(T) = \mathrm{Isom}(\mathcal{G}_T, \mathcal{G}'_T)$ for every morphism $T \to S$. There exists a morphism

$$\mathbf{Isom}(\mathcal{G}, \mathcal{G}') \to \mathbf{Isom}((M, e), (M', e')) \tag{3.2}$$

sending $\alpha\colon \mathcal{G} \to \mathcal{G}'$ to the induced isomorphism when quotienting out $\mathbb{G}_m$. Given $\beta \in \mathrm{Isom}((M, e), (M', e'))$, let $\mathbf{Isom}(\mathcal{G}, \mathcal{G}'; \beta) \subset \mathbf{Isom}(\mathcal{G}, \mathcal{G}')$ be the fiber of (3.2) above β, and similarly let $\mathrm{Isom}(\mathcal{G}, \mathcal{G}'; \beta) = \mathbf{Isom}(\mathcal{G}, \mathcal{G}'; \beta)(S)$. If $(M, e) = (M', e')$, we call elements of $\mathrm{Isom}(\mathcal{G}, \mathcal{G}'; \mathrm{Id}_M)$ *framed isomorphisms* and we say $\mathcal{G}$ and $\mathcal{G}'$ are *framed isomorphic* if $\mathrm{Isom}(\mathcal{G}, \mathcal{G}'; \mathrm{Id}_M) \neq \varnothing$. If $\mathcal{G} = \mathcal{G}'$, write $\mathbf{Aut}(\mathcal{G}) = \mathbf{Isom}(\mathcal{G}, \mathcal{G})$ and $\mathbf{Aut}(\mathcal{G}; \mathrm{Id}) = \mathbf{Isom}(\mathcal{G}, \mathcal{G}; \mathrm{Id}_M)$ and their S-points by $\mathrm{Aut}(\mathcal{G})$ and $\mathrm{Aut}(\mathcal{G}; \mathrm{Id})$.

We first classify theta groups over separably closed fields. Given a type D such that $\#D$ is invertible on S, we define the *standard theta group of type* D over S, following [Mum66, page 294]. In the notation of §3.1, let $\mathcal{G}_D = \mathbb{G}_m \times M_D = \mathbb{G}_m \times K_D \times K_D^\vee$ and define the group law via

$$(\lambda, x, \chi) \cdot (\lambda', x', \chi') = (\lambda\lambda'\chi'(x), x + x', \chi + \chi'). \tag{3.3}$$

(We write the group law on $\mathbb{G}_m$ multiplicatively and on $K_D, K_D^\vee$ additively.) A calculation then shows that, with the obvious inclusion from $\mathbb{G}_m$ and projection to M_D, $\mathcal{G}_D$ is a theta group for (M_D, e_D).

Lemma 3.8. *Let (M, e) be a symplectic module over S and let $\mathcal{G}, \mathcal{G}'$ be theta groups for (M, e). Then there exists an étale surjective $T \to S$ such that $\mathcal{G}_T$ and $\mathcal{G}'_T$ are framed isomorphic, in other words such that $\mathbf{Isom}(\mathcal{G}, \mathcal{G}'; \mathrm{Id})(T) \neq \varnothing$.*

Proof. The usual proof when S is the spectrum of an algebraically closed field generalizes to an arbitrary base scheme S. Indeed, since the statement is local on S we may assume by a standard argument that S is connected. (We explain the standard argument once: we may assume $S = \mathrm{Spec}(A)$ is affine, so there exists a finitely generated $\mathbb{Z}$-algebra $A_0 \subset A$ such that $\mathcal{G}, \mathcal{G}'$ and (M, e) are defined over $S_0 = \mathrm{Spec}(A_0)$, so we may replace S by S_0 and assume S is Noetherian, in which case the connected components of S are open.) By Lemma 3.3 we may assume after étale base change that $(M, e) = (M_D, e_D)$ and $\mathcal{G}' = \mathcal{G}_D$ for some type D. After a further base change, there exist homomorphisms $K_D \to \mathcal{G}$, $K_D^\vee \to \mathcal{G}$ that split the homomorphisms $\mathcal{G}|_{K_D} \to K_D$ and $\mathcal{G}|_{K_D^\vee} \to K_D^\vee$ respectively. Using these splittings, we can write down an explicit framed isomorphism between $\mathcal{G}$ and $\mathcal{G}_D$, as in [Mum66, Corollary of Theorem 1]. □

Lemma 3.9. *Let $\mathcal{G}$ be a theta group for a symplectic module (M, e) over S.*

1. *The forgetful map $\mathbf{Aut}(\mathcal{G}) \to \mathbf{Sp}(M)$ is surjective.*
2. *Given a morphism $T \to S$ and $m \in M(T)$, let α_m be the isomorphism $\mathcal{G}_T \to \mathcal{G}_T$ defined by $\tilde{x} \mapsto e(m, x)\tilde{x}$, where $x \in M(T)$ is the image of $\tilde{x}$. Then the assignment $m \mapsto \alpha_m$ induces an isomorphism of group schemes $M \simeq \mathbf{Aut}(\mathcal{G}; \mathrm{Id})$.*

Proof. We may assume S is connected. Furthermore we may assume by Lemmas 3.3 and 3.8 that $M = M_D$ and $\mathcal{G} = \mathcal{G}_D$ for some type D. Part 1 then follows from [Ols08, Lemma 6.3.7]. (Sketch of proof: it suffices to prove that for every S-scheme T, every $\alpha \in \mathbf{Sp}(M)(T)$ lifts locally to $\mathbf{Aut}(\mathcal{G})$. This follows from the fact that $\alpha^*\mathcal{G}_T$ is a theta group for $(M, e)_T$, so must be étale locally isomorphic to $\mathcal{G}_T$.) Part 2 follows from a direct computation whose proof is identical to the case where S is the spectrum of a field [BL04, Lemma 6.6.6]. The exactness of (3.4) follows from Part 1 and using the isomorphism $M \simeq \mathbf{Aut}(\mathcal{G}; \mathrm{Id})$ of Part 2. □

Consequently, by using the isomorphism $\alpha \colon M \xrightarrow{\sim} \mathbf{Aut}(\mathcal{G}; \mathrm{Id})$ we obtain the exact sequence of finite étale group schemes

$$1 \to M \to \mathbf{Aut}(\mathcal{G}) \to \mathbf{Sp}(M) \to 1. \tag{3.4}$$

This sequence will play a fundamental role in this paper, because of Lemma 3.11.

Corollary 3.10. *Let S be a scheme and D a type with $\#D$ invertible on S. Then the assignment $\mathcal{G} \mapsto \mathbf{Isom}(\mathcal{G}, \mathcal{G}_D)$ induces a bijection between:*

- *Isomorphism classes of theta groups for (M, e) over S, where (M, e) is a symplectic module of type D over S;*
- *$\mathbf{Aut}(\mathcal{G}_D)$-torsors over S.*

Suppose that there exists a theta group $\mathcal{G}$ for the symplectic module (M, e) over S. Then the assignment $\mathcal{G}' \mapsto \mathbf{Isom}(\mathcal{G}', \mathcal{G}; \mathrm{Id}_M)$ induces a bijection between:

- *Framed isomorphism classes of theta groups for (M, e) over S;*
- *$\mathbf{Aut}(\mathcal{G}; \mathrm{Id})$-torsors over S.*

Proof. This follows from Lemmas 3.3 and 3.8 and étale descent, more precisely Lemma 2.2. (Noting that theta groups are always affine over S, so every descent datum is effective by [Poo17, Theorem 4.3.5].) □

Let (M, e) be a finite symplectic module over S of type D. Its isomorphism class corresponds to an element $c_M \in \mathrm{H}^1(S, \mathbf{Sp}(M_D))$ using Lemma 3.5.

Lemma 3.11. *There exists a theta group for (M, e) over S if and only if the class c_M lifts along the map $\mathrm{H}^1(S, \mathbf{Aut}(\mathcal{G}_D)) \to \mathrm{H}^1(S, \mathbf{Sp}(M_D))$.*

Proof. This follows from Lemma 3.5, Corollary 3.10 and the fact that if $\mathcal{G}$ is a theta group for (M', e'), then the pushout of the $\mathbf{Aut}(\mathcal{G}_D)$-torsor $\mathbf{Isom}(\mathcal{G}', \mathcal{G}_D)$ along $\mathbf{Aut}(\mathcal{G}_D) \to \mathrm{Sp}(M_D)$ is isomorphic to $\mathbf{Isom}(M', M_D)$. □

Let (M, e) be a symplectic module over S. Consider the fibered category $\mathrm{ThetaGrp}_{(M,e)}$ where for an S-scheme T, the groupoid $\mathrm{ThetaGrp}_{(M,e)}(T)$ has objects theta groups for $(M, e)_T$, and morphisms given by framed isomorphisms between theta groups. Then $\mathrm{ThetaGrp}_{(M,e)}$ is a stack in the étale topology on S. By Lemma 3.8, it is in fact a gerbe, and by Part 2 of Lemma 3.9, this gerbe is banded by the group scheme M. By Lemma 2.3, this gerbe defines a class $[\mathrm{ThetaGrp}_{(M,e)}] \in \mathrm{H}^2(S, M)$, and this class vanishes if and only if there exists a theta group for (M, e). It is possible to use the formalism of gerbes to show that $[\mathrm{ThetaGrp}_{(M,e)}]$ is the image of c_M under a connecting homomorphism associated to the exact sequence (3.4) for the pair $(M_D, \mathcal{G}_D)$, but we will not need this.

It will sometimes be useful to refine theta groups to finite étale group schemes (this is somewhat implicit in [MB85, Chapitre I, Proposition 5.7]). Let (M, e) be a symplectic module of type $D = (d_1, \dots, d_g)$. Given an integer $n \geq 1$ and a group scheme $G \to S$, let $G[n]$ denote the kernel of the multiplication-by-n morphism $[n] \colon G \to G$.

Lemma 3.12. *Let $n = d_g$ if d_g is odd and $n = 2d_g$ if d_g is even. If $\mathcal{G}$ is a theta group for (M, e), then $\mathcal{G}[n]$ is a finite étale closed subgroup scheme of $\mathcal{G}$ fitting in a central extension $1 \to \mu_n \to \mathcal{G}[n] \to M \to 1$. Moreover, the assignment $\mathcal{G} \to \mathcal{G}[n]$ induces an equivalence between the following groupoids:*

- *Theta groups for (M, e) over S, with morphisms given by isomorphisms of theta groups;*
- *Central extensions of the form $1 \to \mu_n \to \mathcal{H} \to M \to 1$ whose commutator pairing equals e, with morphisms given by isomorphisms of group schemes $\mathcal{H} \to \mathcal{H}'$ that restrict to the identity on μ_n.*

Proof. For every $k \geq 1$, $\mathcal{G}[k]$ is a closed subscheme of $\mathcal{G}$. The identity $(xy)^k = x^k y^k [x, y]^{k(k+1)/2} = e(x, y)^{k(k+1)/2} x^k y^k$ for $x, y \in \mathcal{G}$ and the fact that M is killed by d_g shows that $\mathcal{G}[n]$ is closed under multiplication and inversion, so is indeed a closed subgroup scheme of $\mathcal{G}$. Since d_g is invertible on S, n is also invertible on S and so $\mathcal{G}[n]$ is finite étale. To prove the equivalence of groupoids, we describe an inverse and leave the remaining verifications to the reader. If $\mathcal{H}$ is a central μ_n-extension of M whose commutator pairing equals e, define $\mathcal{G}$ as the (sheaf) quotient of $\mathbb{G}_m \times \mathcal{H}$ by $\{(\lambda, \lambda^{-1}) \colon \lambda \in \mu_n\}$. Then $\mathcal{G}$ is representable by a group scheme which is a theta group for (M, e), and $\mathcal{H} \mapsto \mathcal{G}$ is the desired inverse. □

3.3 Mumford theta groups

Let $A \to S$ be a g-dimensional abelian scheme and let $\lambda \colon A \to A^\vee$ be a polarization whose degree is invertible on S. Then the kernel $A[\lambda]$ is a finite étale S-group scheme. Since λ is self-dual, $A[\lambda]$ is Cartier dual to itself. This self-duality is witnessed by a nondegenerate alternating pairing $e_\lambda \colon A[\lambda] \times A[\lambda] \to \mathbb{G}_m$, called the *Weil pairing*. Therefore the pair $(A[\lambda], e_\lambda)$ is a symplectic module over S, in the sense of Section 3.1. We say (A, λ) is of type D if D has length g and $(A[\lambda], e_\lambda)$ is of type D.

Let $X \to S$ be an A-torsor and let L be a line bundle on X such that $\phi_L = \lambda$, see §2.4 for the notation. If $a \in A(S)$, let $t_a \colon X \to X$ be the translation-by-a morphism. Following Mumford [Mum66, p. 289], let

$\mathcal{G}(L) \to S$ be the group scheme such that for every S-scheme T,

$$\mathcal{G}(L)(T) = \{(a, \varphi) \mid a \in A[\lambda](T),\ \varphi \text{ is an isomorphism of line bundles } L_T \xrightarrow{\sim} t_a^* L_T\}.$$

The group operation is given by $(a, \varphi) \cdot (b, \psi) = (a + b, t_b^* \varphi \circ \psi)$, where $t_b^*\varphi \circ \psi$ is the composite $L \xrightarrow{\psi} t_b^* L \xrightarrow{t_b^*\varphi} t_b^*(t_a^* L) = t_{a+b}^* L$. By [Mum67, §6, Proposition 1] (who assumes $X = A$ but whose proof continues to hold for general X), $\mathcal{G}(L)$ is representable by a group scheme and the forgetful map $(a, \varphi) \mapsto a$ defines a central extension

$$1 \to \mathbb{G}_m \to \mathcal{G}(L) \to A[\lambda] \to 1.$$

Proposition 3.13. *$\mathcal{G}(L)$ is a theta group for $(A[\lambda], e_\lambda)$.*

Proof. It suffices to prove that the commutator pairing of $\mathcal{G}(L)$ is e_λ. To prove this, we may assume S is the spectrum of an algebraically closed field and $X = A$, in which case it is well known, see for example [Mum91, p. 44-46] or [Pol03, Section 12.2]. □

Let M be another line bundle on X with $\phi_M = \lambda$. Let $\gamma\colon L \to M$ be an isomorphism. Then the assignment $(a, \varphi) \mapsto (a, t_a^*(\gamma)\varphi\gamma^{-1})$ induces an isomorphism $F_\gamma\colon \mathcal{G}(L) \to \mathcal{G}(M)$ of group schemes that restricts to the identity on $\mathbb{G}_m$ and induces the identity on $A[\lambda]$. In the notation of Section 3.2, F_γ is a framed isomorphism. Any other isomorphism $\gamma'\colon L \to M$ differs from γ by a nonzero scalar, hence $F_{\gamma'} = F_\gamma$. We conclude that if L and M are isomorphic line bundles on X with $\phi_L = \phi_M = \lambda$, then there is a canonical framed isomorphism between $\mathcal{G}(L)$ and $\mathcal{G}(M)$. Similarly, if M is a line bundle on S and $p\colon X \to S$ is the structure map, the assignment $(a, \varphi) \mapsto (a, \varphi \otimes p^* \mathrm{Id}_M)$ induces an isomorphism $\mathcal{G}(L) \xrightarrow{\sim} \mathcal{G}(L \otimes p^* M)$. We use these observations to extend the scope of the construction of $\mathcal{G}(L)$, as follows.

Let ℓ be an element of $\mathbf{Pic}_{X/S}(S)$ with $\phi_\ell = \lambda$ (see §2.4). Let $f\colon S' \to S$ be an étale surjective morphism such that $f^*\ell = [L]$ for some line bundle L on $X_{S'}$, using the sequence (2.1). Let $p_1, p_2\colon S' \times_S S' \to S'$ denote the two projections. Since $p_1^* f^* \ell = p_2^* f^* \ell$, there exists a line bundle M on $S' \times_S S'$ and an isomorphism $\gamma\colon p_1^* L \xrightarrow{\sim} p_2^* L \otimes p^* M$. By the previous paragraph, this induces a framed isomorphism $F\colon p_1^*\mathcal{G}(L) \xrightarrow{\sim} p_2^*\mathcal{G}(L)$ of theta groups that is independent of the choice of γ and M. Therefore F defines a descent datum. Since theta groups are affine over S, every such descent datum is effective. Therefore, there exists a theta group $\mathcal{G}$ for $(A[\lambda], e_\lambda)$ over S, unique up to unique framed isomorphism, whose base change along f corresponds to $\mathcal{G}(L)$ equipped with the descent datum F. We denote such a group by $\mathcal{G}(\ell)$. Up to unique isomorphism, $\mathcal{G}(\ell)$ is independent of the choice of cover f.

In conclusion, we have just shown that if there exists an A-torsor $X \to S$ and an element $\ell \in \mathbf{Pic}_X^\lambda(S)$, then there exists a theta group $\mathcal{G}(\ell)$ for $(A[\lambda], e_\lambda)$. In the next section, we show that the converse holds (Theorem 3.18).

3.4 A theta group criterion for representing a polarization

Let $A \to S$ be an abelian scheme, let $\lambda\colon A \to A^\vee$ be a polarization whose degree is invertible on S and let X be an A-torsor. Given $a \in A(S)$, recall that we denote the translation-by-a maps $A \to A$ and $X \to X$ by t_a. Given $x \in X(S)$, write $t_x\colon A \to X$ for the unique A-equivariant map sending the zero section to x.

The scheme $\mathbf{Pic}_A^\lambda$ of line bundles representing λ is a torsor under $A^\vee = \mathbf{Pic}_A^0$ by tensoring with degree-zero line bundles. Pulling back the $A^\vee$-action on $\mathbf{Pic}_A^\lambda$ along $\lambda\colon A \to A^\vee$ defines an A-action on $\mathbf{Pic}_A^\lambda$. This action is explicitly given by $a \cdot [L] = \lambda(a) \otimes [L] = \phi_L(a) \otimes [L] = t_a^*[L]$.

Given an element $\ell \in \mathbf{Pic}_X^\lambda(S)$, the assignment $x \mapsto t_x^* \ell$ defines a morphism $\psi_\ell\colon X \to \mathbf{Pic}_A^\lambda$. This morphism is equivariant with respect to the A-action on the source and target, since $\psi_\ell(t_a(x)) = t_{t_a(x)}^* \ell = t_a^* t_x^* \ell = t_a^* \psi_\ell(x)$.

Lemma 3.14. *In the above notation, the assignment $\ell \mapsto \psi_\ell$ induces a bijection*

$$\mathbf{Pic}^\lambda_X(S) \xrightarrow{1:1} \{A\text{-equivariant morphisms } X \to \mathbf{Pic}^\lambda_A\}. \tag{3.5}$$

If $f\colon Y \to X$ is a morphism of A-torsors and $\ell \in \mathbf{Pic}^\lambda_X(S)$, then $\psi_{f^*\ell} = \psi_\ell \circ f$.

Proof. Since both sides of (3.5) satisfy étale descent, to prove bijectivity we may assume (after applying an étale base change) that $X \to S$ has a section. Let x be such a section. Then giving an A-equivariant morphism $X \to \mathbf{Pic}^\lambda_A$ is the same as giving its value at $x \in X(S)$. But $\psi_\ell(x) = t_x^*\ell$ and $t_x^*\colon \mathbf{Pic}^\lambda_X \to \mathbf{Pic}^\lambda_A$ is an isomorphism. So (3.5) is a bijection when $X \to S$ has a section, hence a bijection in general. The final sentence is a computation: $\psi_{f^*\ell}(y) = t_y^* f^*\ell = (f \circ t_y)^*\ell = t_{f(y)}^*\ell = \psi_\ell \circ f$. □

Remark 3.15. *We can construct an explicit inverse to* (3.5)*, as follows. Let $\psi\colon X \to \mathbf{Pic}^\lambda_A$ be A-equivariant. Let $S' \to S$ be an étale surjective morphism such that there exists an element $x \in X(S')$. Let $\ell = (t_x^{-1})^*\psi(x)$. Then ℓ is independent of the choice of x, descends to an element of $\mathbf{Pic}^\lambda_X(S)$ and satisfies $\psi_\ell = \psi$.*

Corollary 3.16. *In the above notation, $\mathbf{Pic}^\lambda_X(S) \neq \varnothing$ if and only if $[X]$ maps to $[\mathbf{Pic}^\lambda_A]$ under the map $\mathrm{H}^1(\lambda)\colon \mathrm{H}^1(S, A) \to \mathrm{H}^1(S, A^\vee)$.*

Proof. By definition of pushout, $[X]$ maps to $[\mathbf{Pic}^\lambda_A]$ under $\mathrm{H}^1(\lambda)$ if and only if there exists an A-equivariant morphism $X \to \mathbf{Pic}^\lambda_A$. Conclude by Lemma 3.14. □

Let $\mathrm{PolTor}_{(A,\lambda)}$ be the category fibered in groupoids such that for every S-scheme T, the groupoid $\mathrm{PolTor}_{(A,\lambda)}(T)$ is defined as follows: objects are pairs (X, ℓ) where $X \to T$ is a torsor under $A_T \to T$ and $\ell \in \mathbf{Pic}^\lambda_X(T)$; morphisms $(X, \ell) \to (X', \ell')$ are isomorphisms of torsors $f\colon X \to X'$ such that $f^*(\ell') = \ell$. This is a stack in the étale topology over S. Since every A-torsor is étale locally trivial and since the translation action of A on $\mathbf{Pic}^\lambda_A$ is transitive, $\mathrm{PolTor}_{(A,\lambda)}$ is a gerbe.

