TALK 6 : Quantitative Inequalities

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Some notation:

- K number field
- C/K curve of genus $g \geq 2$
- $\alpha_0 \in \operatorname{Pic} C$ with $(2g-2)\alpha_0 \sim \omega_C$
- Embed C in J using $i_{\alpha_0} : C \hookrightarrow J$
- Get $\hat{h} = \hat{h}_{\Theta_{\alpha_0}}$ on $J(\bar{K})$ & $C(\bar{K})$.
- Renormalization: $\hat{h}_K := [K:\mathbb{Q}]\,\hat{h}$
- Get $\langle x,y\rangle$, $|x|$, $\cos\theta(x,y)$ etc.

Recall admissible metrics

- $\Delta : C \hookrightarrow C \times C$, $\Delta = \Delta(C)$

There's a unique adelic metric on $\mathcal{O}(\Delta)$ satisfying certain properties, denoted $\bar{\mathcal{O}}(\Delta)_a$ "admissible metric"
This gives adelic metrics $\bar{\mathcal{O}}(P)_a$ & $\bar{\omega}_a$ on $\mathcal{O}(P)$ and ω

Also get $G_{a,v} = -\log \| 1_\Delta \|_{\Delta,v,a}$, a function ~~on~~

$$(C \times C \setminus \Delta)_v^{an} \longrightarrow \mathbb{R}$$

We thus get the

- admissible volume $\bar{\omega}_a^2 \in \mathbb{R}$
- "Zhang's global φ-invariant" $\varphi(C) = \sum_v \varepsilon_v \varphi_v(C)$

where

$$\varphi_v(C) = -\int_{(C^2)^{an}} G_{a,v}\, c_1(\bar{\mathcal{O}}(\Delta)_a) \wedge c_1(\bar{\mathcal{O}}(\Delta)_a)$$

These two are related:

Theorem (Wilms$^{+\ldots}$): $\bar{\omega}_a^2 \geq \frac{g-1}{2g+1}\varphi(C) > 0$

Goal: to cover

Theorem A: Let $P_1, P_2 \in C(K)$ distinct s.t.

- $|P_1| \geq 1.2 \cdot 10^g \, g^{7/3} \sqrt{\bar{\omega}_a^2}$
- $|P_2| \geq 10^5 g^{5/2} |P_1|$

Then $\cos\theta(P_1, P_2) \leq \sqrt{\frac{1.01}{g}}$ "Quantitative Vojta"

Theorem B ("Quantitative Mumford") Let $P_1, P_2 \in C(K)$ be distinct with

- $|P_1| \geq 10^g \, g^{7/3} \sqrt{\bar{\omega}_a^2}$
- $|P_1| \leq |P_2| \leq 1.15 |P_1|$

Then $\cos\theta(P_1, P_2) \leq \frac{1.01}{g}$

We first sketch the proof of Theorem B.

Ingredients:

- Theorem 2.7 (2)

$$h_{\bar{\Theta}(\Delta)_a}(P_1, P_2) = \bar{\Theta}(P_1) \cdot \bar{\Theta}(P_2)$$
$$= \frac{1}{g}\left(|P_1|^2 + |P_2|^2\right) - 2\langle P_1, P_2 \rangle$$
$$- \frac{\bar{\omega}_a^2}{4g(g-1)}$$

- "Global sparsity" (Theorem 3.3): lower bound

$$h_{\bar{\Theta}(\Delta)_a}(P_1, P_2) \geq - C_g \cdot \bar{\omega}_a^2$$

where $C_g = 1.2 \cdot 10^4 g^{1/3}\, 2 \log 2 + 3.96 \cdot 10^{10} g^{11/3} \cdot 2$
$\forall P_1 \neq P_2 \in C(\mathbb{R})$

How to deduce Theorem B: get

$$\cos\theta = \frac{\langle P_1, P_2 \rangle}{|P_1||P_2|} \leq \frac{1}{2g}\left(\frac{|P_2|}{|P_1|} + \frac{|P_1|}{|P_2|}\right) + \frac{1}{2}\left(C_g - \frac{1}{4g(g-1)}\right)\frac{\bar{\omega}_a^2}{|P_1||P_2|}$$

Using all inequalities gives the conclusion.

