TALK 6: QUANTITATIVE BOGOMOLOV

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Setup: C genus $g \geq 2$ curve$/K$

$\alpha_0 \in \operatorname{Pic} C$, $(2g-2)\alpha_0 \sim K_C$

Embed C into J via α_0

Have $\hat{h} : J(\bar{K})_{\mathbb{R}} \longrightarrow \mathbb{R}$ and $\langle \cdot, \cdot \rangle$, $|\cdot|$, $L(P,Q)$

Arithmetic intersection: pairing

$(-)\cdot(-) : \widehat{\operatorname{Pic}}(C)^2 \longrightarrow \mathbb{R}$

Previously: studied rational points of large height, those with $\hat{h}(x) \geq c \cdot \bar{\omega}_a^2$, using quantitative Vojta & Mumford inequalities.

Today: points of small height.

Manin-Mumford: $C(\bar{K}) \cap J(\bar{K})_{tors} = \{P \in C(\bar{K}) : \hat{h}(P) = 0\}$ is finite

Bogomolov: $\exists\, \varepsilon > 0$ s.t. $\{x \in C(\bar{K}) : \hat{h}(x) < \varepsilon\}$ finite.

First proved by Ullmo in '98

Uniform version by Dimitrov-Gao-Habegger, Kühne '21

New proof of uniform version by Yuan '24

<u>Lemma</u> (Quadratic Identity): If $P_1, \ldots, P_n \in C(\bar{K})$, then

$$(D) \quad \frac{4(n+g-1)(g-1)}{n-1} \sum_{i<j} |P_i - P_j|^2$$

$$= \frac{4ng(g-1)}{n-1} \sum_{i<j} \bar{\Theta}(P_i)_a \cdot \bar{\Theta}(P_j)_a + \frac{n^2}{2} \bar{\omega}_a^2 + \left|(2g-2)\sum P_i - n\omega\right|^2$$

~~Follows~~
<u>Proof</u>: Use that $\hat{h} = |\cdot|^2$ is a quadratic form & Parts (A) & (B).

Generalized parallelogram law: if $P_1, \ldots, P_n \in C(\bar{K})$ & $x \in J(\bar{K})_{\mathbb{R}}$

$$\sum_{i<j} |P_i - P_j|^2 = n \sum_i |P_i - x|^2 - \sum_i |P_i - nx|^2$$

~~To~~ Proof of Theorem (1): let $P_1, \ldots, P_n \in C(\bar{K})$ distinct s.t. $|P_i - x|^2 \leq t^2 \bar{\omega}_a^2$. Wlog P_i K-rational.
Suppose ~~If~~ $(*)$ fails, then wlog $n \geq g^2$, so

$$\frac{(n+g-1)(g-1)}{n-1} < g \quad (\dagger)$$

8 We now consider

Theorem 4.1(2) in [YYZ]: let $k > 1$, $\theta \in (0, \frac{\pi}{2})$

s.t. $g^{\cancel{6}} \cos^2\theta > \frac{(g+1)^2}{4^k} + \frac{1}{3.2 \cdot 10^{11} g^{11/3}}$

Then $\forall\ 0 \neq x \in J(K)_{\mathbb{R}}$,

$$\#\{P \in C(K) : |x| \leq |P| \leq k|x|,\ \angle(P, x) \leq \theta\}$$
$$\leq 3.2 \cdot 10^{11} g^{17/3}.$$

Lemma: let $k > 1$, $a_1, \ldots, a_n \in [1, 2]$. Then

$$\text{(F)} \quad \sum a_i^2 \leq \frac{(k+1)^2}{4kn} \left(\sum a_i\right)^2$$

Proof: Sum over $(a_i - 1)(a_i - k) \leq 0$ $\square$

Proof of Theorem 4.1(2) Let $P_1, \ldots, P_n$ s.t.

$$|x| \leq |P_i| \leq k|x|, \quad \angle(P_i, x) \leq \theta$$

We claim:

$$\text{(G)}: \quad \left|(2g-2)\sum P_i - nw\right|^2 \geq \frac{(2g-2)^2 \cos^2\theta}{\gamma - \cos^2\theta} \sum_{i<j} |P_i - P_j|^2$$

where $\gamma = \frac{(k+1)^2}{4k}$. This claim follows from (E), (F) & elementary manipulations.