On the other hand, consider the quotient stack $[A\backslash\mathbf{Pic}^\lambda_A]$. By definition, if T is an S-scheme then the groupoid $[A\backslash\mathbf{Pic}^\lambda_A](T)$ is defined as follows: objects are pairs (X, ψ), where $X \to T$ is a torsor under $A_T \to T$ and $\psi\colon X \to \mathbf{Pic}^\lambda_A$ an A-equivariant morphism; morphisms $(X, \psi) \to (X', \psi')$ are isomorphisms of A-torsors $f\colon X \to X'$ such that $\psi' \circ f = \psi$. Lemma 3.14 immediately implies:

Lemma 3.17. *The assignment $(X, \ell) \mapsto (X, \psi_\ell)$ defines an isomorphism of stacks $\mathrm{PolTor}_{(A,\lambda)} \to [A\backslash\mathbf{Pic}^\lambda_A]$.*

We will now compare these gerbes to the gerbe $\mathrm{ThetaGrp}_{(A[\lambda],e_\lambda)}$ introduced in §3.2. Recall that for an S-scheme T, $\mathrm{ThetaGrp}_{(A[\lambda],e_\lambda)}(T)$ is the groupoid of theta groups for $(A[\lambda], e_\lambda)_T$ with morphisms given by framed isomorphisms. Note that the assignment $(X, \ell) \mapsto \mathcal{G}(\ell)$ of Section 3.3 can be upgraded to a functor $\Theta\colon \mathrm{PolTor}_{(A,\lambda)} \to \mathrm{ThetaGrp}_{(A[\lambda],e_\lambda)}$, by sending a morphism $f\colon (X, \ell) \to (X', \ell')$ to $(f^*)^{-1}\colon \mathcal{G}(\ell) \to \mathcal{G}(\ell')$, where $f^*\colon \mathcal{G}(\ell') \to \mathcal{G}(f^*\ell') = \mathcal{G}(\ell)$ is the unique framed isomorphism of theta groups with the property that, after some étale surjective base change $T \to S$ over which $\ell'_T = [M]$ for line bundle M on X', $(f^*)_T\colon \mathcal{G}(M) \to \mathcal{G}((f_T)^*M)$ is given by $(x, \varphi) \mapsto (x, (f_T)^*\varphi)$.

The next theorem is one of the main technical results of this paper, so we restate our assumptions.

Theorem 3.18. *Let $A \to S$ be an abelian scheme and let $\lambda\colon A \to A^\vee$ be a polarization whose degree is invertible on S. Then the functor $\Theta\colon \mathrm{PolTor}_{(A,\lambda)} \to \mathrm{ThetaGrp}_{(A[\lambda],e_\lambda)}$ sending (X, ℓ) to $\mathcal{G}(\ell)$ is an isomorphism of gerbes. Consequently, there are isomorphisms*

$$[A\backslash\mathbf{Pic}^\lambda_A] \simeq \mathrm{PolTor}_{(A,\lambda)} \simeq \mathrm{ThetaGrp}_{(A[\lambda],e_\lambda)} \tag{3.6}$$

and $(X, \ell) \mapsto \mathcal{G}(\ell)$ induces an equivalence between the following groupoids:

1. *Pairs (X, ℓ), where X is an A-torsor and $\ell \in \mathbf{Pic}_{X/S}(S)$ is an element with $\phi_\ell = \lambda$, with morphisms $(X, \ell) \to (X', \ell')$ given by isomorphisms of torsors $f\colon X \to X'$ such that $f^*\ell' = \ell$;*
2. *Theta groups for $(A[\lambda], e_\lambda)$ over S, with morphisms given by framed isomorphisms.*

Proof. Since the source and target of Θ are gerbes, it suffices to show by Lemma 2.1 that $\Theta_x\colon \mathbf{Aut}(x) \to \mathbf{Aut}(\Theta(x))$ is an isomorphism for every S-scheme T and object x of $\mathrm{PolTor}_{(A,\lambda)}(T)$. Since this can be checked étale locally, we may assume that $T = S$, that $x = (A, [L])$ for some line bundle L on A with $\phi_L = \lambda$, and that $A[\lambda]$ is a constant group scheme. We now explicitly describe the map $\Theta_{(A,[L])}\colon \mathbf{Aut}((A, [L])) \to \mathbf{Aut}(\mathcal{G}(L); \mathrm{Id})$ and show that it is an isomorphism.

First note that $a \mapsto t_a$ induces an isomorphism $A[\lambda] \simeq \mathbf{Aut}((A, [L]))$, so it suffices to prove that the map $a \mapsto \Theta_{(A,[L])}(t_a)$ is an isomorphism $A[\lambda] \to \mathbf{Aut}(\mathcal{G}(L); \mathrm{Id})$. If $a \in A(S)$, then $(x, \varphi) \mapsto (x, t_a^*\varphi)$ defines an isomorphsm $\Phi_a\colon \mathcal{G}(L) \to \mathcal{G}(t_a^*L)$. On the other hand, if $a \in A[\lambda](S)$ then there exists an isomorphism $\gamma\colon L \to t_a^*L$. We have seen in Section 3.3 that the assignment $(x, \varphi) \mapsto (x, t_x^*(\gamma)\varphi\gamma^{-1})$ induces an isomorphism $F_a\colon \mathcal{G}(L) \to \mathcal{G}(t_a^*L)$ which does not depend on the choice of γ. By definition, $\Theta_{(A,[L])}(t_a)$ equals $\Phi_a^{-1} \circ F_a \in \mathrm{Aut}(\mathcal{G}(L); \mathrm{Id})$. We compute

$$(\Phi_a^{-1} \circ F_a)((x, \varphi)) = (x, t_a^*((t_x^*\gamma)\varphi\gamma^{-1})) = (a, \eta) \cdot (x, \varphi) \cdot (a, \eta)^{-1} = e_\lambda(a, x)(x, \varphi).$$

Therefore the map $a \mapsto \Theta_{(A,[L])}(t_a)$ is exactly the isomorphism $\alpha\colon A[\lambda] \to \mathbf{Aut}(\mathcal{G}(L); \mathrm{Id})$ of Lemma 3.9(2). We conclude that Θ is an equivalence and $\mathrm{PolTor}_{(A,\lambda)} \simeq \mathrm{ThetaGrp}_{(A[\lambda], e_\lambda)}$. The first isomorphism of (3.6) follows from Lemma 3.17. The equivalence of groupoids follows from taking S-points of Θ. □

Remark 3.19. *Given a theta group $\mathcal{G}$ for $(A[\lambda], e_\lambda)$ it is possible to explicitly construct a pair (X, ℓ) in $\mathrm{PolTor}_{(A,\lambda)}(S)$ with $\mathcal{G}(\ell) \simeq \mathcal{G}$; we sketch the details. For an S-scheme T, let $X(T)$ be the set of pairs (M, α), where $M \in \mathbf{Pic}_A^\lambda(T)$ and $\alpha \in \mathbf{Isom}(\mathcal{G}(M), \mathcal{G}; \mathrm{Id})(T)$. This is represented by a scheme $X \to S$, and the forgetful map $X \to \mathbf{Pic}_A^\lambda$ is a torsor under $\mathbf{Aut}(\mathcal{G}; \mathrm{Id}) \simeq A[\lambda]$. We can extend this $A[\lambda]$-action on X to an A-action, via the formula $a \cdot (M, \alpha) = (t_a^*M, \alpha \circ \Phi_a^{-1})$. (A similar computation to the proof of Theorem 3.18 shows that this action indeed restricts to the given $A[\lambda]$-action.) This defines an A-equivariant map $X \to \mathbf{Pic}_A^\lambda$, hence a pair (X, ℓ) representing λ by Lemma 3.14.*

Corollary 3.20. *In the notation of Theorem 3.18, the following statements are equivalent:*

- *There exists an A-torsor $X \to S$ and $\ell \in \mathbf{Pic}_{X/S}(S)$ with $\phi_\ell = \lambda$;*
- *There exists a theta group for $(A[\lambda], e_\lambda)$ over S.*
- *The class $[\mathbf{Pic}_A^\lambda] \in \mathrm{H}^1(S, A^\vee)$ lies in the image of the map $\mathrm{H}^1(S, A) \to \mathrm{H}^1(S, A^\vee)$ induced by λ.*

Proof. Combine Theorem 3.18 and Corollary 3.16. □

3.5 Linear theta groups

Under the equivalence of groupoids of Theorem 3.18 between theta groups and pairs (X, ℓ) with $\ell \in \mathbf{Pic}_X(S)$ representing λ, it is natural to ask: when we can choose ℓ to be of the form $[L]$ for some line bundle L on X? In other words, when is λ represented by an actual line bundle on X, not just an element of $\mathbf{Pic}_X(S)$? The answer is given by Theorem 3.24 and uses the concept of linear theta groups.

Let S be a scheme and let $D = (d_1, \dots, d_g)$ be a type with $\#D$ invertible on S. If $\mathscr{F}$ is a quasi-coherent sheaf on S, let $\mathbf{Aut}(\mathscr{F})$ be the sheaf of groups on the étale site of S with $\mathbf{Aut}(\mathscr{F})(T) = \mathrm{Aut}_{\mathcal{O}_T}(f^*\mathscr{F})$ for every morphism $f\colon T \to S$. For example, if $\mathscr{F} = \mathcal{O}_S^{\oplus n}$ then $\mathbf{Aut}(\mathscr{F}) = \mathrm{GL}_{n,S}$.

Definition 3.21. *Let $\mathcal{G}$ be a theta group for a symplectic module (M, e) of type D over S. A* representation *for $\mathcal{G}$ is a quasi-coherent sheaf on $\mathscr{F}$ endowed with a homomorphism $\rho\colon \mathcal{G} \to \mathbf{Aut}(\mathscr{F})$. We say that ρ (or by abuse of notation $\mathscr{F}$) has* weight 1 *if the subgroup $\mathbb{G}_{m,S}$ of $\mathcal{G}$ acts on $\mathscr{F}$ via scalar multiplication, i.e., via restriction of the standard $\mathcal{O}_S$-action on $\mathscr{F}$ to $\mathbb{G}_{m,S} \leq \mathcal{O}_S$. We say $\mathscr{F}$ is a* Schrödinger representation *for $\mathcal{G}$ if it is of weight 1 and if $\mathscr{F}$ is a locally free of rank $\#D = d_1 \cdots d_g$. If such a representation exists, we say that $\mathcal{G}$ is* linear*.*

Example 3.22 (The motivating example)**.** *Let (A, λ) be a polarized abelian scheme of type D over S and L a line bundle on an A-torsor X with $\phi_L = \lambda$. If S is the spectrum of a field k, then the space of global sections $\mathrm{H}^0(X, L)$, endowed with the action $(a, \varphi) \cdot s = t^*_{-a}(\varphi(s))$, is a weight-1 representation of $\mathcal{G}(L)$. By Riemann–Roch, $\dim \mathrm{H}^0(X, L) = \#D$, so $\mathrm{H}^0(X, L)$ is a Schrödinger representation for $\mathcal{G}(L)$. If S is a general base scheme and $\pi\colon A \to S$ the structure morphism, then π_*L again has the structure of a weight-1 representation. By cohomology and base change and the result for fields, it is locally free of rank $\#D$, so π_*L is a Schrödinger representation.*

Let $\mathcal{G}_D$ be the standard theta group of type D for (M_D, e_D) constructed in §3.2. Then we can construct a Schrödinger representation $\mathscr{V}_D$ for $\mathcal{G}_D$, following Mumford [Mum66, p. 297]. Recall that $M_D = K_D \times K_D^\vee$, where K_D is a contant group scheme. Let $\mathscr{V}_D$ be the $\mathcal{O}_S$-module with $\mathscr{V}_D(U) = \{\text{functions } K_D \to \mathcal{O}_S(U)\}$ for every open $U \subset S$. Then $\mathscr{V}_D$ is free of rank $\#K_D = \#D$. Let $\mathcal{G}_D$ act on $\mathscr{V}_D$ via the formula

$$((\lambda, x, \chi) \cdot f)(y) = \lambda\chi(y)f(y + x).$$

A calculation shows that this action is well defined and that $\mathscr{V}_D$ is a Schrödinger representation for $\mathcal{G}_D$.

If S is the spectrum of an algebraically closed field, then every theta group has a unique Schrödinger representation; this is an algebraic version of the Stone–von Neumann theorem [Mum66, page 295, Proposition 3]. Over an arbitrary base, we have:

Proposition 3.23. *Let $\mathcal{G}$ be a theta group for a symplectic module (M, e) over S.*

1. *There exists an étale surjective morphism $T \to S$ such that there exists a Schrödinger representation for $\mathcal{G}_T$.*
2. *If $\mathcal{G}$ is linear and $\mathscr{V}$ is a Schrödinger representation for $\mathcal{G}$, then the assignment $\mathscr{F} \mapsto \mathscr{V} \otimes \mathscr{F}$ defines an equivalence between the category of quasi-coherent sheaves on S and the category of weight-1 representations of $\mathcal{G}$. (The $\mathcal{G}$-action on $\mathscr{V} \otimes \mathscr{F}$ is induced by the given $\mathcal{G}$-action on $\mathscr{V}$ and the trivial one on $\mathscr{F}$.)*
3. *If $\mathscr{V}, \mathscr{V}'$ are Schrödinger representations for $\mathcal{G}$, then there exists an invertible sheaf $\mathscr{L}$ on S and an isomorphism of $\mathcal{G}$-representations $\mathscr{V}' \simeq \mathscr{V} \otimes \mathscr{L}$.*

Proof.

1. We may assume that S is connected and (by Lemmas 3.3 and 3.8) that $\mathcal{G} = \mathcal{G}_D$, in which case we may take $\mathscr{V}_D$ as Schrödinger representation.
2. This is [MB85, p.113, Corollaire 2.4.3].
3. Part 2 implies that $\mathscr{V}' = \mathscr{V} \otimes \mathscr{L}$ for some quasi-coherent $\mathscr{L}$ on S. To show $\mathscr{L}$ is invertible, we may take an étale base change of S, hence assume $\mathcal{G} = \mathcal{G}_D$ for some type D. Now apply [Mum67, §6, Proposition 2].

□

Let $\mathcal{G}$ be a theta group for a symplectic module (M, e) over S. Let $\mathrm{Sch}_{\mathcal{G}}$ be the fibered category over S-schemes such that for an S-scheme T, the groupoid $\mathrm{Sch}_{\mathcal{G}}(T)$ has objects Schrödinger representations for $\mathcal{G}_T$ and morphisms given by $\mathcal{G}_T$-equivariant isomorphisms. Descent for quasi-coherent sheaves [BLR90, Section 6.1, Theorem 4] shows that $\mathrm{Sch}_{\mathcal{G}}$ is a stack in the étale topology, and Proposition 3.23 shows that $\mathrm{Sch}_{\mathcal{G}}$ is a gerbe. Moreover, that same proposition shows that if $\mathscr{V}$ is a Schrödinger representation for $\mathcal{G}$, then $\mathbf{Aut}_{\mathrm{Sch}_{\mathcal{G}}(S)}(\mathscr{V}) \simeq \mathbb{G}_{m,S}$. Hence $\mathrm{Sch}_{\mathcal{G}}$ is a gerbe banded by $\mathbb{G}_m$ and by Lemma 2.3 defines a class $[\mathrm{Sch}_{\mathcal{G}}] \in \mathrm{H}^2(S, \mathbb{G}_m)$ which vanishes if and only if $\mathcal{G}$ is linear.

On the other hand, let $\pi\colon A \to S$ be an abelian scheme, let X be an A-torsor and let $\ell \in \mathbf{Pic}_X(S)$. Let $\mathrm{Ob}(\ell)$ be the category fibered in groupoids such that for an S-scheme T, the groupoid $\mathrm{Ob}(\ell)(T)$ has objects line bundles L on X_T such that $[L] = \ell_T$, and morphisms $L \to L'$ are given by isomorphisms of line bundles. The exact sequence (2.1) shows that $\mathrm{Ob}(\ell)$ is a gerbe, which is again banded by $\mathbb{G}_m$. Hence $\mathrm{Ob}(\ell)$ defines a class $[\mathrm{Ob}(\ell)] \in \mathrm{H}^2(S, \mathbb{G}_m)$, which equals the class $\mathrm{ob}(\ell)$ of ℓ under the connecting homomorphism $\mathbf{Pic}_X(S) \to \mathrm{H}^2(S, \mathbb{G}_m)$ of (2.1).

Theorem 3.24. *Let $\pi\colon A \to S$ be an abelian scheme, $\lambda\colon A \to A^\vee$ a polarization whose degree is invertible on S, X an A-torsor and $\ell \in \mathbf{Pic}_X(S)$ an element with $\phi_\ell = \lambda$ and with theta group $\mathcal{G}(\ell)$. Then the assignment $L \mapsto \pi_* L$ induces an isomorphism $\mathrm{Ob}(\ell) \xrightarrow{\sim} \mathrm{Sch}_{\mathcal{G}(\ell)}$. Consequently, there exists a line bundle L on X with $[L] = \ell$ if and only if the theta group $\mathcal{G}(\ell)$ is linear.*

Proof. The morphism is well defined by Example 3.22. By Lemma 2.1, we just need to verify that if L is an object of $\mathrm{Ob}(\ell)(T)$ for some $T \to S$, then $\mathrm{Aut}(L) \to \mathrm{Aut}((\pi_T)_* L)$ is an isomorphism. This follows from that fact that every automorphism of L is given by multiplication by an element of $\lambda \in \mathbb{G}_m(T)$, and that the induced automorphism of the locally free sheaf $(\pi_T)_* L$ is again multiplication by λ. □

Corollary 3.25. *Let $A \to S$ be an abelian scheme and λ a polarization on A whose degree is invertible on S. Then the following statements are equivalent:*

- *There exists an A-torsor X and line bundle L on X with $\phi_L = \lambda$;*
- *There exists a linear theta group for $(A[\lambda], e_\lambda)$.*

Proof. Combine Theorems 3.18 and 3.24. □

Proof of Theorem Θ. Combine Corollaires 3.20 and 3.25. □

The following bound on the order of the class $[\mathrm{Sch}_{\mathcal{G}}] \in \mathrm{H}^2(S, \mathbb{G}_m)$ will be useful in Section 4.

Proposition 3.26. *Let $\mathcal{G}$ be a theta group for a symplectic module (M, e) of type $D = (d_1, \ldots, d_g)$ over S. Then $(\#D)[\mathrm{Sch}_{\mathcal{G}}] = 0$ in $\mathrm{H}^2(S, \mathbb{G}_m)$, where we recall that $\#D = d_1 \cdots d_g$.*

Proof. This is a result of Polishchuk [Pol02, Proposition 2.1]. □

We record a case where the existence problem of theta groups is trivial:

Proposition 3.27. *Let k be a field of cohomological dimension ≤ 1 in the sense of [Poo17, Definition 1.4.3]. (For example, k is finite or the function field of a curve over an algebraically closed field.) Then for every symplectic module (M, e) over k, there exists a linear theta group for (M, e).*

Proof. Below Lemma 3.11, we have defined a class $[\mathrm{ThetaGrp}_{(M,e)}] \in \mathrm{H}^2(k, M)$ which vanishes if and only if there exists a theta group for (M, e). By definition of cohomological dimension, $\mathrm{H}^2(k, M) = 0$ and so there exists a theta group $\mathcal{G}$ for (M, e). Moreover, since $\mathrm{H}^2(k, \mathbb{G}_m) = 0$, the class $[\mathrm{Sch}_{\mathcal{G}}]$ vanishes and so $\mathcal{G}$ is linear. □

3.6 Symmetric line bundles representing λ

In this section, we prove an analogue of Theorem 3.18 for symmetric line bundles on symmetrized torsors (Theorem 3.31). We start by recalling the well-known situation for symmetric line bundles on A.

Let $A \to S$ be an abelian scheme. Denote the inversion map by $[-1]\colon A \to A$. Recall that a line bundle L on A is *symmetric* if $L \simeq [-1]^*L$. Since $[-1]^*$ acts as inversion on $\mathbf{Pic}^0_A$ and as the identity on $\mathbf{NS}_A$, taking $[-1]^*$-fixed points of the exact sequence (2.3) for $X = A$ results in the exact sequence

$$1 \to A^\vee[2] \to \mathbf{Pic}^{\mathrm{sym}}_A \to \mathbf{NS}_A \to 1, \tag{3.7}$$

see [PR11, Section 3.2]. A line bundle L on A is symmetric if and only if its class $[L] \in \mathbf{Pic}_A(S)$ lies in $\mathbf{Pic}^{\mathrm{sym}}_A(S)$. Write $\mathbf{Pic}^{\mathrm{sym},\lambda}_A$ for the fiber of an element $\lambda \in \mathbf{NS}_A(S)$ under the projection map. This sequence shows that $\mathbf{Pic}^{\mathrm{sym},\lambda}_A$ is a torsor under $A^\vee[2]$, which has an S-point if and only if there exists a symmetric line bundle L on A representing λ. The class of $\mathbf{Pic}^{\mathrm{sym},\lambda}_A$ maps to the class of $\mathbf{Pic}^{\lambda}_A$ under the map $\mathrm{H}^1(S, A^\vee[2]) \to \mathrm{H}^1(S, A^\vee)$.