How to prove global sparsity (theorem 3.3)?

$$h_{\overline{\mathcal{O}}(\Delta)_a}(P_1, P_2) = \sum_{v \in M_K} \varepsilon_v \underline{G_{a,v}(P_1, P_2)}$$

bound each one from below using φ_v-invariant

For finite v, using combinatorics of reduction graphs

for infinite v, use ~~hard~~ ~~our~~ new analytic input (Part II)

Proof of Theorem A: define adelic Vojta line bundle

$$\overline{L} = d_1 p_1^* \overline{\omega}_a + d_2 p_2^* \overline{\omega}_a$$
$$+ d\left((2g-2) \overline{\mathcal{O}}(\Delta)_a - p_1^* \overline{\omega}_a - p_2^* \overline{\omega}_a \right)$$

We then similarly get

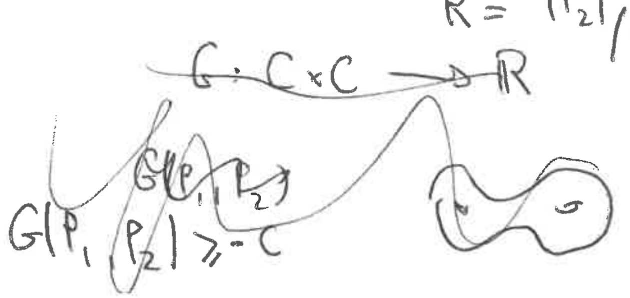
$$h_{\overline{L}}(P_1, P_2) = \frac{2g-2}{g}\left(d_1 |P_1|^2 + d_2 |P_2|^2\right) - (4g-4) d \langle P_1, P_2 \rangle$$
$$+ \frac{d_1 + d_2 - 2gd}{4g(g-1)} \overline{\omega}_a^2$$

Choose

$$\eta = \frac{1}{10^5 g}$$

$$R = |P_2| / |P_1|$$

$$\begin{cases} d_1 \approx \sqrt{g+\eta} \cdot \frac{1}{R} \cdot d \\ d_2 \approx \sqrt{g+\eta} \cdot R \cdot d \end{cases}$$



Then

$$h_L(P_1,P_2) \approx (\cdot\cdot)\Big(\overset{\text{explicit}}{\sqrt{\tfrac{1+g}{g}}} - \cos\theta\Big) + (\cdot\cdot)$$

Key point: need explicit lower bound for $h_L(P_1,P_2)$.
Need existence of "small sections" of L

<u>Proposition</u> If $d_1 \geq gd$, $d \geq d_2$ & $d_1 d_2 > gd^2$, then $\exists\, n \in \mathbb{Z}_{>0}$ & $0 \neq s \in H^0(C\times C, nL)$ s.t.

$$-\sum \varepsilon_v \log \|s\|_{n\bar{L}, v, \sup} \geq -nc\,\overline{\omega}_a^2$$

where $c = g(g+1)d_1 d^2 / 4(g-1)(d_1 d_2 - gd^2)$.

Proof of this:

Let $\bar{L}(c) =$ adelic line bundle $\bar{L}$ where metrics are changed by $-\log\|\cdot\|_{\bar{L}(c)} = -\log\|\cdot\|_{\bar{L}} - \log|c|_v$

~~It suffices to find~~ It suffices to show $\widehat{\mathrm{vol}}(\bar{L}(c)) > 0$.

We use "Siu's inequality": If $\bar{L}_1, \bar{L}_2$ nef on C^2, then
$$\widehat{\mathrm{vol}}(\bar{L}_1 - \bar{L}_2) \geq \bar{L}_1^3 - 3\bar{L}_1^2\cdot\bar{L}_2$$

Apply it to $\bar{L}_1 = ((d_1-d)p_1^*\overline{\omega}_a + d(g-1)p_2^*\overline{\omega}_a + d(2g-2)\bar{\Theta}(\Delta)_a)(c)$
$\bar{L}_2 = (gd-d_2)\,p_2^*\overline{\omega}_a$ gives the result.