curve of field

C/K genus $g \geq 2$

Thm (Vojta)

Let $P_1 \neq P_2 \in C(K)$ Assume

$$|P_1| \geq 1.2 \cdot 10^9 g^{\frac{7}{3}} \sqrt{\omega_a^2}$$

$$|P_2| \geq 10^5 g^{5/2} |P_1|$$

$$\Rightarrow \cos\theta(P_1, P_2) \leq \sqrt{\frac{1.01}{g}}$$

Thm (Mumford)

Let $P_1 \neq P_2 \in C(\overline{K})$ Assume

$$|P_1| \geq 10^9 g^{\frac{7}{3}} \sqrt{\omega_a^2}$$

$$|P_1| \leq |P_2| \leq 1.15|P_2|$$

$$\Rightarrow \cos\theta(P_1, P_2) \leq \frac{1.01}{g}$$

Thm (Bogomolov) $J = \mathrm{Jac}(C)$

1) for all $0 \leq r \leq \sqrt{\frac{1}{8g}}$, $x \in J(\overline{K})_{\mathbb{R}}$,

$$\#\{P \in C(\overline{K}) : |P - x| \leq r\sqrt{\omega_a^2}\} \leq \frac{3.2 \cdot 10^{11} g^{\frac{11}{3}}}{1 - 8g r^2}\left(1 + \log\frac{1}{1 - 8g r^2}\right)$$

2) Let $\kappa > 1$, $\theta \in [0, \pi/2)$ s.t.

$$g \cos^2\theta > \frac{(\kappa+1)^2}{4\kappa} + \frac{1}{3.2 \cdot 10^{11} g^{\frac{11}{3}}}$$

Then for any $0 \neq x \in J(\overline{K})_{\mathbb{R}}$

$$\#\{P \in C(\overline{K}) : |x| \leq |P| \leq \kappa|x|,\ \theta(P, x) \leq \theta\} \leq 3.2 \cdot 10^{11} g^{11/3}$$

Main theorem:

Let K field of char 0, C a curve of genus $g \geq 2$ over K, $J = \mathrm{Jac}(C)$. For any line bundle $\deg(\alpha) = 1$ on C and any finite rank subgroup $\Lambda \subset J(K)$ we have

$$\#\left(C(K) - \alpha \cap \Lambda\right) \leq 10^{13} g^{8} \cdot \min\left(1 + \frac{n}{5\sqrt{g}}, 1 + \frac{3\log g}{g}\right)^{rk(\Lambda)}$$

Here finite rank means there exists $\Lambda_0 \leq J(K)$ with $rk(\Lambda)$ generators and n s.t. $\Lambda \leq \frac{1}{n}\Lambda_0$

Can reduce to the case $K = \#$ field by a spreading out ~~argument~~ + specialization argument

The case $rk(\Lambda) = 1$ is quantitative Manin–Mumford

Taking wlog α_0 s.t. $(2g-2)\alpha_0 = \omega$ by weakening $rk\,\Lambda$ to $rk\,\Lambda - 1$

Since $C(K) - \alpha \cap \Lambda \subseteq (C(K) - \alpha_0) \cap (\Lambda' \cup \{x_0\})$

with $\Lambda' = A_1 + \langle \Lambda, \alpha - \alpha_0 \rangle$

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3 types of points

$$(C \cap \Lambda)_{small} = \left\{ |P| \leq \sqrt{\frac{\omega_a^2}{16\delta}} \right\}$$

$$(C \cap \Lambda)_{medium} = \left\{ \sqrt{\frac{\omega_a^2}{16\delta}} < |P| \leq 1.2 \cdot 10^{9} \delta^{7/3} \sqrt{\omega_a^2} \right\}$$

$$(C \cap \Lambda)_{large} = \left\{ |P| > 1.2 \cdot 10^{9} \delta^{7/3} \sqrt{\omega_a^2} \right\}$$

1: small points: Apply Bogomolov w/ ~~the~~ $z = \sqrt{\frac{1}{16\delta}}$

gives $\leq 6.4 \cdot 10^{11} \delta^{\frac{17}{3}} (1 + 10^{-6} \log 2)$

$n = d\Lambda$

Medium points

For $P \in \mathcal{J}(K)$ define $\mathrm{Cap}(P, \theta)$ as
$\{X \in \Lambda_{\mathbb{R}} : |X| = 1,\ \angle(x, P) \leq \theta\}$

Define $(C \cap \Lambda)_{z,z'} = \{P \in C \cap \Lambda : z < |P| \leq z'\}$

we can split medium points into

$$\log_K \left(\frac{1.2 \cdot 10^9 \cdot 8^{7/3} \cdot \sqrt{\omega_n^2}}{\frac{1}{\sqrt{168}} \cdot \sqrt{\omega_n^2}} \right) \leq 4 \ll 11 \log_K 10 + \frac{17}{6} \log_K 8$$

and regions with $z' = Kz$

For any $s \in S^{n-1}$ Bogomolov for $x = rs$ implies that $\#\{P \in (C \cap \Lambda)_{z,Kz},\ s \in \mathrm{Cap}(P, \theta)\} \leq 3.2 \cdot 10^{11}\, 8^{\frac{17}{3}}$. Integrating over S^{n-1} gives upper bound

$$3.2 \cdot 10^{11} \cdot 8^{\frac{17}{3}} \cdot \frac{\mathrm{Vol}(S^{n-1})}{\mathrm{Vol}(\mathrm{Cap}(P, \theta))}$$

calculate $\mathrm{Vol}(S^{n-1}) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$, $\mathrm{Vol}(\mathrm{Cap}(P, \theta)) > \frac{\pi^{\frac{n-1}{2}} \sin^{n-1}\theta}{\Gamma\frac{n+1}{2}}$