In contrast to $\mathbf{Pic}^{\lambda}_A$, the class of $\mathbf{Pic}^{\mathrm{sym},\lambda}_A$ is known to have a very concrete alternative description. Consider the nondegenerate pairing $e_2\colon A[2] \times A^\vee[2] \to \mu_2$ induced by the identification $A^\vee[2] = A[2]^\vee$, and let $e_2^\lambda\colon A[2] \times A[2] \to \mu_2$ be the alternating pairing defined by $e_2^\lambda(x, y) = e_2(x, \lambda(y))$. We say a map of S-schemes $q\colon A[2] \to \mu_2$ is a *quadratic refinement* of e_2^λ if $q(x+y)q(x)q(y) = e_2^\lambda(x, y)$ for all $x, y \in A[2]$. The scheme of quadratic refinments for e_2^λ is a torsor under $A^\vee[2]$. It is well known (see [Pol03, Section 13.1] and [PR11, Proposition 3.6]) that this torsor is isomorphic to $\mathbf{Pic}^{\mathrm{sym},\lambda}_A$. Consequently, there exists a symmetric line bundle on A representing λ if and only if there exists a quadratic refinement $q\colon A[2] \to \mu_2$ of e_2^λ. This condition is often satisfied in practice (for example, when k is finite); see [PR11, Proposition 3.12] for a list of sufficient conditions.

We now consider the weaker question whether there exists a symmetric line bundle on a symmetric A-torsor representing λ.

Definition 3.28. *Let X be an A-torsor. An* inversion *on X is a morphism of S-schemes $\tau\colon X \to X$ such that $\tau^2 = \mathrm{Id}_X$ and such that $\tau(a + x) = (-a) + x$ for all $a \in A(T)$, $x \in X(T)$ and S-schemes T. If such a τ exists, we say X is* symmetric. *A* symmetrized A-torsor *is a pair (X, τ), where X is an A-torsor and τ is an inversion. A morphism between symmetrized A-torsors $(X, \tau), (X', \tau')$ is an isomorphism of A-torsors $X \to X'$ intertwining τ and τ'.*

If (X, τ) is a symmetrized torsor, then X^τ is a torsor under $A[2]$, and the assignment $(X, \tau) \mapsto X^\tau$ induces an equivalence of categories between the category of symmetrized torsors for A and the category of $A[2]$-torsors. An A-torsor X is symmetric if and only if its class $[X] \in \mathrm{H}^1(S, A)$ lies in the image of the map $\mathrm{H}^1(S, A[2]) \to \mathrm{H}^1(S, A)$, if and only if $2[X] = 0$.

Given a symmetrized torsor (X, τ) for A, let $\mathbf{Pic}^\tau_X \subset \mathbf{Pic}_X$ denote the fixed points of $\tau^*\colon \mathbf{Pic}_X \to \mathbf{Pic}_X$. We say a line bundle L on X is *τ-symmetric* if it defines an element of $\mathbf{Pic}^\tau_X(S)$. If $(X, \tau) = (A, [-1])$, then $\mathbf{Pic}^\tau_X = \mathbf{Pic}^{\mathrm{sym}}_A$ and τ-symmetric line bundles are the same as symmetric line bundles. Similarly to (3.7), taking τ^*-fixed points of the exact sequence (2.3) results in the exact sequence

$$1 \to A^\vee[2] \to \mathbf{Pic}^\tau_X \to \mathbf{NS}_A \to 1.$$

Let $\lambda\colon A \to A^\vee$ be a polarization whose degree is invertible on S. Write $\mathbf{Pic}^{\tau,\lambda}_X$ for the fiber of $\mathbf{Pic}^\tau_X \to \mathbf{NS}_X \simeq \mathbf{NS}_A$ above $\lambda \in \mathbf{NS}_A(S)$. We will prove a criterion for the existence of an element $\ell \in \mathbf{Pic}^{\tau,\lambda}_X(S)$ for some symmetrized A-torsor (X, τ) in terms of theta groups.

Let $\mathrm{PolTor}^{\mathrm{sym}}_{(A,\lambda)}$ be the stack such that for each S-scheme T, the groupoid $\mathrm{PolTor}^{\mathrm{sym}}_{(A,\lambda)}(T)$ has: objects given by triples (X,τ,ℓ), where (X,τ) is a symmetrized torsor for $A_T \to T$ and $\ell \in \mathbf{Pic}^{\tau,\lambda}_X(T)$; morphisms $(X,\tau,\ell) \to (X',\tau',\ell')$ given by isomorphisms of symmetrized torsors $f\colon (X,\tau) \to (X',\tau')$ such that $f^*(\ell') = \ell$. This is a stack in the étale topology over S. On the other hand, consider the quotient stack $[A[2]\backslash \mathbf{Pic}^{\mathrm{sym},\lambda}_A]$, where $A[2]$ acts on $\mathbf{Pic}^{\mathrm{sym},\lambda}$ by translation of line bundles. Similarly to Lemma 3.17 and using the notation of that lemma, we have:

Lemma 3.29. *The assignment $(X,\tau,\ell) \mapsto (X^\tau, \psi_\ell|_{X^\tau})$ defines an isomorphism of stacks $\mathrm{PolTor}^{\mathrm{sym}}_{(A,\lambda)} \to [A[2]\backslash \mathbf{Pic}^{\mathrm{sym},\lambda}_A]$.*

Proof. This follows from taking τ-fixed points on both sides of Lemma 3.14 and using the identity $\psi_{\tau^*\ell} = [-1]^* \circ \psi_\ell \circ \tau$ for all $\ell \in \mathbf{Pic}^\lambda_X(S)$. □

The next definition has been considered before, see for example [MS23, Remark 5.22] and [Pol02, Section 1].

Definition 3.30. *Let (M,e) be a symplectic module over S and $\mathcal{G}$ a theta group for (M,e). An* inversion *on $\mathcal{G}$ is an element $\iota \in \mathrm{Aut}(\mathcal{G})$ such that ι maps to $-\mathrm{Id}_M$ under the map $\mathrm{Aut}(\mathcal{G}) \to \mathrm{Sp}(M)$ of (3.4). If an inversion on $\mathcal{G}$ exists, we say that $\mathcal{G}$ is* symmetric*. A* symmetrized theta group *for (M,e) is a pair $(\mathcal{G},\iota)$ where $\mathcal{G}$ is a theta group for (M,e) and ι an inversion on $\mathcal{G}$. An isomorphism (resp. framed isomorphism) between symmetrized theta groups $(\mathcal{G},\iota) \to (\mathcal{G}',\iota')$ is an isomorphism (resp. framed isomorphism) $f\colon \mathcal{G} \to \mathcal{G}'$ of theta groups such that $f \circ \iota = \iota' \circ f$.*

If (M,e) is a symplectic module over S, let $\mathrm{ThetaGrp}^{\mathrm{sym}}_{(M,e)}$ be the fibered category such that for any S-scheme T, the groupoid $\mathrm{ThetaGrp}^{\mathrm{sym}}_{(M,e)}(T)$ has objects symmetrized theta groups for $(M,e)_T$, and morphisms given by framed isomorphisms.

Given a triple (X,τ,ℓ) in $\mathrm{PolTor}^{\mathrm{sym}}_{(A,\lambda)}(S)$, we can upgrade the theta group $\mathcal{G}(\ell)$ (defined in §3.3) to a symmetrized theta group, as follows. First suppose ℓ is represented by a line bundle L on X that satisfies $\tau^*L \simeq L$. (This can always be achieved after an étale surjective base change.) Then the assignment $(a,\varphi) \mapsto (-a, \tau^*\varphi)$ induces an isomorphism $\mathcal{G}(L) \to \mathcal{G}(\tau^*L)$. A choice of isomorphism $\tau^*L \to L$ induces a framed isomorphism $\mathcal{G}(\tau^*L) \to \mathcal{G}(L)$ independent of this choice (see §3.3). Their composition is an inversion $\iota_L\colon \mathcal{G}(L) \to \mathcal{G}(L)$ on $\mathcal{G}(L)$. In general, there exists a unique inversion $\iota_\ell\colon \mathcal{G}(\ell) \to \mathcal{G}(\ell)$ such that if $T \to S$ is an étale surjective map with $\ell_T = [L]$ satisfying $\tau^*L \simeq L$, then $(\iota_\ell)_T = \iota_L$. We define the functor $\Theta^{\mathrm{sym}}\colon \mathrm{PolTor}^{\mathrm{sym}}_{(A,\lambda)} \to \mathrm{ThetaGrp}_{(A[\lambda],e_\lambda)}$ by sending (X,τ,ℓ) to $(\mathcal{G}(\ell),\iota_\ell)$, and with morphisms defined similarly to the functor $\Theta\colon \mathrm{PolTor}_{(A,\lambda)} \to \mathrm{ThetaGrp}_{(A[\lambda],e_\lambda)}$ of Theorem 3.18.

Theorem 3.31. *Let $A \to S$ be an abelian scheme and let $\lambda\colon A \to A^\vee$ be a polarization whose degree is invertible on S. Then the functor $\Theta^{\mathrm{sym}}\colon \mathrm{PolTor}^{\mathrm{sym}}_{(A,\lambda)} \to \mathrm{SymThetaGrp}_{(A[\lambda],e_\lambda)}$ sending (X,τ,ℓ) to $(\mathcal{G}(\ell),\iota_\ell)$ is an isomorphism of stacks. Consequently, there are isomorphisms*

$$[A[2]\backslash \mathbf{Pic}^{\mathrm{sym},\lambda}_A] \simeq \mathrm{PolTor}^{\mathrm{sym}}_{(A,\lambda)} \simeq \mathrm{SymThetaGrp}_{(A[\lambda],e_\lambda)} \tag{3.8}$$

and $(X,\tau,\ell) \mapsto (\mathcal{G}(\ell),\iota_\ell)$ induces an equivalence between the following groupoids:

1. *Triples (X,τ,ℓ), where (X,τ) is a symmetrized A-torsor and $\ell \in \mathbf{Pic}^\tau_X(S)$ is an element with $\phi_\ell = \lambda$, with isomorphisms $(X,\tau,\ell) \to (X',\tau',\ell')$ given by isomorphisms of symmetrized torsors $f\colon (X,\tau) \to (X',\tau')$ such that $f^*\ell' = \ell$;*

2. *Symmetrized theta groups for $(A[\lambda],e_\lambda)$ over S, with isomorphisms given by framed isomorphisms.*

Proof. This follows from taking "fixed points" of the isomorphism of Theorem 3.18 under a certain duality operation. More precisely, let T be an S-scheme and (X, ℓ) an object of $\mathrm{PolTor}_{(A,\lambda)}(T)$. Let X' be the A-torsor whose underlying S-scheme equals X, but where the A-action $+' \colon A \times X' \to X'$ is given by $a +' x = (-a) + x$, where on the right hand side we use the A-action on X. Let ℓ' equal $\ell \in \mathbf{Pic}^{\lambda}_X(S)$, seen as an element of $\mathbf{Pic}^{\lambda}_{X'}(S)$. Define the functor $\Phi\colon \mathrm{PolTor}_{(A,\lambda)} \to \mathrm{PolTor}_{(A,\lambda)}$ by sending (X, ℓ) to (X', ℓ') and leaving the morphisms unchanged. Unwinding the definition of an inversion on an A-torsor, we find that the stack $\mathrm{PolTor}^{\mathrm{sym}}_{(A,\lambda)}$ is equivalent to the stack $\mathrm{PolTor}^{\Phi}_{(A,\lambda)}$ of pairs $((X, \ell), \tau)$, where (X, ℓ) is an object of $\mathrm{PolTor}_{(A,\lambda)}$ and τ is an isomorphism $(X, \ell) \xrightarrow{\sim} \Phi((X, \ell))$. On the other hand, given a theta group $\mathcal{G}$ for $A[\lambda]$, let $\mathcal{G}'$ be the theta group whose underlying group scheme equals $\mathcal{G}$ but whose projection map $\mathcal{G}' \to A[\lambda]$ equals the composite $\mathcal{G} \to A[\lambda] \xrightarrow{-\mathrm{Id}} A[\lambda]$. Let $\Psi\colon \mathrm{ThetaGrp}_{(A[\lambda],e_\lambda)} \to \mathrm{ThetaGrp}_{(A[\lambda],e_\lambda)}$ be the functor sending $\mathcal{G}$ to $\mathcal{G}'$ and leaving the morphisms unchanged. Then $\mathrm{ThetaGrp}^{\mathrm{sym}}_{(A[\lambda],e_\lambda)}$ is equivalent to the stack $\mathrm{ThetaGrp}^{\Psi}_{(A[\lambda],e_\lambda)}$ of pairs $(\mathcal{G}, \iota)$, where $\mathcal{G}$ is an object of ThetaGrp and ι an isomorphism $\mathcal{G} \xrightarrow{\sim} \Psi(\mathcal{G})$. Since the functors $\Theta \circ \Phi$ and $\Psi \circ \Theta$ are isomorphic, the functor Θ of Theorem 3.18 induces an isomorphism of stacks $\mathrm{PolTor}^{\Phi}_{(A,\lambda)} \xrightarrow{\sim} \mathrm{ThetaGrp}^{\Psi}_{(A[\lambda],e_\lambda)}$. After identifying $\mathrm{PolTor}^{\Phi}_{(A,\lambda)}$ with $\mathrm{PolTor}^{\mathrm{sym}}_{(A,\lambda)}$ and $\mathrm{ThetaGrp}^{\Psi}_{(A[\lambda],e_\lambda)}$ with $\mathrm{ThetaGrp}^{\mathrm{sym}}_{(A[\lambda],e_\lambda)}$, this isomorphism equals Θ^{sym}, proving that Θ^{sym} is an isomorphism. The first isomorphism of (3.8) follows from Lemma 3.29. The equivalence of groupoids follows from taking S-points of Θ^{sym}. □

Corollary 3.32. *Let A/S be an abelian scheme and λ a polarization whose degree is invertible on S. The following statements are equivalent:*

1. *There exists a symmetrized A-torsor (X, τ) and a τ-symmetric $\ell \in \mathbf{Pic}_X(S)$ with $\phi_\ell = \lambda$.*
2. *There exists a symmetric theta group for $(A[\lambda], e_\lambda)$.*
3. *The class $[\mathbf{Pic}^{\mathrm{sym},\lambda}_A] \in \mathrm{H}^1(S, A^\vee[2])$ lies in the image of $\mathrm{H}^1(\lambda)\colon \mathrm{H}^1(S, A[2]) \to \mathrm{H}^1(S, A^\vee[2])$.*

The following statements are also equivalent:

1. *There exists a symmetrized A-torsor (X, τ) and a τ-symmetric line bundle L on X with $\phi_L = \lambda$.*
2. *There exists a linear symmetric theta group for $(A[\lambda], e_\lambda)$.*

Proof. The equivalence between the first three statements follows from the isomorphisms of Theorem 3.31. Together with Theorem 3.24, it implies the equivalence between the last two statements. □

The following lemma will be useful in Section 4.1. It uses concepts and notation of Section 3.5.

Lemma 3.33. *Let $\mathcal{G}$ be a symmetric theta group for a symplectic module (M, e) over S. Then $8[\mathrm{Sch}_{\mathcal{G}}] = 0$ in $\mathrm{H}^2(S, \mathbb{G}_m)$.*

Proof. This follows from results of Polishchuk, specifically by combining [Pol02, Theorem 1.4 and Proposition 2.2]. □

4 Proofs of Theorems A and B

We use the formalism of the previous section to show that the answer to Question 1 is yes for many pairs (A, λ). Theorem A follows from Corollary 4.4 and Theorem B follows from Corollary 4.12. In Section 4.2, which might be of independent interest, we answer a question of Chidambaram affirmatively [Chi24]. Section 4.4 surveys all variants of Question 1 and logical implications between these.

4.1 Symplectic modules of odd order

Let S be a scheme and $D = (d_1, \dots, d_g)$ a type such that d_g is invertible on S. In Sections 3.1 and 3.2 we have constructed group schemes $M_D, \mathcal{G}_D$ and a sequence of finite étale group schemes

$$1 \to M_D \to \mathbf{Aut}(\mathcal{G}_D) \to \mathbf{Sp}(M_D) \to 1 \tag{4.1}$$

which is exact by Lemma 3.9. We now introduce an automorphism of this sequence, which will show that it splits when $\#D$ is odd. In the notation of (3.3), let $\iota \in \mathrm{Aut}(\mathcal{G}_D)$ be the element defined by the formula $\iota(\lambda, x, \chi) = (\lambda, -x, -\chi)$. Then ι lifts $-\mathrm{Id}_{M_D}$, in other words is an inversion on $\mathcal{G}_D$ in the notation of Section 3.6. Conjugation by ι defines an automorphism $\Phi\colon \mathbf{Aut}(\mathcal{G}_D) \to \mathbf{Aut}(\mathcal{G}_D)$ that restricts to $-\mathrm{Id}$ on M_D and induces the identity on $\mathbf{Sp}(M_D)$. In other words, we obtain a commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & M_D & \longrightarrow & \mathbf{Aut}(\mathcal{G}_D) & \longrightarrow & \mathbf{Sp}(M_D) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle -\mathrm{Id}} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ 1 & \longrightarrow & M_D & \longrightarrow & \mathbf{Aut}(\mathcal{G}_D) & \longrightarrow & \mathbf{Sp}(M_D) & \longrightarrow & 1 \end{array} \tag{4.2}$$

Let $\mathbf{Aut}(\mathcal{G}_D, \iota)$ be the fixed point subgroup scheme of $\Phi\colon \mathbf{Aut}(\mathcal{G}_D) \to \mathbf{Aut}(\mathcal{G}_D)$.

Lemma 4.1. *Suppose that $\#D = d_1 \cdots d_g$ is odd. Then the restriction $\mathbf{Aut}(\mathcal{G}_D, \iota) \to \mathbf{Sp}(M_D)$ is an isomorphism. Consequently, the sequence (4.1) splits.*

Proof. The claim can be checked after an étale base change, so we may assume all the group schemes in (4.1) are split and identify them with their S-points. Given an element $m \in M_D$, let $\alpha_m \in \mathrm{Aut}(\mathcal{G}_D)$ be the framed automorphism constructed in Lemma 3.9. Let $g \in \mathrm{Sp}(M_D)$ be an element and $\tilde{g} \in \mathrm{Aut}(\mathcal{G}_D)$ an arbitrary lift. Then $\Phi(\tilde{g}) = \alpha_m \tilde{g}$ for some $m \in M_D$. Since $\Phi(\alpha_n \tilde{g}) = \alpha_{m-2n}(\alpha_n \tilde{g})$ for all $n \in M_D$ and since $\#M_D$ is odd, there exists a unique $n \in M_D$ such that $\alpha_n \tilde{g} \in \mathrm{Aut}(\mathcal{G}_D, \iota)$, namely $n = \frac{1}{2}m$. This shows $\mathrm{Aut}(\mathcal{G}_D, \iota) \to \mathrm{Sp}(M_D)$ is a bijection, hence an isomorphism. □

Theorem 4.2. *Let (M, e) be a symplectic module of type D over a scheme S. Assume that $\#D$ is odd. Then there exists a symmetric theta group for (M, e). If $(\mathcal{G}, \iota)$ and $(\mathcal{G}', \iota')$ are symmetrized theta groups for (M, e), then there exists a unique framed isomorphism between $(\mathcal{G}, \iota)$ and $(\mathcal{G}', \iota')$. Every symmetric theta group for (M, e) is linear.*

Proof. By Lemma 4.1, the class of (M, e) in $\mathrm{H}^1(S, \mathbf{Sp}(M_D))$ (using Lemma 3.5) lifts to a class in $\mathrm{H}^1(S, \mathbf{Aut}(\mathcal{G}_D, \iota))$, hence to a class $c \in \mathrm{H}^1(S, \mathbf{Aut}(\mathcal{G}_D))$. By Lemma 3.11, this implies that there exists a theta group for (M, e). The fact that c comes from a class in $\mathrm{H}^1(S, \mathbf{Aut}(\mathcal{G}_D, \iota))$ means that there exists a symmetric theta group for (M, e). This proves existence. To prove uniqueness, let $(\mathcal{G}, \iota), (\mathcal{G}', \iota')$ be symmetrized theta groups for (M, e). To prove that there exists a unique framed isomorphism between them, we may apply an étale base change and hence by Lemma 3.8 we may assume that $\mathcal{G} = \mathcal{G}'$ and that M is constant. Then $\iota' = \alpha_m \circ \iota$ for some $m \in M(S)$, where $\alpha\colon M \to \mathbf{Aut}(\mathcal{G}; \mathrm{Id})$ is the isomorphism of Lemma 3.9. The formula $\iota' \circ \alpha_n = \alpha_n \circ (\alpha_{m-2n}\iota)$ shows that there exists a unique $n \in M(S)$ such that α_n is an isomorphism $(\mathcal{G}, \iota) \xrightarrow{\sim} (\mathcal{G}', \iota')$, namely $n = \frac{1}{2}m$.

It suffices to prove that every symmetric theta group $\mathcal{G}$ for (M, e) is linear. Equivalently, by the discussion in §3.5, we need to show the class $c = [\mathrm{Sch}_{\mathcal{G}}] \in \mathrm{H}^2(S, \mathbb{G}_m)$ vanishes. By Proposition 3.26, c has odd order. On the other hand, Lemma 3.33 shows that $8c = 0$. We conclude that $c = 0$ and hence that $\mathcal{G}$ is linear. □

Remark 4.3. *It is possible to explicitly construct a symmetric theta group for (M, e): let $b\colon M \times M \to \mathbb{G}_m$ be an alternating pairing with $b^2 = e$ (such a b exists since $\#M$ is odd) and let $\mathcal{G}$ be the group $\mathbb{G}_m \times M$ endowed with multiplication $(\lambda, m) \cdot (\lambda', n) = (\lambda\lambda' b(m, n), m + n)$. A computation shows that $\mathcal{G}$ is a theta group for (M, e) and $\iota(\lambda, m) = (\lambda, -m)$ is an inversion on $\mathcal{G}$, so $\mathcal{G}$ is symmetric. Theorem 4.2 shows that $\mathcal{G}$ is linear, but we do not know how to construct a Schrödinger representation for $\mathcal{G}$ explicitly.*

Theorem A follows from the next stronger result.

Corollary 4.4. *Let $A \to S$ be an abelian scheme and let $\lambda\colon A \to A^\vee$ be a polarization of type D whose degree is odd and invertible on S. Then there exists a symmetrized A-torsor (X, τ) and a τ-symmetric line bundle L on X with $\phi_L = \lambda$. If (X', τ', L') is another such triple, there exists a unique isomorphism of symmetrized torsors $f\colon (X,\tau) \to (X', \tau')$ that satisfies $f^*[L'] = [L]$ in $\mathbf{Pic}_X(S)$.*

Proof. Combine Theorems 4.2 and 3.24 with the equivalence of groupoids of Theorem 3.31. □

4.2 Realizing symplectic modules as 2-torsion of Jacobians

In this section and most of the remainder of the paper, we assume the base S is the spectrum of a field k. We now show that, under certain assumptions on the characteristic, every symplectic module of type $(2), (2,2)$ or $(2,2,2)$ over k admits a theta group. The crucial observation is to realize every such symplectic module as the 2-torsion in a principally polarized abelian variety. More precisely, we prove:

Proposition 4.5. *Let k be an infinite field of characteristic not 2 and let $g \in \{1,2,3\}$. If $g = 3$, additionally assume that k has characteristic zero. Let D be the type $(2, \cdots, 2)$ of length g. Then for every symplectic module (M, e) of type D over k, there exists a genus-g curve C with Jacobian J and an isomorphism $(M, e) \simeq (J[2], e_2)$, where e_2 is the Weil pairing on $J[2]$.*

Since the proof is much easier for $g = 1, 2$, we start by treating those cases.

Proof of Theorem 4.5 when $g = 1, 2$: First suppose $g = 1$. Let (M, e) be a symplectic module of type (2) over k. The Gal_k-action on the three nonzero elements of $M(k^{\mathrm{sep}}) \simeq (\mathbb{Z}/2\mathbb{Z})^2$ determines (after labeling these elements) a homomorphism $\mathrm{Gal}_k \to S_3$ to the symmetric group. This homomorphism corresponds to an étale cubic k-algebra K. There exists a monic separable polynomial $f(x) \in k[x]$ of degree 3 such that $K \simeq k[x]/(f(x))$. Let E/k be the elliptic curve with Weierstrass equation $y^2 = f(x)$. Since the nonzero elements of $E[2](k^{\mathrm{sep}})$ are in bijection with the roots of f, there is an isomorphism of finite k-groups $M \simeq E[2]$. Since there is a unique nondegenerate alternating pairing on an $\mathbb{F}_2$-vector space of dimension 2, every such isomorphism must intertwine e with e_2.

Now consider the case $g = 2$, where roughly the same principles apply. Let $b_1, \ldots, b_6$ be the standard basis of $\mathbb{F}_2^6$, let $\Sigma\colon \mathbb{F}_2^6 \to \mathbb{F}_2$ be the map which sums all the coordinates and let $\Delta\colon \mathbb{F}_2 \to \mathbb{F}_2^6$ be the diagonal embedding. Let $N = \ker(\Sigma)/\operatorname{image}(\Delta)$. The coordinate-permuting action of S_6 on $\mathbb{F}_2^6$ induces an action of S_6 on N, hence defines a homomorphism $\rho\colon S_6 \to \mathrm{GL}(N)$. The standard bilinear pairing $\mathbb{F}_2^6 \times \mathbb{F}_2^6 \to \mathbb{F}_2$ (in which the basis $b_1, \ldots, b_6$ is orthonormal) restricts to an alternating pairing on $\ker(\Sigma)$ whose radical equals $\operatorname{image}(\Delta)$, so induces an alternating bilinear pairing $e_N\colon N \times N \to \mathbb{F}_2 \simeq \{\pm 1\}$. Since the original pairing on $\mathbb{F}_2^6$ is S_6-invariant, e_N is also S_6-invariant, hence ρ lands in $\mathrm{Sp}(N, e_N) \subset \mathrm{GL}(N)$. A computation (using that the kernel of ρ must be $\{1\}$, A_6 or S_6 and that the orders of S_6 and $\mathrm{Sp}_4(\mathbb{F}_2)$ are equal) shows that ρ is an isomorphism $S_6 \xrightarrow{\sim} \mathrm{Sp}(N, e_N) \simeq \mathrm{Sp}_4(\mathbb{F}_2)$.

Now let (M, e) be a symplectic module of type $(2,2)$ over k. View (N, e_N) as a symplectic module over k^{sep}, and fix an isomorphism $(M, e)_{k^{\mathrm{sep}}} \xrightarrow{\sim} (N, e_N)$. This choice encodes the Galois action on (M, e) as a homomorphism $\varphi\colon \mathrm{Gal}_k \to \mathrm{Sp}(N, e_N)$. The composition $\rho^{-1} \circ \varphi\colon \mathrm{Gal}_k \to S_6$ corresponds to a (marked) Gal_k-set S. Let $f(x) \in k[x]$ be a monic separable polynomial of degree 6 such that there is an isomorphism of Gal_k-sets between S and the roots of f. Let C/k be the projective genus-2 hyperelliptic curve with affine equation $y^2 = f(x)$, and let J be its Jacobian variety. Given a root ω_i of $f(x)$, let $P_i = (\omega_i, 0) \in C(k^{\mathrm{sep}})$ be the associated point. By [PS97, Proposition 6.2], the assignment $e_i \mapsto P_i$ induces an isomorphism $N \to J[2](k^{\mathrm{sep}})$ which by [PS97, Section 7] intertwines e_N with the Weil pairing e_2. Under this isomorphism, the Galois action

$\rho\colon \mathrm{Gal}_k \to \mathrm{Sp}(N, e_N)$ is identified with the Galois action on $J[2](k^{\mathrm{sep}})$, so it determines an isomorphism of symplectic modules $(M, e) \simeq (J[2], e_2)$ over k. □

To prove the $g = 3$ case of Proposition 4.5 we need some preparatory results and notation. Let $\Phi \subset \mathbb{R}^7$ be a root system of type E_7 and let $\Lambda = \mathbb{Z}\Phi$ be the associated root lattice, which comes equipped with a bilinear positive definite pairing $(\cdot,\cdot)\colon \Lambda \times \Lambda \to \mathbb{Z}$ with the property that $(\alpha, \alpha) = 2$ for every root $\alpha \in \Phi$. Basic properties of root lattices and a construction of Λ may be found in [Lur01, Section 2.2]. Let $\Lambda^\vee = \mathrm{Hom}(\Lambda, \mathbb{Z})$. The pairing $(\cdot,\cdot)$ induces a map $\Lambda \to \Lambda^\vee$ which is injective with cokernel isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Let $N = \mathrm{image}(\Lambda/2\Lambda \to \Lambda^\vee/2\Lambda^\vee) \simeq \mathbb{F}_2^6$. The pairing $(\Lambda/2\Lambda) \times (\Lambda/2\Lambda) \to \{\pm 1\}, (x, y) \mapsto (-1)^{(x,y)}$ induces a perfect alternating pairing on N, denoted by e_N. View (N, e_N) as a symplectic module over k where N is a constant group scheme. Let $W \leq \mathrm{Aut}(\Lambda, (\cdot,\cdot))$ be the Weyl group of Φ. Let $W^1 = \ker(\det) \leq W$ be the index-2 subgroup of W. There is a decomposition $W = W^1 \times \{\pm 1\}$. The W-action on Λ induces a W-action on N, hence determines a homomorphism $\rho\colon W \to \mathrm{Sp}(N, e_N) \simeq \mathrm{Sp}_6(\mathbb{F}_2)$. This map is surjective with kernel $\{\pm 1\}$, and the restriction of ρ to W^1 is an isomorphism; see [Bou68, p. 229, Exercise 4].

Assume k is of characteristic zero and let $V = \Lambda \otimes_{\mathbb{Z}} k$. Then W acts faithfully on V. Let $k[V] = \mathrm{Sym}^\bullet(V^\vee)$ be the ring of polynomial functions on V, let $k[V]^W$ be the subring of W-invariant polynomials, let $V /\!\!/ W = \mathrm{Spec}(k[V]^W)$ and let $\pi\colon V \to V /\!\!/ W$ be the morphism induced by the inclusion $k[V]^W \subset k[V]$. Let $U \subset V$ be the open subset on which G acts freely. Then U is W-stable, U/W is an open subset of $V /\!\!/ G$, and the restriction of π to U is a G-torsor $U \to U/W$. For a tuple $b = (p_2, p_6, p_8, p_{10}, p_{12}, p_{14}, p_{18}) \in k^7$, let $C_b \subset \mathbb{P}^2$ be the projective curve with affine equation

$$y^3 = x^3y + p_{10}x^2 + x(p_2y^2 + p_8y + p_{14}) + p_6y^2 + p_{12}y + p_{18}. \tag{4.3}$$

To state the following proposition, note that since $(N, e_N) \simeq (M_D, e_D)$ for $D = (2, 2, 2)$, Lemma 3.5 shows that $(M, e) \mapsto \mathbf{Isom}((M, e), (N, e_N))$ induces a bijection between symplectic modules of type $(2, 2, 2)$ and $\mathrm{Sp}(N, e_N)$-torsors.

Proposition 4.6. *There exists an isomorphism of k-algebras $k[V]^W \simeq k[p_2, p_6, p_8, p_{10}, p_{12}, p_{14}, p_{18}]$ (where the elements p_i are algebraically independent) that induces an isomorphism of varieties $V /\!\!/ W \simeq \mathbb{A}^7$ and that satisfies the following properties:*

1. *If $b = (p_2, \ldots, p_{18}) \in (U/W)(k)$, then C_b is a smooth projective curve of genus 3; let J_b denote its Jacobian variety.*
2. *If $b \in (U/W)(k)$, let $\mathcal{T}_b$ be the push-out of the W-torsor $\pi^{-1}(b)$ along $\rho\colon W \to \mathrm{Sp}(N, e_N)$ and let (M_b, e_b) denote the symplectic module whose associated $\mathrm{Sp}(N, e_N)$-torsor is isomorphic to $\mathcal{T}_b$. Then there is an isomorphism of symplectic modules $(J_b[2], e_2) \simeq (M_b, e_b)$ over k.*

Proof. In [Lag24], a family of projective curves $C \to V /\!\!/ W$ is constructed for every root lattice of type A, D, E (beware that the notation in that paper is different to ours: what we call V is denoted by $\mathfrak{t}$). In the E_7 case, the explicit description (4.3) follows from [Lag24, Proposition 3.13(3)]. The smoothness of C_b when $b \in (U/W)(k)$ follows from [Lag24, Lemma 3.14]. The isomorphism of symplectic modules of Part 2 follows from the description of the monodromy of $J_b[2]$ in [Lag24, Proposition 3.22]. □

Proof of Proposition 4.5 when $g = 3$. Using Proposition 4.6 and its notation, it suffices to prove that every symplectic module of type $(2, 2, 2)$ over k is isomorphic to (M_b, e_b) for some $b \in (U/W)(k)$. Equivalently, it suffices to prove that every $\mathrm{Sp}(N, e_N)$-torsor over k is isomorphic to $\mathcal{T}_b$ for some $b \in (U/W)(k)$. Equivalently, using the decomposition $W = W^1 \times \{\pm 1\}$ and the fact that $\rho\colon W \to \mathrm{Sp}(N, e_N)$ is an isomorphism when restricted to W^1, it suffices to show that every W-torsor is isomorphic to $\pi^{-1}(b)$ for some $b \in (U/W)(k)$. This follows from the "versal torsors trick" [GMS03, Part 1, §5.4]; for completeness, we give a proof here.

Let T be a W-torsor. Let $U' = U \times^W T = (U \times_k T)/W$, where W acts via $w \cdot (u,t) = (wu, w^{-1}t)$. Define a W-action on U' via the assignment on representatives $w \cdot [(u,t)] = [(wu,t)]$. Let $\pi' \colon U' \to U'/W$ be the quotient morphism. The projection $U \times_k T \to U$ induces an isomorphism $U'/W \simeq U/W$ which we use to identify U'/W with U/W. The set of elements $b \in (U/W)(k)$ satisfying $T \simeq \pi^{-1}(b)$ as W-torsors equals $\pi'(U'(k)) \subset (U'/W)(k) = (U/W)(k)$. Therefore it suffices to prove that $U'(k)$ is nonempty. But U' is a nonempty open subset of $V' = V \times^W T$, which is the variety obtained by twisting the affine space V along the cocycle determined by the image of $[T] \in \mathrm{H}^1(k,W)$ under $\mathrm{H}^1(k,W) \to \mathrm{H}^1(k, \mathbf{Aut}(V))$. Since the W-action on V is linear, this map factors through $\mathrm{H}^1(k, \mathrm{GL}_V)$, which is trivial by Hilbert's Theorem 90. Therefore V' is isomorphic to V and U' is a nonempty open subset of the affine space V'. Since k is infinite, we conclude that $U'(k)$ is nonempty. □

Remark 4.7. *Reformulating the $g = 3$ case of Proposition 4.5, we have proved that for every field k of characteristic zero and Galois representation $\rho\colon \mathrm{Gal}_k \to \mathrm{Sp}_6(\mathbb{F}_2)$, there exists a genus-3 curve C/k of the form (4.3) with Jacobian J such that the Galois representation associated to $J[2](k^{\mathrm{sep}})$ is isomorphic to ρ. This answer a question of Chidambaram [Chi24, Question 1.2] affirmatively, for every field of characteristic zero. It seems likely that a similar proof would answer the same question affirmatively for every field of characteristic $\neq 2$, using a generalization of Proposition 4.6, but we have not pursued this.*

Remark 4.8. *The proof of the $g = 1, 2$ cases of Proposition 4.5 can also be written in the language of root lattices and correspond to the cases where Φ has type A_2 and A_5, in which case W is isomorphic to S_3 and S_6 respectively.*

Corollary 4.9. *Let k be a field of characteristic not 2 and let $g \in \{1,2,3\}$. Let D be the type $(2,\ldots,2)$ of length g. If $g = 3$, additionally assume that k has characteristic zero. Let (M,e) be a symplectic module of type D over k. Then there exists a symmetric linear theta group for (M,e).*

Proof. Suppose k is finite. By Proposition 3.27, there exists a linear theta group $\mathcal{G}$ for (M,e). Since $\mathrm{Id}_M = -\mathrm{Id}_M$, $\mathrm{Id}_{\mathcal{G}}$ is an inversion on $\mathcal{G}$ and so $\mathcal{G}$ is also symmetric. So we may assume that k is infinite. By Proposition 4.5, there exists a smooth projective curve C of genus g over k with Jacobian J and an isomorphism $(M,e) \simeq (J[2], e_2)$. The pair $(J[2], e_2)$ arises from the polarization $2\lambda\colon J \to J^\vee$, where λ is the canonical principal polarization on J. Let $L = (1,\lambda)^*\mathcal{P}$ be the pullback of the Poincaré bundle $\mathcal{P}$ on $J \times J^\vee$. Then L is a symmetric line bundle on A with $\phi_L = 2\lambda$, by [MFK94, Proposition 6.10]. Hence $\mathcal{G}(L)$ is a symmetric linear theta group for $(J[2], e_2) \simeq (M,e)$. □

4.3 Proof of Theorem B

We now combine the results of Sections 4.1 and 4.2 to obtain Theorem B whose proof is given at the end of this section. We first need to discuss direct sums of symplectic modules and their interaction with theta groups.

Let $(M_1,e_1), (M_2,e_2)$ be two symplectic modules over a field k. Then the direct sum $(M,e) = (M_1,e_1) \oplus (M_2,e_2)$ is the symplectic module with underlying group scheme $M = M_1 \oplus M_2$ and pairing given by $e((m_1,m_2),(n_1,n_2)) = e_1(m_1,n_1)e_2(m_2,n_2)$. Let $1 \to \mathbb{G}_m \to \mathcal{G} \to M \to 1$ be a theta group for (M,e). Then the restriction $\mathcal{G}_1 = \mathcal{G}|_{M_1}$ of $\mathcal{G}$ to $M_1 \subset M$ is a theta group for (M_1,e_1), similarly $\mathcal{G}_2 = \mathcal{G}|_{M_2}$ is a theta group for (M_2,e_2). Every framed isomorphism restricts to framed isomorphisms on both pieces, so the association $\mathcal{G} \mapsto (\mathcal{G}_1, \mathcal{G}_2)$ can be upgraded to a functor (in the notation of §3.2):

$$\mathrm{ThetaGrp}_{(M_1,e_1)\oplus(M_2,e_2)}(k) \to \mathrm{ThetaGrp}_{(M_1,e_1)}(k) \times \mathrm{ThetaGrp}_{(M_2,e_2)}(k). \tag{4.4}$$

Lemma 4.10. *1. The functor (4.4) is an equivalence of groupoids.*

2. $\mathcal{G}$ is symmetric if and only if $\mathcal{G}_1$ and $\mathcal{G}_2$ are symmetric.

3. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are linear then $\mathcal{G}$ is linear; the converse holds when $\#M_1$ and $\#M_2$ are coprime.

Proof. For $i = 1, 2$, let $\mathcal{G}_i$ be a theta group for (M_i, e_i) and let $\pi_i \colon M \to M_i$ be the projection. The pullback $\pi_i^*\mathcal{G}_i$ is a central extension $1 \to \mathbb{G}_m \to \pi_i^*\mathcal{G}_i \xrightarrow{\beta_i} M \to 1$. Let $\mathcal{G}$ be the Baer sum of $\pi_1^*\mathcal{G}_1$ and $\pi_2^*\mathcal{G}_2$; explicitly, $\mathcal{G}$ is the quotient of

$$\{(g_1, g_2) \in \pi_1^*\mathcal{G}_1 \times \pi_2^*\mathcal{G}_2 \colon \beta_1(g_1) = \beta_2(g_2)\}$$

by the subgroup $\{(\lambda, \lambda^{-1}) \colon \lambda \in \mathbb{G}_m\}$. A calculation (which we omit) shows that $\mathcal{G}$ is a theta group for (M, e) and provides a quasi-inverse to the functor $\mathcal{G} \mapsto (\mathcal{G}|_{M_1}, \mathcal{G}|_{M_2})$, proving Part 1. Part 2 follows from the fact that $\iota \mapsto (\iota_1, \iota_2)$ induces a bijection between the set of inversions on $\mathcal{G}$ and the set of pairs on inversions on $\mathcal{G}_1$ and $\mathcal{G}_2$. Polishchuk has shown (see [Pol03, Proposition 2.2]) that $[\mathrm{Sh}_{\mathcal{G}}] = [\mathrm{Sh}_{\mathcal{G}_1}] + [\mathrm{Sh}_{\mathcal{G}_2}]$ in $\mathrm{H}^2(k, \mathbb{G}_m)$. Since $[\mathrm{Sh}_{\mathcal{G}_i}]$ is killed by $\#M_i$ (Proposition 3.26), Part 3 follows. □

Given a symplectic module (M, e) and a prime p, let $M(p)$ be the submodule of elements of p-power order, and let $e(p)$ be the restriction of e to $M(p) \times M(p)$. Then $(M(p), e(p))$ is a symplectic module and $(M, e) \simeq \oplus_p (M(p), e(p))$ and so combining Theorem 4.2 and Lemma 4.10 shows:

Corollary 4.11. *In the above notation, (M, e) admits a theta group if and only if $(M(2), e(2))$ does. The same statement holds with "theta group" replaced by "linear theta group", or by "symmetric theta group", or by "linear symmetric theta group".*

Combining Corollaries 4.11 and 4.9 immediately shows:

Corollary 4.12. *Let (M, e) be a symplectic module over k. Suppose that $(M(2), e(2))$ has type $(1, \dots, 1)$, $(1, \dots, 1, 2)$, $(1, \dots, 1, 2, 2)$, or $(1, \dots, 2, 2, 2)$. In the last case, additionally assume that k has characteristic zero. Then there exists a symmetric linear theta group for (M, e).*

Proof of Theorem B. In the notation of the theorem, $(A[\lambda], e_\lambda)$ satisfies the assumptions of Corollary 4.12, so there exists a linear theta group for $(A[\lambda], e_\lambda)$. We conclude by Theorem Θ. □

4.4 A survey of known cases

For convenience of the reader and for future reference, we collect known positive results towards Question 1 and its variants, either proven in this paper or already present in the literature. Let A be a g-dimensional abelian variety over a field k. Let $\lambda \colon A \to A^\vee$ be a polarization of type $D = (d_1, \dots, d_g)$, and assume $\#D$ is invertible in k. Consider the following properties that (A, λ) could satisfy:

(1) There exists a symmetric line bundle L on A with $\phi_L = \lambda$.