So quotient is $\le \dfrac{\sqrt{2\pi n}^{-1}}{\sin^{n-1}\theta}$

numerator doesn't $K = 2$,

at $\theta = \arccos\sqrt{\frac{1.13}{8}}$, $\sin\theta = \sqrt{1 - \frac{1.13}{8}}$

So get $\sqrt{2\pi n}^{-1} \cdot \left(1 - \frac{1.13}{8}\right)^{-\frac{n-1}{2}} \le\ll \left(1 + \sqrt{\frac{1.13}{8}}\right)^{n-1}$ $O(\frac{1}{\sqrt{n}})$

Large points

large points

let $A(n,\theta)$ = maximal number of points P_i in $\mathbb{R}^n$ s.t. the $\mathrm{con}(P_i, \theta/2)$ do not overlap

Kabatyanskii–Levenshtein gave

$A(n) \leq \frac{\sqrt{2\pi n}}{\sin^{n-1}\frac{\theta}{2}}$ but will need better a $\theta \simeq \theta/2$

Choose large $S \subseteq (C \cap \Lambda)_{\text{large}}$ s.t.
for all $P \in (C \cap \Lambda)_{\text{large}}$ there is a $Q \in S$ s.t. $\angle(P,Q) \leq \theta$ and $|Q| \leq |P|$.
for $Q_1 \neq Q_2 \in S$, $\angle(Q_1, Q_2) > \theta$, so $\#S \leq A(n,\theta)$
Pf: greedy algorithm

Now

$$\mathrm{Ri}(C \cap \Lambda_{\text{large}}) = \bigcup_{Q \in S} \{P \in C \cap \Lambda, P \geq Q, \angle(P,Q) < \alpha\}$$

$$= \bigcup_{Q \in S} \bigcup_{i \geq 1} (P \in C \cap \Lambda, k^{i-1}|Q| \leq |P| \leq k^i|Q| \}_{\ldots})$$

and $\theta = \arccos\left(\sqrt{\frac{1.01}{8}}\right)$ by quantitative Hoeffding? Simplex

to go to $i = \log_{k}(10^5 g^{5/2})$

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Now take $K = 7.15$ so we can apply Mueller

$\angle(P_1, P_2) \geq \arccos\left(\frac{100.1}{8}\right)$

do orthogonal decomposition

$\frac{P_i}{|P_i|} = a_i \frac{Q}{|Q|} + x_i$ with $a_i \in \mathbb{R}$, $x_i \in Q^{\perp}$.

Then $a_1 a_2 + \langle x_1, x_2 \rangle = \langle \frac{P_1}{|P_1|}, \frac{P_2}{|P_2|} \rangle \leq \cos(\angle(P_1, P_2)) \leq \frac{2\ldots}{5}$

for $a_i = \langle \frac{P_i}{|P_i|}, \frac{Q}{|Q|} \rangle = \cos\angle(P_i, Q) \leq \sqrt{\frac{100.1}{8}}$

So $\langle x_1, x_2 \rangle < 0$ and $\angle(x_1, x_2) > \frac{\pi}{2}$

So $\#\{P \in C\Lambda' \Lambda : K^{i-1}|Q| \leq P \leq K^i|Q|, \angle(P, Q) \ldots\} \leq$
$A(n-1, \frac{\pi}{2}) \leq n$ by a result of Rankin ('55)

combining gets

$\tau(C\Lambda\Lambda)_{\log} \leq 3.4 \cdot 10^6 \log 3 \cancel{\left(\frac{1+5}{4\sqrt{5}}\right)^{n-1} A}$

$\cancel{\log 3 + \ldots}$

$(100 + \frac{\pi}{2} \log 8)\, n \cdot A(n, \theta)$

Ilan uses two different sphere packing bounds on $A(n,\theta)$, one for $n \le 141$ which follows from results of Rankin.

$$A(n,\theta) \le \frac{(1+\sqrt{\cos\theta})\, n^{3/2}}{\sqrt{\cos\theta}\,(1-\cos\theta)^{\frac{n-1}{2}}} \quad \text{for } \theta < [illegible]\,\pi$$

for $n \geq 142$ uses an upper bound from ~~Kabatjansk~~ Kabatjanskiĭ - Levenstein

this involves the ODE

$$(1-x^2)\frac{d^2 f}{\{dx\}^2} - (n-1)x\frac{df}{dx} + m(m+n-2)f = 0$$

and ~~its larg~~ the largest root of the polynomial solution

this leads to

$$A(n,\theta) \le 6\binom{m+n-2}{n-2} \quad \text{for } \beta = \arccos\sqrt{\frac{70.7}{8}}$$

$$m = \left\lceil \frac{1-\sin\beta}{2\sin\beta}\, n \right\rceil$$