(2) There exists a line bundle L on A with $\phi_L = \lambda$.

(3) There exists a symmetrized A-torsor (X, τ) and a τ-symmetric line bundle L on X with $\phi_L = \lambda$ (see §3.6).

(4) There exists an A-torsor X and a line bundle L on X with $\phi_L = \lambda$.

(5) There exists a symmetrized A-torsor (X, τ) and element $\ell \in \mathbf{Pic}_X^{\tau,\lambda}(k)$.

(6) There exists an A-torsor X and element $\ell \in \mathbf{Pic}_X^{\lambda}(k)$.

These properties are connected by the following implications:

$$\begin{array}{ccccc} (1) & \Longrightarrow & (3) & \Longrightarrow & (5) \\ \Downarrow & & \Downarrow & & \Downarrow \\ (2) & \Longrightarrow & (4) & \Longrightarrow & (6) \end{array}$$

Theorems Θ and Corollary 3.32 show that the validity of Properties (3), (4), (5) and (6) only depends on the isomorphism class of the symplectic module $(A[\lambda], e_\lambda)$. This is not true for Property (2), since there exist principally polarized abelian varieties for which (2) does not hold (see the example of Jacobians in the introduction). This is also not true for Property (1) for similar reasons, see [PR11, p. 1317, Example 3.20].

Complementing the criteria we have given in Section 3, we state some conditions which guarantee that one of the above properties holds. Let 2^{n_i} be the largest power of 2 dividing d_i and let $D_2 = (2^{n_1}, \ldots, 2^{n_g})$.

- Property (1) holds if and only if $(A[2], e_2^\lambda)$ admits a quadratic refinement (see the beginning of §3.6). This is the case whenever one of the following conditions is satisfied [PR11, Proposition 3.12]: k is $\mathbb{R}$ or a finite field, or $n_2 \geq 1$ (in other words d_2 is even).
- Property (2) holds if k is a local field [PS99, §4, Lemma 1].
- Property (3) holds if $D_2 = (1, \ldots, 1)$, $(1, \ldots, 1, 2)$, $(1, \ldots, 1, 2, 2)$. It also holds if k has characteristic zero and $D_2 = (1, \ldots, 1, 2, 2, 2)$. (Theorem B)
- Property (3) holds if k has cohomological dimension ≤ 1 (Proposition 3.27).

The first bullet point shows that if (A, λ) does not satisfy (1), then $D_2 = (1, 1, 2^{n_3}, \ldots, 2^{n_g})$.

Proposition 4.13. *Suppose* $\dim A \leq 2$. *Then Property* (3) *holds for* (A, λ).

Proof. When $\dim A = 1$, Property (1) holds. If $\dim A = 2$, then (1) holds except if $D_2 = (1, 1)$, in which case (3) holds by Theorem B. ☐

Remark 4.14. *We do not know whether Property* (3) *always holds when* $\dim A = 3$. *This would involve analyzing pairs* (A, λ) *of type* D *with* $D_2 = (1, 1, 2^n)$ *for* $n \geq 2$.

By considering Jacobians of curves, one can show that (2) does not necessarily imply (1), that (3) does not imply (1) and that (4) does not imply (2). We do not know of a pair (A, λ) for which (6) holds but (3) does not.

5 Proof of Theorem C

In this section, we construct pairs (A, λ) of type $(1, \ldots, 1, 2, \ldots, 2)$ for which the answer to Question 1 is no. In §5.1 we reformulate the existence of theta groups in terms of Galois cohomology and lifting problems. In §5.2, we show that for every type $(2, \ldots, 2)$ of length ≥ 4, there exists a symplectic module (M, e) over some field k that does not admit a theta group. To find such an example of the form $(A[\lambda], e_\lambda)$, we recall Siegel modular varieties in §5.3-5.4. In §5.5 we use a calculation of Totaro to reduce our problem to a group theory computation. We carry out this computation in §5.6-A. Explicitly, Theorem C follows from Propositions 5.18 and 5.19.

5.1 Reformulation in terms of Galois cohomology

Let k be a field. Let $D = (d_1, \dots, d_g)$ be a type in the sense of Definition 3.2 and suppose d_g is invertible in k. In Section 3 we have constructed group schemes $M_D, \mathcal{G}_D$ and a sequence of finite k-groups

$$1 \to M_D \to \mathbf{Aut}(\mathcal{G}_D) \to \mathbf{Sp}(M_D) \to 1 \tag{5.1}$$

which is exact by Lemma 3.9. Since M_D is an abelian normal subgroup of $\mathbf{Aut}(\mathcal{G}_D)$, the conjugation action $g \cdot m = gmg^{-1}$ of $\mathbf{Aut}(\mathcal{G}_D)$ on M_D factors through an action of $\mathbf{Sp}(M_D)$. A computation shows that this agrees with the standard action of $\mathbf{Sp}(M_D)$ on M_D via the defining embedding $\mathbf{Sp}(M_D) \hookrightarrow \mathbf{Aut}(M_D)$. We have seen (Lemma 3.11) that a symplectic module (M, e) of type D admits a theta group if and only if its class $[(M, e)] \in \mathrm{H}^1(k, \mathbf{Sp}(M_D))$ lifts to $\mathrm{H}^1(k, \mathbf{Aut}(\mathcal{G}_D))$. We now show that when k contains sufficiently many roots of unity, we may formulate this condition in terms of lifting Galois representations.

Let $n = d_g$ if n is odd and $n = 2d_g$ if n is even. Let $\mathcal{H}_D = \mathcal{G}_D[n] = \{x \in \mathcal{G}_D \colon x^n = 1\}$ be the subscheme of elements of order n. Lemma 3.12 shows that $\mathcal{H}_D$ is a subgroup scheme fitting in a central extension $1 \to \mu_n \to \mathcal{H}_D \to M_D \to 1$.

Lemma 5.1. *Suppose that k contains a primitive nth root of unity. Then M_D, $\mathbf{Sp}(M_D)$ and $\mathbf{Aut}(\mathcal{G}_D)$ are constant group schemes.*

Proof. The assumption implies M_D is constant, hence $\mathbf{Sp}(M_D)$ is constant too. A calculation shows that in the notation of (3.3) we have $\mathcal{H}_D \simeq \mu_n \times M_D$ (as schemes, not as groups). Therefore $\mathcal{H}_D$ is constant, hence $\mathbf{Aut}(\mathcal{H}_D)$ is constant too. By Lemma 3.12, $\mathbf{Aut}(\mathcal{G}_D) \simeq \mathbf{Aut}(\mathcal{H}_D)$, so $\mathbf{Aut}(\mathcal{G}_D)$ is constant. □

Suppose that k contains a primitive nth root of unity. Then (5.1) is an extension of constant group schemes, so can be interpreted as an extension of (abstract) groups

$$1 \to M_D \to \mathrm{Aut}(\mathcal{G}_D) \to \mathrm{Sp}(M_D) \to 1. \tag{5.2}$$

Since M_D is an abelian normal subgroup, this extension defines a class $c_D \in \mathrm{H}^2(\mathrm{Sp}(M_D), M_D)$ in group cohomology, where $\mathrm{Sp}(M_D)$ acts on M_D via the defining representation $\mathrm{Sp}(M_D) \hookrightarrow \mathrm{Aut}(M_D)$. If (M, e) is a symplectic module of type D over k, its class $[(M, e)] \in \mathrm{H}^1(k, \mathbf{Sp}(M_D))$ (under the bijection of Lemma 3.5) can be interpreted as a conjugacy class of continuous homomorphisms $\rho_M \colon \mathrm{Gal}_k \to \mathrm{Sp}(M_D)$.

Lemma 5.2. *Assume k contains a primite nth root of unity, where $n = d_g$ if d_g is odd and $n = 2d_g$ if d_g is even. Let $\rho_M \colon \mathrm{Gal}_k \to \mathrm{Sp}(M_D)$ be a homomorphism corresponding to a symplectic module (M, e) of type D over k. Then there exists a theta group for (M, e) if and only if ρ_M lifts to a homomorphism $\mathrm{Gal}_k \to \mathrm{Aut}(\mathcal{G}_D)$ if and only if $\rho_M^* c_D = 0$ in $\mathrm{H}^2(k, M_D)$.*

Proof. The first equivalence follows from Lemma 3.11. The second equivalence follows from the fact that a lifting for ρ_M exists if and only if the pullback of (5.2) along ρ_M splits if and only if $\rho_M^*(c_D) = 0$. □

The sequence (5.2) is not always split:

Proposition 5.3. *Let D be the type $(2, \dots, 2)$ of length $g \geq 3$. Then the class $c_D \in \mathrm{H}^2(\mathrm{Sp}(M_D), M_D)$ of the extension (5.2) is nonzero.*

Proof. The group $\mathcal{H}_D$ is an almost extraspecial 2-group of order 2^{2+2g} (see [BCRR18, Section 3]). By Lemma 3.12, $\mathrm{Aut}(\mathcal{G}_D)$ can be identified with the subgroup of group automorphisms of $\mathcal{H}_D$ that induce the identity on the center μ_4. The nonvanishing of c_D is therefore the content of [Gri73, p. 407, Corollary 2]. □

Remark 5.4. *In the appendix of [Sch00], it is claimed that $c_D \neq 0$ when $D = (2, 2)$, but no proof is given.*

5.2 Symplectic modules not admitting a theta group

In preparation of the proof of Theorem C, we show the following (a priori) weaker statement, using recent work of Merkurjev–Scavia [MS25].

Proposition 5.5. *Suppose that $D = (2, \ldots, 2)$ has length $g \geq 4$. Then there exists a field k of characteristic zero and a symplectic module (M, e) of type D over k such that there exists no theta group for (M, e).*

This statement is weaker, since the symplectic module (M, e) not admitting a theta group might not be of the form $(A[\lambda], e_\lambda)$ for some polarized abelian variety (A, λ). Nevertheless, the intermediate results used to prove Proposition 5.5 will also be used to prove Theorem C.

Let G be a finite group, M a finite G-module and k a field. We say an element $c \in \mathrm{H}^i(G, M)$ is *negligible* over k if for every field extension K/k and every continuous homomorphism $f\colon \mathrm{Gal}_K \to G$, the pullback $f^*(c) \in \mathrm{H}^i(\mathrm{Gal}_K, M) = \mathrm{H}^i(K, M)$ is trivial. The subset of such elements forms a subgroup denoted by $\mathrm{H}^i(G, M)_{\mathrm{neg},k} \subset \mathrm{H}^i(G, M)$. Under minor assumptions on k, Merkurjev–Scavia give an explicit description of $\mathrm{H}^2(G, M)_{\mathrm{neg},k}$. Given $m \in M$, let $G_m = \mathrm{Stab}_G(m)$ and define the composite map

$$\varphi_m \colon \mathrm{H}^2(G_m, \mathbb{Z}) \to \mathrm{H}^2(G_m, M) \to \mathrm{H}^2(G, M), \tag{5.3}$$

where the first map is induced by the G_m-module homomorphism $\mathbb{Z} \to M$ sending 1 to m, and the second map is corestriction along $G_m \subset G$. Given a finite group H, let $e(H)$ be its exponent, namely the least common multiple of the orders of its elements.

Theorem 5.6 (Merkurjev–Scavia)**.** *Suppose that k contains a primitive root of unity of order $e(G)e(M)$. Then $\mathrm{H}^2(G, M)_{\mathrm{neg},k}$ is generated by the images of φ_m where m ranges over all elements of M.*

Proof. This follows from [MS25, Corollary 4.2], after observing that if $m \in M$ and $\chi \in \mathrm{H}^2(G_m, \mathbb{Z})$, the cup product class $m \cup \chi \in \mathrm{H}^2(G_m, M)$ equals the image of χ under the homomorphism $\mathrm{H}^2(G_m, \mathbb{Z}) \to \mathrm{H}^2(G_m, M)$ induced by $\mathbb{Z} \to M, 1 \mapsto m$. □

Let D be the type $(2, \ldots, 2)$ of length g. We will apply Theorem 5.6 to the group $\mathrm{Sp}(M_D) \simeq \mathrm{Sp}_{2g}(\mathbb{F}_2)$ acting on $M_D \simeq \mathbb{F}_2^{2g}$.

Lemma 5.7. *Let $m \in M_D$ be nonzero, let $G_m = \mathrm{Stab}_{\mathrm{Sp}(M_D)}(m)$ and suppose that $g \geq 4$. Then the abelianization of G_m is trivial.*

Proof. For ease of notation, denote the form e_D on M_D by $\langle \cdot, \cdot \rangle$. Let $e_1, \ldots, e_g, f_g, \ldots, f_1$ be an $\mathbb{F}_2$-basis for M_D. Since all nondegenerate alternating forms on M_D are conjugate, we may assume that $\langle e_i, f_j \rangle = \delta_{ij}$ and $\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0$ for all $1 \leq i, j \leq g$. Since $\mathrm{Sp}_{2g}(\mathbb{F}_2)$ acts transitively on the nonzero elements of $\mathbb{F}_2^{2g}$, we may also assume $m = e_1$. Let $F_1 = \mathbb{F}_2 e_1$ and $F_2 = F_1^\perp = \mathbb{F}_2\{e_1, \ldots, e_g, f_g, \ldots, f_2\}$. Then G_m preserves the flag $F_1 \subset F_2 \subset M_D$. Let U be the subgroup of G_m of those elements that induce the identity map on F_2/F_1. Let L be the subgroup of G_m that preserves the subspaces $\mathbb{F}_2 e_1$, $\mathbb{F}_2\{e_2, \ldots, e_g, f_g, \ldots, f_2\}$ and $\mathbb{F}_2 f_1$. Then $G_m = L \ltimes U$ and $L \simeq \mathrm{Sp}_{2g-2}(\mathbb{F}_2)$. As explicit matrix subgroups of $\mathrm{Sp}_{2g}(\mathbb{F}_2)$, we have

$$G_m = \left\{ \left(\begin{array}{c|ccc|c} 1 & * & \cdots & * & * \\ \hline 0 & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & * & \cdots & * & * \\ \hline 0 & 0 & \cdots & 0 & 1 \end{array} \right) \right\}, \; L = \left\{ \left(\begin{array}{c|ccc|c} 1 & 0 & \cdots & 0 & 0 \\ \hline 0 & * & \cdots & * & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & * & \cdots & * & 0 \\ \hline 0 & 0 & \cdots & 0 & 1 \end{array} \right) \right\}, \; U = \left\{ \left(\begin{array}{c|ccc|c} 1 & * & \cdots & * & * \\ \hline 0 & 1 & \cdots & 0 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & * \\ \hline 0 & 0 & \cdots & 0 & 1 \end{array} \right) \right\}.$$

It is well known (see for example [BCRR18, Appendix]) that $\mathrm{Sp}_{2n}(\mathbb{F}_2)$ is perfect for all $n \geq 3$. Since $g \geq 4$, $\mathrm{Sp}_{2g-2}(\mathbb{F}_2)$ is perfect and hence $[L, L] = L$. We calculate that $[U, U]$ is of order 2, generated by the element $I + E$, where E is the matrix with 1 in the top right corner and zeroes everywhere else. The conjugation action of L on U induces an action on $U/[U, U] \simeq \mathbb{F}_2^{2g-2}$ and can be identified with the standard action of $\mathrm{Sp}_{2g-2}(\mathbb{F}_2)$ in its defining representation. When $g \geq 2$, there are no covariants for this action. In conclusion, we have shown that the homology groups $\mathrm{H}_1(L, \mathbb{Z})$ and $\mathrm{H}_0(L, \mathrm{H}_1(U, \mathbb{Z}))$ both vanish. Using the low terms of the Hochschild–Serre spectral sequence $\mathrm{H}_p(L, \mathrm{H}_q(U, \mathbb{Z})) \Rightarrow \mathrm{H}_{p+q}(G_m\mathbb{Z})$, it follows that $G_m^{\mathrm{ab}} = \mathrm{H}^1(G_m, \mathbb{Z})$ vanishes too. □

Proof of Proposition 5.5. We analyze the maps φ_m of (5.3) for the pair $(G, M) = (\mathrm{Sp}(M_D), M_D)$. If $m = 0$, then $\varphi_m = 0$, since the first map $\mathrm{H}^2(G_m, \mathbb{Z}) \to \mathrm{H}^2(G_m, M_D)$ is induced by the zero map $\mathbb{Z} \to M_D$. If $m \neq 0$, then the abelianization G_m^{ab} is trivial by Lemma 5.7, so

$$\mathrm{H}^2(G_m, \mathbb{Z}) \simeq \mathrm{H}^1(G_m, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G_m, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G_m^{\mathrm{ab}}, \mathbb{Q}/\mathbb{Z}) = 0.$$

Let k be a field of characteristic zero containing a primitive root of unity of order $e(\mathrm{Sp}(M_D))e(M_D)$. Using Theorem 5.6, we conclude that $\mathrm{H}^2(\mathrm{Sp}(M_D), M_D)_{\mathrm{neg},k} = \{0\}$. On the other hand, Proposition 5.3 shows that $c_D \in \mathrm{H}^2(\mathrm{Sp}(M_D), M_D)$ is nonzero. We conclude that c_D is not negligible. In other words, after possibly enlarging k, there exists a homomorphism $\rho\colon \mathrm{Gal}_k \to \mathrm{Sp}(M_D)$ that does not lift to a homomorphism $\mathrm{Gal}_k \to \mathrm{Aut}(\mathcal{G}_D)$. Using Lemma 3.5, ρ corresponds to a symplectic module (M, e). Since k contains a primitive 4th root of unity, Lemma 5.2 shows that there does not exist a theta group for (M, e). □

Remark 5.8. *The assumption $g \geq 4$ in Proposition 5.5 is optimal. Indeed, if $g = 3$, then $c_D \neq 0$ by Proposition 5.3. However, Corollary 4.9 shows that if k is of characteristic zero, then every symplectic module of type D over every field extension of k admits a theta group. Therefore, by Lemma 5.2, if k contains a primitive 4th root of unity, then c_D is a nonzero element of $\mathrm{H}^2(\mathrm{Sp}(M_D), M_D)_{\mathrm{neg},k}$.*

5.3 Paramodular groups

Let $D = (d_1, \ldots, d_g)$ be a type of length g, viewed as a diagonal $g \times g$ matrix. Define the block $2g \times 2g$-matrix

$$J_D = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$$

and let $(-,-)_D$ be the alternating form on $\mathbb{Z}^{2g}$ with Gram matrix J_D. Let Λ_D be $\mathbb{Z}^{2g}$, thought of as being equipped with the pairing $(-,-)_D$. Define the paramodular group

$$\mathrm{Sp}_{2g}^D(\mathbb{Z}) = \mathrm{Sp}(\Lambda_D) = \{g \in \mathrm{GL}(\Lambda_D)\colon g \text{ preserves } (-,-)_D\}.$$

More generally, we view Sp_{2g}^D as a group scheme over $\mathbb{Z}$, where for a ring R we let $\mathrm{Sp}_{2g}^D(R)$ be the subgroup of $\mathrm{GL}(\Lambda_D \otimes R)$ preserving $(-,-)_D$.

Let D_g be the type $(1, \ldots, 1)$ of length g. Then we simply write $\mathrm{Sp}_{2g}(R)$ for $\mathrm{Sp}_{2g}^{D_g}(R)$ and we recover the usual symplectic group. Denote the standard basis of $\mathbb{Z}^{2g}$ by $e_1, \ldots, e_g, f_1, \ldots, f_g$. Consider the linear map $\alpha\colon \Lambda_D \to \Lambda_{D_g}$ sending e_i to $d_i e_i$ and f_i to f_i. Then α intertwines $(-,-)_D$ with $(-,-)_{D_g}$, hence conjugation by α induces an isomorphism $\mathrm{Sp}_{2g}^D(R) \xrightarrow{\sim} \mathrm{Sp}_{2g}(R)$ for every ring R in which d_g is invertible. In particular, we may view $\mathrm{Sp}_{2g}^D(\mathbb{Z})$ as a subgroup of $\mathrm{Sp}_{2g}^D(\mathbb{Q}) \simeq \mathrm{Sp}_{2g}(\mathbb{Q})$ commensurable with $\mathrm{Sp}_{2g}(\mathbb{Z})$.

Define $\Lambda_D^\vee = \{x \in \Lambda_D \otimes \mathbb{Q}\colon (x, y) \in \mathbb{Z} \text{ for all } y \in \Lambda_D\}$. Then $\Lambda_D \subset \Lambda_D^\vee$ has finite index and the quotient $\Lambda_D^\vee/\Lambda_D$ is a finite abelian group. Using the standard $\mathbb{Z}$-basis $\{e_1, \ldots, e_g, f_1, \ldots, f_g\}$ of Λ_D, $\Lambda_D^\vee$ has $\mathbb{Z}$-basis

$\frac{1}{d_1}e_1, \ldots, \frac{1}{d_g}e_g, \frac{1}{d_1}f_1, \ldots, \frac{1}{d_g}f_g$. The $\mathbb{Q}$-valued bilinear pairing $(-,-)_D \colon \Lambda_D^\vee \times \Lambda_D^\vee \to \mathbb{Q}$ induces a bilinear pairing $b_D \colon \Lambda_D^\vee/\Lambda_D \times \Lambda_D^\vee/\Lambda_D \to \mathbb{Q}/\mathbb{Z}$.

For each $n \geq 1$, let $\zeta_n = e^{2\pi i/n} \in \mathbb{C}$. Recall the standard symplectic module M_D of type D from Section 3.1, which we consider here as a (constant) group scheme over $\mathbb{C}$. In the notation of that section, consider the homomorphism

$$\beta \colon M_D = (\mathbb{Z}/d_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/d_g\mathbb{Z}) \times \mu_{d_1} \times \cdots \times \mu_{d_g} \to \Lambda_D^\vee/\Lambda_D$$

which maps $1 \in \mathbb{Z}/d_i\mathbb{Z}$ to $\frac{1}{d_i}e_i$ and $\zeta_{d_i} \in \mu_{d_i}$ to $\frac{1}{d_i}f_i$. Then a direct computation shows that β is an isomorphism that intertwines the pairing e_D with the pairing $e^{2\pi i b_D}$.

Since every element of $\mathrm{Sp}_{2g}^D(\mathbb{Z})$ induces an automorphism of $\Lambda_D^\vee/\Lambda_D$ preserving b_D, we obtain a group homomorphism $\mathrm{Sp}_{2g}^D(\mathbb{Z}) \to \mathrm{Aut}((\Lambda_D^\vee/\Lambda_D, b_D))$. Conjugating by β, we obtain a homomorphism $\mathrm{red}_D \colon \mathrm{Sp}_{2g}^D(\mathbb{Z}) \to \mathrm{Sp}(M_D)$.

Lemma 5.9. *The reduction map* $\mathrm{red}_D \colon \mathrm{Sp}_{2g}^D(\mathbb{Z}) \to \mathrm{Sp}(M_D)$ *is surjective.*

Proof. This is the main result of [Bra93]. ☐

Denote the kernel of this reduction map by $\Gamma(D)$. We then obtain an exact sequence

$$1 \to \Gamma(D) \to \mathrm{Sp}_{2g}^D(\mathbb{Z}) \to \mathrm{Sp}(M_D) \to 1. \tag{5.4}$$

If $n \geq 1$ is an integer, define the type $nD = (nd_1, \ldots, nd_g)$. Note that $(\cdot,\cdot)_{nD} = n(\cdot,\cdot)_D$ and $\mathrm{Sp}_{2g}^{nD}(\mathbb{Z}) = \mathrm{Sp}_{2g}^D(\mathbb{Z})$ for all n. We call a subgroup $\Gamma \subset \mathrm{Sp}_{2g}^D(\mathbb{Z})$ a congruence subgroup if $\Gamma(nD) \subset \Gamma$ for some $n \geq 1$. For later use, we will summarize some group-theoretic properties of finite-index subgroups of $\mathrm{Sp}_{2g}^D(\mathbb{Z})$.

Proposition 5.10. *If $g \geq 2$, then every finite-index subgroup of* $\mathrm{Sp}_{2g}^D(\mathbb{Z})$ *is a congruence subgroup.*

Proof. This follows from the resolution of the congruence subgroup problem for Sp_{2g} for $g \geq 2$ [BMS67]. ☐

Proposition 5.11. *Let $g \geq 2$ and let $\Gamma \subset \mathrm{Sp}_{2g}^D(\mathbb{Z})$ be a finite-index subgroup. Then the abelianization Γ^{ab} is finite and $\mathrm{H}^2(\Gamma, \mathbb{Z})$ is a finitely generated abelian group.*

Proof. Since Γ is an arithmetic group, it is finitely presented [PRR23, §4.4, Theorem 4.8]. Therefore Γ^{ab} is a finitely generated abelian group. Since Sp_{2g} has real rank ≥ 2, Γ satisfies Kazhdan's property (T) [Zim84, Theorem 7.1.4]. Consequently [Zim84, Theorem 7.1.7], there are no nontrivial homomorphisms $\Gamma \to \mathbb{R}$. Therefore the finitely generated group Γ^{ab} must be finite.

Since Γ is finitely presented, Hopf's formula [Bro82, Chapter II, §5, Theorem 5.3] shows that $\mathrm{H}_2(\Gamma, \mathbb{Z})$ is finitely generated. By the universal coefficient theorem and the finiteness of Γ^{ab}, we conclude that $\mathrm{H}^2(\Gamma, \mathbb{Z})$ is also finitely generated. ☐

Proposition 5.12 (Borel)**.** *Let $g \geq 3$ and let $\Gamma \subset \mathrm{Sp}_{2g}^D(\mathbb{Z})$ be a finite-index subgroup. Then $\mathrm{H}^2(\Gamma, \mathbb{Q}) \simeq \mathbb{Q}$ and the restriction map $\mathrm{H}^2(\mathrm{Sp}_{2g}^D(\mathbb{Z}), \mathbb{Q}) \to \mathrm{H}^2(\Gamma, \mathbb{Q})$ is an isomorphism.*

Proof. This follows from results of Borel [Bor74] with optimized stable range, see [Tsh19, Theorem 2]. (The result in [Tsh19] is only stated for finite-index subgroups of $\mathrm{Sp}_{2g}(\mathbb{Z})$, but the proof applies more generally to finite-index subgroups of $\mathrm{Sp}_{2g}^D(\mathbb{Z})$.) ☐

Combining Propositions 5.11 and 5.12 we obtain, for every $g \geq 3$ and finite-index subgroup $\Gamma \subset \mathrm{Sp}_{2g}(\mathbb{Z})$, an exact sequence

$$1 \to \mathrm{H}^1(\Gamma, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^2(\Gamma, \mathbb{Z}) \to \mathbb{Z} \to 1 \tag{5.5}$$

induced by the sequence $1 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 1$. Since $\mathrm{H}^1(\Gamma, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Hom}(\Gamma^{\mathrm{ab}}, \mathbb{Q}/\mathbb{Z})$, computing $\mathrm{H}^2(\Gamma, \mathbb{Z})$ boils down to computing the abelianization of Γ.

To compute such abelianizations, it will be useful to consider p-adic and adelic variants of congruence subgroups. Let $g \geq 2$ and let $\Gamma \subset \mathrm{Sp}_{2g}^D(\mathbb{Z})$ be a finite-index subgroup. For each $n \geq 1$, let Γ_n denote the image of Γ under the reduction map $\mathrm{Sp}_{2g}^D(\mathbb{Z}) = \mathrm{Sp}_{2g}^{nD}(\mathbb{Z}) \to \mathrm{Sp}(M_{nD})$. Denote the profinite completion of Γ by $\hat{\Gamma}$. By Proposition 5.10, the natural map $\hat{\Gamma} \to \varprojlim_n \Gamma_n$ (where n ranges over positive integers) is an isomorphism. For example, if $\Gamma = \mathrm{Sp}_{2g}^D(\mathbb{Z})$, then by Lemma 5.9 we have $\hat{\Gamma} \simeq \mathrm{Sp}_{2g}^D(\hat{\mathbb{Z}})$, where $\hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$. Given a topological group G (such as a discrete or profinite group), denote by G^{der} the closure of the commutator subgroup of G and let $G^{\mathrm{ab}} = G/G^{\mathrm{der}}$.

Lemma 5.13. *Let $g \geq 2$ and let $\Gamma \subset \mathrm{Sp}_{2g}^D(\mathbb{Z})$ be a finite-index sugroup. Then the natural map $\Gamma^{\mathrm{ab}} \to (\hat{\Gamma})^{\mathrm{ab}}$ is an isomorphism of finite groups.*

Proof. This follows from the identity $\widehat{\Gamma^{\mathrm{ab}}} \simeq (\hat{\Gamma})^{\mathrm{ab}}$ and the fact that Γ^{ab} is finite by Proposition 5.11. □

Lemma 5.14. *Let $g \geq 2$ be an integer and p a prime.*

1. $\mathrm{Sp}_{2g}(\mathbb{Z})^{\mathrm{ab}}$ *is trivial when* $g \geq 3$.
2. *If* $g \geq 3$ *or* $p \geq 3$*, then* $\mathrm{Sp}_{2g}(\mathbb{Z}_p)^{\mathrm{ab}}$ *is trivial.*
3. *If* $R = \mathbb{Z}$ *or* $\mathbb{Z}_p$*, let* $\mathrm{Sp}_{2g}(R, p) = \ker(\mathrm{Sp}_{2g}(R) \to \mathrm{Sp}_{2g}(R/pR))$*. If* p *is odd, then* $\mathrm{Sp}_{2g}(\mathbb{Z}, p)^{\mathrm{ab}} \simeq \mathrm{Sp}_{2g}(\mathbb{Z}_p, p)^{\mathrm{ab}} \simeq (\mathbb{Z}/p\mathbb{Z})^{2g^2+g}$.

Proof. Parts 1 and 2 are classical, see for example [LSTX17, Lemma 1 and Proposition 1]. The third part follows from [Sat10, Proposition 10.1 and Corollary 10.2], the fact that $\widehat{\mathrm{Sp}_{2g}(\mathbb{Z}, p)} \simeq \mathrm{Sp}_{2g}(\mathbb{Z}_p, p) \times \prod_{\ell \neq p} \mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$ and Lemma 5.13. □

5.4 Moduli of polarized abelian varieties

Let $D = (d_1, \ldots, d_g)$ be a type of length g. Let $\mathcal{A}_D \to \mathrm{Spec}(\mathbb{C})$ be the moduli stack of g-dimensional polarized abelian varieties of type D [dJ93, Section 1]. It is classical (see [Ols12, Theorem 2.1.11] or [MFK94, Chapter 7, Proposition 7.9 and Theorem 7.10]) that $\mathcal{A}_D$ is a smooth Deligne–Mumford stack whose coarse space is a quasi-projective variety. Let $\mathcal{A}_{\Gamma(D)} \to \mathrm{Spec}(\mathbb{C})$ be the moduli stack of triples (A, λ, α), where (A, λ) is a polarized abelian variety of type D and α is an isomorphism of symplectic modules $(A[\lambda], e_\lambda) \xrightarrow{\sim} (M_D, e_D)$. Then $\mathcal{A}_{\Gamma(D)} \to \mathcal{A}_D$ is a torsor under the group $\mathrm{Sp}(M_D)$, so is a smooth Deligne–Mumford stack too. If $d_1 \geq 3$, then objects of $\mathcal{A}_{\Gamma(D)}$ have no nontrivial automorphisms (see [BL04, Corollary 5.1.10]) and $\mathcal{A}_{\Gamma(D)}$ is representable by a smooth quasi-projective variety over $\mathbb{C}$.

Let $\Gamma \subset \mathrm{Sp}_{2g}^D(\mathbb{Z})$ be subgroup containing $\Gamma(nD)$ for some n with $nd_1 \geq 3$. Then the finite group $\Gamma/\Gamma(nD)$ acts on the variety $\mathcal{A}_{\Gamma(nD)}$; we denote the corresponding quotient stack by $\mathcal{A}_\Gamma$. If Γ is torsion-free (i.e., every element of finite order equals the identity) then this $\Gamma/\Gamma(nD)$-action is free, and $\mathcal{A}_\Gamma$ is again a smooth variety. It carries a universal abelian scheme $A_\Gamma^{\mathrm{univ}} \to \mathcal{A}_\Gamma$ equipped with a polarization of type D.

Let $\mathfrak{h}_g = \{Z \in \mathrm{Mat}_{g\times g}(\mathbb{C})\colon Z^t = Z \text{ and } \mathrm{Im}(Z) \text{ is positive definite}\}$ be the Siegel upper-half space of genus g. The group $\mathrm{Sp}^D_{2g}(\mathbb{R})$ acts on $\mathfrak{h}_g$ via the association

$$g \cdot Z = (\alpha Z + \beta)(\gamma Z + \delta)^{-1} \quad \text{if } g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Sp}^D_{2g}(\mathbb{Q}),\ Z \in \mathfrak{h}_g$$

Let $\mathcal{A}^{\mathrm{an}}_\Gamma$ be the analytification of $\mathcal{A}_\Gamma$. To every $Z \in \mathfrak{h}_g$, we can associate a polarized abelian variety (A_Z, λ_Z) with a symplectic basis (see [BL04, Section 8.1]), and this association defines an isomorphism of orbifolds

$$\mathcal{A}^{\mathrm{an}}_\Gamma \simeq [\Gamma\backslash\mathfrak{h}_g].$$

If Γ is torsion-free, then $\mathcal{A}^{\mathrm{an}}_\Gamma$ is a connected complex manifold whose universal cover $\mathfrak{h}_g$ is contractible. Therefore, $\mathcal{A}^{\mathrm{an}}_\Gamma$ is a classifying space for its fundamental group Γ (equivalently, a $K(\Gamma, 1)$ space). Consequently, if $\mathscr{F}$ is a locally constant sheaf of finite commutative groups on the étale site of $\mathcal{A}_\Gamma$, then there are canonical isomorphisms

$$\mathrm{H}^2(\mathcal{A}_\Gamma, \mathscr{F}) \simeq \mathrm{H}^2(\mathcal{A}^{\mathrm{an}}_\Gamma, \mathscr{F}^{\mathrm{an}}) \simeq \mathrm{H}^2(\Gamma, M_{\mathscr{F}}), \tag{5.6}$$

where H^2 denotes étale, singular and group cohomology respectively, $\mathscr{F}^{\mathrm{an}}$ denotes the analytification of $\mathscr{F}$, and $M_{\mathscr{F}}$ denotes the Γ-representation corresponding to the local system $\mathscr{F}$ on $\mathcal{A}^{\mathrm{an}}_\Gamma$. The first isomorphism is the comparison isomorphism between étale and singular cohomology [SGA73, Exposé IX, Théorème 4.4]; the second isomorphism is the comparison isomorphism between the singular cohomology of a classifying space and the group cohomology of its fundamental group [Bro82, Section III.1, p. 59].

5.5 Twisted Chow groups and negligible étale cohomology

Let X be a smooth and geometrically integral separated scheme of finite type over a field k with function field $k(X)$. Let $\mathscr{F}$ be a locally constant étale sheaf of finite commutative groups which is killed by a positive integer n that is invertible in k.

Definition 5.15. *Say a class $c \in \mathrm{H}^i(X, \mathscr{F})$ is* negligible *if for every field extension K/k and k-morphism $f\colon \mathrm{Spec}(K) \to X$, $f^*(c) = 0$ in $\mathrm{H}^i(\mathrm{Spec}(K), f^*\mathscr{F})$.*

(Compare this definition with the one made in §5.2 in the context of group cohomology; morally this definition coincides with the group cohomology one when "$X = BG$".) Write $\mathrm{H}^i(X, \mathscr{F})_{\mathrm{neg}}$ for the subgroup of negligible classes. Reinterpreting and generalizing the computation of Merkurjev–Scavia in group cohomology [MS25], Totaro [Tot25] has determined $\mathrm{H}^2(X, \mathscr{F})_{\mathrm{neg}}$ when $\mathscr{F}$ has finite monodromy. He states this (and much more) in the language of twisted Chow groups; we extract the concrete statement that we need for our purposes.

Let G be a finite group and $\pi\colon Y \to X$ a G-torsor with Y connected. Suppose that $\pi^*\mathscr{F}$ is constant and write M for the corresponding finite abelian group with G-action. For each $m \in M$, let $G_m = \mathrm{Stab}_G(m)$ and consider the quotient $Y_m = Y/G_m$ and the factorization $Y \to Y_m \xrightarrow{\pi_m} X$ of π. Define the composition

$$\varphi_m\colon\ \mathrm{Pic}(Y_m) \otimes \mathbb{Z}/n\mathbb{Z} \to \mathrm{H}^2(Y_m, \mu_n) \to \mathrm{H}^2(Y_m, \pi_m^*\mathscr{F}(1)) \to \mathrm{H}^2(X, \mathscr{F}(1)), \tag{5.7}$$

where:

- $\mathrm{Pic}(Y_m) \otimes \mathbb{Z}/n\mathbb{Z} \to \mathrm{H}^2(Y_m, \mu_n)$ is the connecting map coming from the Kummer sequence $1 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 1$;
- $\mu_n \to \pi_m^*\mathscr{F}(1) := \pi_m^*\mathscr{F} \otimes_{\mathbb{Z}/n\mathbb{Z}} \mu_n$ is the twist of the map $\mathbb{Z}/n\mathbb{Z} \to \pi_m^*\mathscr{F}$ of etale sheaves on Y_m corresponding to the G_m-module map $\mathbb{Z}/n\mathbb{Z} \to M$ sending 1 to m; and

- $\mathrm{H}^2(Y_m, \pi_m^*\mathscr{F}(1)) \to \mathrm{H}^2(Y, \mathscr{F}(1))$ is the transfer (or corestriction) map associated to the finite etale cover $\pi_m\colon Y_m \to X$.

Theorem 5.16 (Totaro)**.** *In the above notation,* $\ker(\mathrm{H}^2(X, \mathscr{F}(1)) \to \mathrm{H}^2(k(X), \mathscr{F}(1)))$ *equals the subgroup generated by the images of* φ_m*, where* m *ranges over the elements of* M*.*

Proof. Totaro defines the twisted Chow group $\mathrm{CH}^1(X, \mathscr{F})$ and cycle map $\mathrm{cl}_{X,\mathscr{F}}\colon \mathrm{CH}^1(X, \mathscr{F}) \to \mathrm{H}^2(X, \mathscr{F}(1))$ whose image equals the kernel of $\mathrm{H}^2(X, \mathscr{F}(1)) \to \mathrm{H}^2(k(X), \mathscr{F}(1))$, see [Tot25, Lemma 8.2] On the other hand, he shows [Tot25, Theorem 8.1] that $\mathrm{CH}^1(X, \mathscr{F})$ is generated by the images of maps $\psi_m\colon \mathrm{Pic}(Y_m) = \mathrm{CH}^1(Y_m) \to \mathrm{CH}^1(Y_m, \mathscr{F}) \to \mathrm{CH}^1(Y, \mathscr{F})$, as m ranges over M. Unwinding the definition of these maps shows that $\mathrm{cl}_{X,\mathscr{F}} \circ \psi_m = \varphi_m$. ☐

Corollary 5.17. *In the above notation, the following are equivalent for a class* $c \in \mathrm{H}^2(X, \mathscr{F}(1))$*:*

1. *c lies in the subgroup generated by the images of* φ_m*, where* m *ranges over the elements of* M*;*
2. *c is negligible in the sense of Definition 5.15;*
3. *if* $\eta\colon \mathrm{Spec}(k(X)) \to X$ *denotes the generic point, then* $\eta^* c = 0$*.*

Proof. The equivalence (1) ⇔ (3) is Theorem 5.16, and the implication (2) ⇒ (3) follows from the definition. It remains to show (1) ⇒ (2). To this end, it suffices to show for each $m \in M$ that the image of φ_m lands in $\mathrm{H}^2(X, \mathscr{F}(1))_{\mathrm{neg}}$. Fix such an m, let $\ell \in \mathrm{Pic}(Y_m)$ and let $f\colon \mathrm{Spec}(K) \to X$ be a k-morphism for some field K/k. The pullback of $\pi_m\colon Y_m \to X$ along f is an étale K-algebra $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$ and comes equipped with a morphism $f_m\colon \mathrm{Spec}(L) \to Y_m$ lifting f. By compatibility between restriction and corestriction, $f^*\varphi_m(\ell)$ equals the image of $f_m^*\ell \in \mathrm{Pic}(\mathrm{Spec}(L))$ under a map $\mathrm{Pic}(\mathrm{Spec}(L)) \otimes \mathbb{Z}/n\mathbb{Z} \to \mathrm{H}^2(\mathrm{Spec}(K), \mathscr{F}(1))$. Since $\mathrm{Pic}(\mathrm{Spec}(L)) = 0$, we conclude that $f^*\varphi_m(\ell) = 0$, as desired. ☐

We now apply these considerations to our situation of interest. Let $D = (d_1, \dots, d_g)$ be a type and let $\Gamma \leq \mathrm{Sp}_{2g}^D(\mathbb{Z})$ be a torsion-free arithmetic subgroup. Let $\mathcal{A}_\Gamma \to \mathrm{Spec}(\mathbb{C})$ be the corresponding Siegel modular variety constructed in Section 5.4. Assume that the restriction of the reduction map $\mathrm{red}_D\colon \mathrm{Sp}_{2g}^D(\mathbb{Z}) \to \mathrm{Sp}(M_D)$ to Γ is surjective. Restricting the sequence (5.4) to Γ, we obtain the exact sequence

$$1 \to \Gamma \cap \Gamma(D) \to \Gamma \xrightarrow{\mathrm{red}_\Gamma} \mathrm{Sp}(M_D) \to 1. \tag{5.8}$$

Recall that $c_D \in \mathrm{H}^2(\mathrm{Sp}(M_D), M_D)$ denotes the class of the extension (5.2); let $c_{D,\Gamma} \in \mathrm{H}^2(\Gamma, M_D)$ denote the pullback of c_D along $\mathrm{red}_\Gamma\colon \Gamma \to \mathrm{Sp}(M_D)$. View M_D as a locally constant étale sheaf on $\mathcal{A}_\Gamma$, trivialized along the $\mathrm{Sp}(M_D)$-cover $\mathcal{A}_{\Gamma\cap\Gamma(D)} \to \mathcal{A}_\Gamma$ and with monodromy given by the $\mathrm{Sp}(M_D)$-action on M_D. Under the comparison isomorphisms (5.6), $c_{D,\Gamma}$ corresponds to a class $c_{D,\Gamma}^{\mathrm{et}} \in \mathrm{H}^2(\mathcal{A}_\Gamma, M_D)$.

Fix an element $m \in M_D$. Let $\Gamma_m \subset \Gamma$ be the stabilizer of m; this is the preimage of $\mathrm{Stab}_{\mathrm{Sp}(M_D)}(m)$ under the reduction map $\Gamma \to \mathrm{Sp}(M_D)$. Consider the composition

$$\varphi_{\Gamma,m}\colon \mathrm{H}^2(\Gamma_m, \mathbb{Z}) \to \mathrm{H}^2(\Gamma_m, M_D) \to \mathrm{H}^2(\Gamma, M_D), \tag{5.9}$$

where the first map is induced by the Γ_m-module map $\mathbb{Z} \to M_D$ sending 1 to m, and the second map is corestriction. Denote the generic fiber of the universal abelian scheme $A_\Gamma^{\mathrm{univ}} \to \mathcal{A}_\Gamma$ by A_Γ^{gen}; it is an abelian variety over the function field $\mathbb{C}(\mathcal{A}_\Gamma)$ equipped with a polarization λ of type D.

Proposition 5.18. *Suppose that* $c_{D,\Gamma}$ *does not lie in the subgroup generated by the images of* $\varphi_{\Gamma,m}$*, where* m *ranges over the elements of* M_D*. Then* $c_{D,\Gamma}^{\mathrm{et}} \in \mathrm{H}^2(\mathcal{A}_\Gamma, M_D)$ *is not negligible and the answer to Question 2 is no for the pair* $(A_\Gamma^{\mathrm{gen}}, \lambda)$ *over* $\mathbb{C}(\mathcal{A}_\Gamma)$*.*

Proof. The module M_D is killed by $n := d_g$. Since we are working over $\mathbb{C}$, we may identify μ_n with $\mathbb{Z}/n\mathbb{Z}$, hence $M_D(1)$ with M_D. Consider the Chern class map $c_1 \colon \operatorname{Pic}(\mathcal{A}_{\Gamma_m}) \to \mathrm{H}^2(\mathcal{A}^{\mathrm{an}}_{\Gamma_m}, \mathbb{Z}) \simeq \mathrm{H}^2(\Gamma_m, \mathbb{Z})$ arising from the exponential sequence and the comparison isomorphism (5.6). The maps defining φ_m of (5.7) and $\varphi_{\Gamma,m}$ of (5.9) fits into a commutative diagram:

$$\begin{array}{ccccccc} \operatorname{Pic}(\mathcal{A}_{\Gamma_m}) & \longrightarrow & \mathrm{H}^2(\mathcal{A}_{\Gamma_m}, \mathbb{Z}/n\mathbb{Z}) & \longrightarrow & \mathrm{H}^2(\mathcal{A}_{\Gamma_m}, M_D) & \longrightarrow & \mathrm{H}^2(\mathcal{A}_{\Gamma}, M_D) \\ \downarrow{\scriptstyle c_1} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ \mathrm{H}^2(\Gamma_m, \mathbb{Z}) & \longrightarrow & \mathrm{H}^2(\Gamma_m, \mathbb{Z}/n\mathbb{Z}) & \longrightarrow & \mathrm{H}^2(\Gamma_m, M_D) & \longrightarrow & \mathrm{H}^2(\Gamma, M_D) \end{array}$$

The commutativity of this diagram follows from the fact that the comparison isomorphisms (5.6) identify c_1 with the étale cycle class map and corestriction in étale cohomology with corestriction in group cohomology.

It follows that the image of φ_m in $\mathrm{H}^2(\mathcal{A}_\Gamma, M_D) \simeq \mathrm{H}^2(\Gamma, M_D)$ is contained in the image of $\varphi_{\Gamma,m}$ for every $m \in M_D$. The assumptions and Theorem 5.16 therefore imply that $c^{\mathrm{et}}_{D,\Gamma}$ is not negligible and that its image c in $\mathrm{H}^2(k, M_D)$ is nonzero, where $k = \mathbb{C}(\mathcal{A}_\Gamma)$. By definition, c is the pullback of the class $c_D \in \mathrm{H}^2(\mathrm{Sp}(M_D), M_D)$ along the homomorphism $\rho \colon \mathrm{Gal}_k \to \mathrm{Sp}(M_D)$ corresponding to the symplectic module $(A^{\mathrm{gen}}_\Gamma[\lambda], e_\lambda)$ over k. The fact that c is nonzero means (by Lemma 5.2) that there does not exist a theta group for $(A^{\mathrm{gen}}_\Gamma[\lambda], e_\lambda)$. By Theorem Θ, this implies that the answer to Question 2 is no for $(A^{\mathrm{gen}}_\Gamma, \lambda)$. □

In the remainder of the paper, we will perform group theory computations to verify the assumptions of Proposition 5.18 for various D and Γ.

5.6 Polarized abelian varieties not admitting theta groups

Let $n, k \geq 0$ be integers and $g = n + k$. Let $D = (1, \ldots, 1, 2, \ldots, 2)$, where 1 occurs n times and 2 occurs k times. Fix a prime $p \geq 3$ and let $\Gamma = \ker(\mathrm{Sp}^D_{2g}(\mathbb{Z}) \to \mathrm{Sp}^D_{2g}(\mathbb{Z}/p\mathbb{Z}))$. The arithmetic subgroup Γ is torsion-free. The restriction of the reduction map $\mathrm{Sp}^D_{2g}(\mathbb{Z}) \to \mathrm{Sp}(M_D)$ to Γ is surjective, and we have an exact sequence

$$1 \to \Gamma(pD) \to \Gamma \to \mathrm{Sp}(M_D) \to 1$$

Proposition 5.19. *Suppose that $n \geq 3$ and $k \geq 4$. Then the class $c_{D,\Gamma} \in \mathrm{H}^2(\Gamma, M_D)$ does not lie in the subgroup generated by the images of $\varphi_{\Gamma,m}$ of (5.9) where m ranges over the elements of M_D.*

The proof of this proposition is given at the end of this subsection, after some preparations.

Proposition 5.20. *If $n \geq 3$ and $k \geq 2$, then $\Gamma(pD)^{\mathrm{ab}} \otimes \mathbb{Z}/2\mathbb{Z} = 0$.*

Proof. Let $\Gamma(pD)_2 = \ker(\mathrm{Sp}^D_{2g}(\mathbb{Z}_2) \to \mathrm{Sp}(M_D))$ and $\Gamma(pD)_p = \ker(\mathrm{Sp}_{2g}(\mathbb{Z}_p) \to \mathrm{Sp}_{2g}(\mathbb{Z}/p\mathbb{Z}))$. By Lemma 5.9, we have an isomorphism of profinite groups

$$\widehat{\Gamma(pD)} \simeq \Gamma(pD)_2 \times \Gamma(pD)_p \times \prod_{\ell \neq 2,p} \mathrm{Sp}_{2g}(\mathbb{Z}_\ell).$$

By Lemma 5.13, it suffices to compute $\widehat{\Gamma(pD)}^{\mathrm{ab}}$. By Lemma 5.14, $\mathrm{Sp}_{2g}(\mathbb{Z}_\ell)^{\mathrm{ab}} = 1$ for all $\ell \neq 2, p$ and $\Gamma(pD)^{\mathrm{ab}}_p \simeq (\mathbb{Z}/p\mathbb{Z})^{2g^2+g}$. Therefore it suffices to prove that $\Gamma(pD)^{\mathrm{ab}}_2 = \{1\}$. This is a tedious exercise in group theory; we postpone the proof to the appendix (Theorem A.1). □

Proposition 5.21. *If $n \geq 3$ and $k \geq 3$, then $c_{D,\Gamma} \neq 0$.*

Proof. The five-term exact sequence associated to the Hochschild–Serre spectral sequence $\mathrm{H}^p(\mathrm{Sp}(M_D), \mathrm{H}^q(\Gamma(pD), M_D)) \Rightarrow \mathrm{H}^{p+q}(\Gamma, M)$ has the form

$$\cdots \to \mathrm{H}^1(\Gamma(pD), M_D)^{\mathrm{Sp}(M_D)} \to \mathrm{H}^2(\mathrm{Sp}(M_D), M_D) \xrightarrow{f} \mathrm{H}^2(\Gamma, M_D).$$

Since the $\Gamma(pD)$-action on M_D is trivial, we have $\mathrm{H}^1(\Gamma(pD), M_D) = \mathrm{Hom}(\Gamma(pD)^{\mathrm{ab}}, M_D)$, which is trivial by Proposition 5.20 and the fact that M_D is killed by 2. Therefore the final map f is injective. By definition, f sends c_D to $c_{D,\Gamma}$. Since $k \geq 3$, $c_D \neq 0$ by Proposition 5.3, so $c_{D,\Gamma} = f(c_D)$ is nonzero. □

Proposition 5.22. *Let $m \in M_D$ be nonzero and let $\Gamma_m \subset \Gamma$ be the stabilizer of m. Assume $n \geq 3$ and $k \geq 4$. Then:*

1. $\Gamma_m^{\mathrm{ab}} \otimes (\mathbb{Z}/2\mathbb{Z}) = 0$.
2. *The cokernel of the restriction map* $\mathrm{res}\colon \mathrm{H}^2(\Gamma, \mathbb{Z}) \to \mathrm{H}^2(\Gamma_m, \mathbb{Z})$ *is finite of odd order.*

Proof. 1. Let $G_m \subset \mathrm{Sp}(M_D)$ be the stabilizer of m. Since $\Gamma \to \mathrm{Sp}(M_D)$ is surjective, $\Gamma_m/\Gamma(pD) \simeq G_m$. The five-term exact sequence in group homology associated to the spectral sequence $\mathrm{H}_p(G_m, \mathrm{H}_q(\Gamma(pD), \mathbb{F}_2)) \Rightarrow \mathrm{H}_{p+q}(\Gamma_m, \mathbb{F}_2)$ looks like

$$\cdots \to \mathrm{H}_0(\mathrm{Sp}(M_D), \mathrm{H}_1(\Gamma(pD), \mathbb{F}_2)) \to \mathrm{H}_1(\Gamma_m, \mathbb{F}_2) \to \mathrm{H}_1(G_m, \mathbb{F}_2) \to 1.$$

By Proposition 5.20, $\mathrm{H}_1(\Gamma(pD), \mathbb{F}_2) = \Gamma(pD)^{\mathrm{ab}} \otimes \mathbb{F}_2 = 0$. By Lemma 5.7 and the assumption $k \geq 4$, $\mathrm{H}_1(G_m, \mathbb{F}_2) = 0$. Therefore $\Gamma_m \otimes \mathbb{F}_2 = \mathrm{H}_1(\Gamma_m, \mathbb{F}_2) = 0$.

2. The exact sequences (5.5) for Γ and Γ_m are connected by a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{H}^1(\Gamma, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \mathrm{H}^2(\Gamma, \mathbb{Z}) & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\
 & & \downarrow \alpha & & \downarrow \mathrm{res} & & \downarrow \beta & & \\
1 & \longrightarrow & \mathrm{H}^1(\Gamma_m, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \mathrm{H}^2(\Gamma_m, \mathbb{Z}) & \longrightarrow & \mathbb{Z} & \longrightarrow & 1
\end{array}$$

By the snake lemma, it suffices to prove that the cokernels of α and β are finite of odd order. For the cokernel of α, this follows from Part 1 and the fact that $\mathrm{H}^1(\Gamma_m, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(\Gamma_m^{\mathrm{ab}}, \mathbb{Q}/\mathbb{Z})$. To analyze β, let $\mathrm{cores}\colon \mathrm{H}^2(\Gamma_m, \mathbb{Z}) \to \mathrm{H}^2(\Gamma, \mathbb{Z})$ the corestriction map. Then $\mathrm{cores} \circ \mathrm{res} = [\Gamma : \Gamma_m]$. Since the action of $\mathrm{Sp}(M_D)$ on the nonzero elements of M_D is transitive, $[\Gamma : \Gamma_m] = [\mathrm{Sp}(M_D) : G_m] = \#M_D - 1 = 2^{2k} - 1$. Therefore $\beta(1) \in \mathbb{Z}$ must be an integer n dividing $2^{2k} - 1$. In particular, n must be odd. So $\mathrm{coker}(\beta) \simeq \mathbb{Z}/n\mathbb{Z}$ is finite of odd order.

□

Proof of Proposition 5.19. By Proposition 5.21, $c_{D,\Gamma} \neq 0$. Hence it suffices to show that $\varphi_{\Gamma,m} = 0$ for each $m \in M_D$. Let $m \in M_D$ be an element. If $m = 0$, then the first map in (5.9) is induced by the zero map $\mathbb{Z} \to M_D$, so evidently $\varphi_{\Gamma,m} = 0$ in this case. So we may assume $m \neq 0$. Let $\mathrm{res}_m\colon \mathrm{H}^2(\Gamma, \mathbb{Z}) \to \mathrm{H}^2(\Gamma_m, \mathbb{Z})$ be the restriction map. We first claim that $\varphi_{\Gamma,m} \circ \mathrm{res}_m = 0$. To prove this, note that $\varphi_{\Gamma,m} \circ \mathrm{res}_m$ is $\mathrm{H}^2(\Gamma, -)$ applied to the composition of Γ-module maps

$$\mathbb{Z} \to \mathbb{Z}[\Gamma/\Gamma_m] \to M_D, \tag{5.10}$$

where $\mathbb{Z}[\Gamma/\Gamma_m]$ is the permutation module on the Γ-set Γ/Γ_m, $\mathbb{Z} \to \mathbb{Z}[\Gamma/\Gamma_m]$ sends 1 to $\sum g \cdot \Gamma_m$ (where g ranges over coset representatives in Γ/Γ_m), and $\mathbb{Z}[\Gamma/\Gamma_m] \to M_D$ sends $g \cdot \Gamma_m$ to $g \cdot m$. So the composite (5.10) sends 1 to $m'' = \sum_{m' \in \Gamma \cdot m} m'$. Since m'' is an $\mathrm{Sp}(M_D)$-invariant element of M_D and since there are no such nonzero elements, we must have $m'' = 0$ and the composite (5.10) must be zero. Therefore the composite $\varphi_m \circ \mathrm{res}_m$ is also zero, as claimed.

We conclude that $\varphi_{\Gamma,m}\colon \mathrm{H}^2(\Gamma_m,\mathbb{Z}) \to \mathrm{H}^2(\Gamma, M_D)$ factors through the cokernel of $\mathrm{res}_m\colon \mathrm{H}^2(\Gamma,\mathbb{Z}) \to \mathrm{H}^2(\Gamma_m,\mathbb{Z})$. Since M_D is killed by 2, the target of $\varphi_{\Gamma,m}$ is killed by 2. By Proposition 5.22, the cokernel of res_m has odd order. Combining the last two sentences shows that $\varphi_{\Gamma,m} = 0$. □

Proof of Theorem C. Combine Propositions 5.18 and 5.19. □

A Group theory computations

In this appendix, we finish the proof of Proposition 5.20 by computing the abelianization of an explicit 2-adic congruence subgroup. Let $n, k \geq 0$ be integers and let $g = n + k$. Consider the type $D = (1, \ldots, 1, 2, \ldots, 2)$, where 1 occurs n times and 2 occurs k times. Consider the profinite groups $\mathrm{Sp}^D_{2g}(\mathbb{Z}_2)$ and $\Gamma = \ker(\mathrm{Sp}^D_{2g}(\mathbb{Z}_2) \to \mathrm{Sp}(M_D))$. (We deviate from our standard notation that Γ is a congruence subgroup of $\mathrm{Sp}^D_{2g}(\mathbb{Z})$.)

Theorem A.1. *If $n \geq 3$ and $k \geq 2$, then Γ^{ab} is trivial.*

The proof of this theorem is given at the end of this section, after some preliminary explicit lemmas. In the words of Mumford [Mum83, p. 202], there is nothing very difficult in any of these. In the standard basis $e_1, \ldots, e_g, f_1, \ldots, f_g$ of $\Lambda_D \otimes \mathbb{Z}_2 = \mathbb{Z}_2^{2g}$, we have

$$\begin{aligned}\mathrm{Sp}^D_{2g}(\mathbb{Z}_2) &= \{\gamma \in \mathrm{GL}_{2g}(\mathbb{Z}_2)\colon \gamma^t J_D \gamma = J_D\} \\ &= \left\{ \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \in \mathrm{GL}_{2g}(\mathbb{Z}_2)\colon X^tDZ \text{ and } Y^tDW \text{ symmetric}, X^tDW - Z^tDY = D \right\}.\end{aligned}$$

We will break each $g \times g$-block X, Y, Z, W into subblocks, and we will write $X = \left(\begin{smallmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{smallmatrix}\right)$, where X_{11} has size $n \times n$, X_{12} has size $n \times k$, X_{21} has size $k \times n$ and X_{22} has size $n \times n$. Similarly we have Y_{ij}, Z_{ij} and W_{ij} for $1 \leq i, j \leq 2$. The condition $J_D g J_D^{-1} = g^{-t} \in \mathrm{GL}_{2g}(\mathbb{Z}_2)$ implies that $DXD^{-1}, DYD^{-1}, DZD^{-1}, DWD^{-1}$ have entries in $\mathbb{Z}_2$. This implies that all the entries of $X_{12}, Y_{12}, Z_{12}, W_{12}$ are in $2\mathbb{Z}_2$.

We record two automorphisms of $\mathrm{Sp}^D_{2g}(\mathbb{Z}_2)$. The first one is $\gamma \mapsto (\gamma^*)^{-1} = (J_D^{-1}\gamma^t J_D)^{-1}$, and the second one is conjugation by $h = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right) \in \mathrm{Sp}^D_{2g}(\mathbb{Z}_2)$. Explicitly, we have

$$\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}^* = \begin{pmatrix} D^{-1}W^tD & -D^{-1}Y^tD \\ -D^{-1}Z^tD & D^{-1}X^tD \end{pmatrix}, \quad h\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}h^{-1} = \begin{pmatrix} W & -Z \\ -Y & X \end{pmatrix}. \tag{A.1}$$

Recall that $M_D = \Lambda_D^\vee/\Lambda_D$ and that the reduction map $\mathrm{Sp}^D_{2g}(\mathbb{Z}_2) \to \mathrm{Sp}(M_D) \simeq \mathrm{Sp}_{2k}(\mathbb{F}_2)$ is surjective. In the coordinates of M_D fixed in Section 5.3, this reduction map is given by

$$\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \mapsto \begin{pmatrix} X_{22} & Y_{22} \\ Z_{22} & W_{22} \end{pmatrix} \bmod 2.$$

The restriction of the symplectic form to the submodule Λ_1 with basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ has Gram matrix J_{D_1}, where $D_1 = (1, \ldots, 1)$ has length n. Similarly, the restriction of the form to the submodule Λ_2 spanned by $\{e_{n+1}, \ldots, e_g, f_{n+1}, \ldots, f_g\}$ has Gram matrix J_{D_2}, where $D_2 = (2, \ldots, 2)$ has length k. The subgroup of elements of $\mathrm{Sp}^D_{2g}(\mathbb{Z}_2)$ that preserve Λ_1 and Λ_2 is isomorphic to $\mathrm{Sp}(\Lambda_1) \times \mathrm{Sp}(\Lambda_2) \simeq \mathrm{Sp}_{2n}(\mathbb{Z}_2) \times \mathrm{Sp}_{2k}(\mathbb{Z}_2)$. This defines an embedding $i\colon \mathrm{Sp}_{2n}(\mathbb{Z}_2) \times \mathrm{Sp}_{2k}(\mathbb{Z}_2) \hookrightarrow \mathrm{Sp}^D_{2g}(\mathbb{Z}_2)$ whose image equals those matrices for which $X_{12}, X_{21}, Y_{12}, Y_{21}, Z_{12}, Z_{21}, W_{12}, W_{21}$ are all zero. Let $\mathrm{Sp}_{2k}(\mathbb{Z}_2, 2) = \ker(\mathrm{Sp}_{2k}(\mathbb{Z}_2) \to \mathrm{Sp}_{2k}(\mathbb{F}_2))$. The restriction of i to Γ is an embedding $\mathrm{Sp}_{2n}(\mathbb{Z}_2) \times \mathrm{Sp}_{2k}(\mathbb{Z}_2, 2) \hookrightarrow \Gamma$.

Lemma A.2. *If $n \geq 3$, $\mathrm{Sp}_{2n}(\mathbb{Z}_2)^{\mathrm{ab}}$ is trivial. If $k \geq 2$, the commutator subgroup of $\mathrm{Sp}_{2k}(\mathbb{Z}_2, 2)$ equals*

$$\mathrm{Sp}_{2k}(\mathbb{Z}_2, 4, 8) := \{A = (a_{ij}) \in \mathrm{Sp}_{2k}(\mathbb{Z}_2)\colon A \equiv I \bmod 4 \text{ and } a_{i,i+k} \equiv a_{i+k,i} \equiv 0 \bmod 8 \text{ for all } i = 1, \ldots, k\}.$$

Proof. The first claim is a special case of Lemma 5.14. The second part follows from the isomorphism $\widehat{\mathrm{Sp}_{2g}(\mathbb{Z}, 2)} \simeq \mathrm{Sp}_{2g}(\mathbb{Z}_2, 2) \times \prod_{p\geq 3} \mathrm{Sp}_{2g}(\mathbb{Z}_p)$, Lemmas 5.13 and 5.14, and [Sat10, Proposition 10.1]. □

Let $L \subset \Gamma$ be the subgroup of elements with $Y = Z = 0$. Let

$$L' = \{X \in \mathrm{GL}_g(\mathbb{Z}_2)\colon X_{12} \equiv 0 \bmod 2, X_{22} \equiv 1 \bmod 2\}. \tag{A.2}$$

Then the assignment $X \mapsto \left(\begin{smallmatrix} X & 0 \\ 0 & D^{-1}X^{-t}D \end{smallmatrix}\right)$ defines an isomorphism of groups $\alpha\colon L' \xrightarrow{\sim} L$. For a ring R and integers $a, b \geq 1$, let $\mathrm{Mat}_{a,b}(R)$ be the R-module of $a \times b$-matrices.

Lemma A.3. *If $n \geq 3$ and $k \geq 2$, then $L \subset \Gamma^{\mathrm{der}}$.*

Proof. By considering block diagonal elements of $\mathrm{Sp}_{2n}(\mathbb{Z}_2)$, we see that every element of L of the form

$$\alpha\left(\begin{pmatrix} X_{11} & 0 \\ 0 & I \end{pmatrix}\right), \quad X_{11} \in \mathrm{GL}_n(\mathbb{Z}_2) \tag{A.3}$$

lies in the image of $i\colon \mathrm{Sp}_{2n}(\mathbb{Z}_2) \to \Gamma$. By Lemma A.2 and the assumption $n \geq 3$, every element in this image lies in Γ^{der}, so every element of the form (A.3) lies in Γ^{der}. Similarly, every element of the form

$$\alpha\left(\begin{pmatrix} I & 0 \\ 0 & X_{22} \end{pmatrix}\right), \quad X_{22} \in \ker(\mathrm{GL}_k(\mathbb{Z}_2) \to \mathrm{GL}_k(\mathbb{Z}/4\mathbb{Z})) \tag{A.4}$$

lies in Γ^{der}. Next, we consider strictly upper triangular elements. If $X \in \mathrm{GL}_n(\mathbb{Z}_2)$ and $Y \in 2\mathrm{Mat}_{n,k}(\mathbb{Z}_2)$, we calculate the commutator

$$\left[\begin{pmatrix} X & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix}\right] = \begin{pmatrix} I & XY - Y \\ 0 & I \end{pmatrix}$$

in L'. We claim that the $\mathbb{Z}$-span of the set $\{XY - Y\colon X \in \mathrm{GL}_n(\mathbb{Z}_2), Y \in 2\mathrm{Mat}_{n,k}(\mathbb{Z}_2)\}$ equals $2\mathrm{Mat}_{n,k}(\mathbb{Z}_2)$. To prove this, we may assume Y consists of a single column, in which case it can be explicitly checked by taking X to be upper and lower triangular matrices (and using that $n \geq 2$). We conclude that every element of the form $\alpha\left(\left(\begin{smallmatrix} I & * \\ 0 & I \end{smallmatrix}\right)\right) \in L$ lies in L^{der}. By an identical argument, every element of the form $\alpha\left(\left(\begin{smallmatrix} I & 0 \\ * & I \end{smallmatrix}\right)\right) \in L$ lies in L^{der}. Now let $\alpha\left(\left(\begin{smallmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{smallmatrix}\right)\right) \in L$ be a general element. Then the condition $X_{12} \equiv 0 \bmod 2$ implies that X_{11} is invertible modulo 2, so $X_{11} \in \mathrm{GL}_n(\mathbb{Z}_2)$. Therefore, left multiplying by $\alpha\left(\left(\begin{smallmatrix} I & 0 \\ -X_{21} & I \end{smallmatrix}\right)\left(\begin{smallmatrix} X_{11}^{-1} & 0 \\ 0 & I \end{smallmatrix}\right)\right) \in \Gamma^{\mathrm{der}}$ and right multiplying by $\alpha\left(\left(\begin{smallmatrix} I & -X_{11}^{-1}X_{12} \\ 0 & I \end{smallmatrix}\right)\right) \in \Gamma^{\mathrm{der}}$ shows that

$$\alpha\left(\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}\right) \equiv \alpha\left(\begin{pmatrix} I & 0 \\ 0 & X_{22} - X_{21}X_{11}^{-1}X_{12} \end{pmatrix}\right) \mod \Gamma^{\mathrm{der}}. \tag{A.5}$$

To prove the lemma, it therefore suffices to show that every element of L the form $\alpha\left(\left(\begin{smallmatrix} I & 0 \\ 0 & * \end{smallmatrix}\right)\right)$ lies in Γ^{der}. We calculate using (A.5) that for all $X \in \mathrm{Mat}_{n,k}(\mathbb{Z}_2)$ and $Y \in \mathrm{Mat}_{k,n}(\mathbb{Z}_2)$:

$$\alpha\left(\left[\begin{pmatrix} I & 2X \\ 0 & I \end{pmatrix}, \begin{pmatrix} I & 0 \\ Y & I \end{pmatrix}\right]\right) = \alpha\left(\begin{pmatrix} * & -4XYX \\ 2YXY & I - 2YX \end{pmatrix}\right) \equiv \alpha\left(\begin{pmatrix} I & 0 \\ 0 & I - 2YX + 8(\cdots) \end{pmatrix}\right) \mod \Gamma^{\mathrm{der}},$$

where $(\cdots)$ denotes an element of $\mathrm{Mat}_{k,k}(\mathbb{Z}_2)$. By combining this identity with the elements of the form (A.4), it suffices to prove that the set $\{I - 2YX\colon X \in \mathrm{Mat}_{n,k}(\mathbb{Z}_2),\ Y \in \mathrm{Mat}_{k,n}(\mathbb{Z}_2)\}$ generates a subgroup of $\mathrm{GL}_k(\mathbb{Z}_2)$ that surjects onto $\ker(\mathrm{GL}_k(\mathbb{Z}/4\mathbb{Z}) \to \mathrm{GL}_k(\mathbb{Z}/2\mathbb{Z}))$. This is true, as can be seen by taking X and Y to be $\mathbb{Z}_2$-multiples of elementary matrices. □

Let $U \subset \Gamma$ be the subgroup of strictly upper triangular elements, i.e., those elements with $X = W = I$ and $Z = 0$. Define the $\mathbb{Z}_2$-module

$$U' = \{Y \in \mathrm{Mat}_{g,g}(\mathbb{Z}_2)\colon Y_{11}, Y_{22} \text{ symmetric, } Y_{12} = 2Y_{21}^t \text{ and } Y_{22} \equiv 0 \bmod 2\}$$

Then the assignment $Y \mapsto \left(\begin{smallmatrix} I & Y \\ 0 & I \end{smallmatrix}\right)$ is an isomorphism of abelian groups $\beta\colon U' \to U$. Let $U^{\mathrm{opp}} \subset \Gamma$ be the subgroup of strictly lower triangular matrices, i.e., those with $X = W = I$ and $Y = 0$.

Lemma A.4. *If $n \geq 3$ and $k \geq 2$, then $U \subset \Gamma^{\mathrm{der}}$ and $U^{\mathrm{opp}} \subset \Gamma^{\mathrm{der}}$.*

Proof. Denote the submodule of symmetric matrices of $\mathrm{Mat}_{n,n}(\mathbb{Z}_2)$ by $\mathrm{Sym}_n(\mathbb{Z}_2)$. By considering strictly upper triangular elements in $\mathrm{Sp}_{2n}(\mathbb{Z}_2)$ and $\mathrm{Sp}_{2k}(\mathbb{Z}_2)$ and using Lemma A.2, we see that every element of the form

$$\beta\left(\begin{pmatrix} Y_{11} & 0 \\ 0 & Y_{22} \end{pmatrix}\right), \quad Y_{11} \in \mathrm{Sym}_n(\mathbb{Z}_2), Y_{22} \in 8\,\mathrm{Sym}_k(\mathbb{Z}_2) \tag{A.6}$$

lies in Γ^{der}. We now calculate commutators between elements of L and U. If $X \in L'$ and $Y \in U'$, then

$$[\alpha(X), \beta(Y)] = \begin{pmatrix} I & XYW^{-1} - Y \\ 0 & I \end{pmatrix}, \tag{A.7}$$

where $W = D^{-1}X^{-t}D$. We claim that every element of U is a product of elements of the form (A.6) and (A.4). To prove this claim, note that

$$XYW^{-1} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \begin{pmatrix} X_{11}^t & 2X_{21}^t \\ \frac{1}{2}X_{12}^t & X_{22}^t \end{pmatrix}.$$

To consider off-diagonal blocks, we compute that if $X_{12} = 0$, $X_{21} = 0$, $X_{22} = 0$, $Y_{11} = 0$ and $Y_{22} = 0$, then

$$XYW^{-1} - Y = \begin{pmatrix} 0 & X_{11}Y_{12} - Y_{12} \\ Y_{21}X_{11}^t - Y_{21} & 0 \end{pmatrix}. \tag{A.8}$$

By an argument similar to the proof of Lemma A.3, the $\mathbb{Z}$-span of $\{YX^t - Y\colon X \in \mathrm{GL}_n(\mathbb{Z}_2),\, Y \in \mathrm{Mat}_{k,n}(\mathbb{Z}_2)\}$ equals $\mathrm{Mat}_{k,n}(\mathbb{Z}_2)$. So every element of U' of the form $\left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)$ is of the form (A.8). To consider the lower-diagonal block, we compute that if $X_{11} = I$, $X_{12} = 0$, $X_{22} = I$, $Y_{12} = 0$, $Y_{21} = 0$ and $Y_{22} = 0$, then

$$XYW^{-1} - Y = \begin{pmatrix} * & * \\ * & 2X_{21}Y_{11}X_{21}^t \end{pmatrix}. \tag{A.9}$$

By considering elementary matrices, we find that the $\mathbb{Z}$-span of the set $\{2XYX^t\colon X \in \mathrm{Mat}_{k,n}(\mathbb{Z}_2),\, Y \in \mathrm{Sym}_n(\mathbb{Z}_2)\}$ equals $2\,\mathrm{Sym}_k(\mathbb{Z}_2)$. Combining all elements of the form (A.6), (A.8) and (A.9) proves the claim, hence proves that $U \subset \Gamma^{\mathrm{der}}$. Applying the second automorphism displayed in (A.1) to this containment also proves that $U^{\mathrm{opp}} \subset \Gamma^{\mathrm{der}}$. ☐

Lemma A.5. *Let H be the closure of the subgroup of Γ generated by L, U and U^{opp}. Then $H = \Gamma$.*

Proof. The proof is inspired by the methods of [Ros78]. Recall that we have an embedding $i\colon \mathrm{Sp}_{2n}(\mathbb{Z}_2) \times \mathrm{Sp}_{2k}(\mathbb{Z}_2, 2) \hookrightarrow \Gamma$ which we use to view $\mathrm{Sp}_{2n}(\mathbb{Z}_2) \times \mathrm{Sp}_{2k}(\mathbb{Z}_2, 2)$ as a subgroup of Γ.

Claim 1: $\mathrm{Sp}_{2n}(\mathbb{Z}_2) \subset H$ and $\mathrm{Sp}_{2k}(\mathbb{Z}_2, 2) \subset H$. Indeed, [BMS67, Corollary 12.5] shows that $\mathrm{Sp}_{2n}(\mathbb{Z})$ is generated by elements in U and U^{opp}, and [Mum83, p. 207, Proposition A3] shows $\mathrm{Sp}_{2k}(\mathbb{Z}, 2)$ is generated by elements in U, U^{opp} and L. Since the inclusions $\mathrm{Sp}_{2n}(\mathbb{Z}) \subset \mathrm{Sp}_{2n}(\mathbb{Z}_2)$ and $\mathrm{Sp}_{2k}(\mathbb{Z}, 2) \subset \mathrm{Sp}_{2k}(\mathbb{Z}_2, 2)$ are dense, the claim is proven.

We will prove the lemma by induction on n. If $n = 0$, then $\Gamma = \mathrm{Sp}_{2k}(\mathbb{Z}_2, 2)$, so the lemma is true by Claim 1. Therefore we may suppose $n \geq 1$. Let $\gamma \in \Gamma$ be an element and let $u = \left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right)$ be the first column of γ, where $v, w \in \mathbb{Z}_2^g$. We write $v = \left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right)$ and $w = \left(\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right)$, where $v_1, w_1 \in \mathbb{Z}_2^n$ and $v_2, w_2 \in \mathbb{Z}_2^k$.

Claim 2: There exists an element $\delta_1 \in \mathrm{Sp}_{2n}(\mathbb{Z}_2)$ such that $\delta_1 \cdot \left(\begin{smallmatrix} v_1 \\ w_1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 & \cdots & 0 \end{smallmatrix}\right)^t$ Indeed, since $\mathrm{Sp}_{2n}(\mathbb{Z}_2)$ acts transitively on primitive vectors (in other words, elements of $\mathbb{Z}_2^{2n} \setminus (2\mathbb{Z}_2^{2n})$) it suffices to prove that $\left(\begin{smallmatrix} v_1 \\ w_1 \end{smallmatrix}\right)$ is primitive. Examining the $(g+1)$-th row of γ^* (using the first formula of (A.1)) shows that $u' = \begin{pmatrix} -w_1^t & -2w_2^t & v_1^t & 2v_2^t \end{pmatrix}$ appears as a row of an element of $\mathrm{Sp}_{2g}^D(\mathbb{Z}_2) \subset \mathrm{GL}_{2g}(\mathbb{Z}_2)$, so u' must be primitive. This implies that $\left(\begin{smallmatrix} v_1 \\ w_1 \end{smallmatrix}\right)$ is primitive, proving the claim.

Claim 3: Suppose $v_1 = \left(\begin{smallmatrix} 1 & 0 & \cdots & 0 \end{smallmatrix}\right)^t$. Then there exists an element $\delta_2 \in U^{\mathrm{opp}}$ such that $\delta_2 \cdot u$ has first n coordinates equal to v_1 and last k coordinates equal to zero. Indeed, let $Z_{21} \in \mathrm{Mat}_{k,n}(\mathbb{Z}_2)$ be the matrix with first column equal to $-w_2$ and all other columns equal to zero, and let $Z = \begin{pmatrix} 0 & 2Z_{21}^t \\ Z_{21} & 0 \end{pmatrix}$. Then $\delta_2 = \left(\begin{smallmatrix} I & 0 \\ Z & I \end{smallmatrix}\right)$ satisfies the conclusion of the claim.

Claim 4: There exists an element $\delta \in H$ such that $\delta \cdot u = \left(\begin{smallmatrix} 1 & 0 & \cdots & 0 \end{smallmatrix}\right)^t$. Indeed, by Claims 1 and 2, there exists a $\delta_1 \in H$ such that the first n coordinates of $\delta_1 \cdot u$ equal $1, 0, \ldots, 0$. By Claim 3, there exists a $\delta_2 \in H$ such that the first n coordinates of $\delta_2 \cdot (\delta_1 \cdot u)$ equal $1, 0, \ldots, 0$ and the last k coordinates equal $0, \ldots, 0$. Applying Claims 1 and 2 again, there is an element $\delta_3 \in \mathrm{Sp}_{2n}(\mathbb{Z}_2) \subset H$ such that $u' = \delta_3 \cdot (\delta_2\delta_1 \cdot u)$ has first n coordinates $1, 0, \ldots, 0$ and last g coordinates equal to 0. Let $v' \in \mathbb{Z}_2^g$ be the vector formed of the first g coordinates of u'. The explicit description (A.2) shows that there exists an element $X \in L'$ such that $X \cdot v' = \left(\begin{smallmatrix} 1 & 0 & \cdots & 0 \end{smallmatrix}\right)^t$. Then the element $\alpha(X) \in L$ satisfies $\alpha(X) \cdot u' = \left(\begin{smallmatrix} 1 & 0 & \cdots & 0 \end{smallmatrix}\right)^t$. Taking $\delta = \alpha(X)\delta_3\delta_2\delta_1 \in H$ proves the claim.

Using Claim 4, it suffices to prove that every element $\gamma \in \Gamma$ whose first column equals $\left(\begin{smallmatrix} 1 & 0 & \cdots & 0 \end{smallmatrix}\right)^t$ lies in H. Let γ be such an element. Recall that $e_1, \ldots, e_g, f_1, \ldots, f_g$ denotes the standard basis of $\Lambda_D \otimes \mathbb{Z}_2 = \mathbb{Z}_2^{2g}$. By assumption, $\gamma \cdot e_1 = e_1$. Therefore γ also preserves the submodule $\langle e_1 \rangle^\perp = \langle e_1, \ldots, e_g, f_2, \ldots, f_g \rangle$. In terms of matrices, this means that γ is of the form

$$\left(\begin{array}{c:c|c:c} 1 & * & * & * \\ \hdashline 0 & X_1 & * & Y_1 \\ \hline 0 & 0 & * & 0 \\ \hdashline 0 & Z_1 & * & W_1 \end{array}\right),$$

where we have divided γ into $g \times g$-blocks using the solid lines, and where each $g \times g$-block is further divided into blocks of size 1×1, $1 \times (g-1)$, $(g-1) \times 1$ and $(g-1) \times (g-1)$ by the dashed lines. The quotient $N = \langle e_1 \rangle^\perp / \langle e_1 \rangle$ has $\mathbb{Z}_2$-basis given by the images of $\{e_2, \ldots, e_g, f_2, \ldots, f_g\}$, and the symplectic form on $\Lambda_D \otimes \mathbb{Z}_2$ induces a symplectic form on N of type $D' = (1, \ldots, 1, 2, \ldots, 2)$, where 1 occurs $n-1$ times and 2 occurs k times. The map γ induces a map $\gamma' \in \mathrm{GL}(N)$ that preserves the induced symplectic form. In the above basis of N, $\gamma' = \left(\begin{smallmatrix} X_1 & Y_1 \\ Z_1 & W_1 \end{smallmatrix}\right)$. Writing $\Gamma' = \ker(\mathrm{Sp}_{2g}^{D'}(\mathbb{Z}_2) \to \mathrm{Sp}(M_{D'}))$, we see that $\gamma' \in \Gamma'$. Let $\gamma'' \in \Gamma$ be the element that acts on $\langle e_2, \ldots, e_g, f_2, \ldots, f_g \rangle$ via γ' and acts trivially on $\langle e_1, f_1 \rangle$. By the induction hypothesis, γ' lies in the analogue $H' \subset \Gamma'$ of the subgroup $H \subset \Gamma$. Therefore $\gamma'' \in H$, and the matrix $(\gamma'')^{-1}\gamma$ is of the form

$$\left(\begin{array}{c:c|c:c} 1 & * & * & * \\ \hdashline 0 & I & * & 0 \\ \hline 0 & 0 & * & 0 \\ \hdashline 0 & 0 & * & I \end{array}\right).$$

This matrix is block upper triangular, so lies in the subgroup generated by L and U, which both lie in H. This shows that $(\gamma'')^{-1}\gamma \in H$, so $\gamma \in H$, proving the lemma. □

Proof of Theorem A.1. Combine Lemmas A.3, A.4 and A.5. □

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