K # field, C/K genus $g \geq 2$, $\alpha_0 \in \mathrm{Pic}^0(C)$ s.t. ①

$(2g-2)\alpha_0 \sim K_C$, $j_{\alpha_0} : C \hookrightarrow \mathrm{Jac}(C)$. Have Néron-Tate
$$P \longmapsto [P - \alpha_0]$$

pairing $\langle\,,\,\rangle_{NT}$ and $\hat{h}$: caution - YYZ normalize both by $[K:\mathbb{Q}]$.

<u>Goal</u>

$$\begin{array}{c} \omega = \mathcal{O}(K_C) \\ | \\ C \end{array} \rightsquigarrow h_{\overline{\omega}_a} \qquad \begin{array}{c} \mathcal{O}(\Delta) \\ | \\ C \times C \end{array} \rightsquigarrow h_{\overline{\mathcal{O}(\Delta)}_a}$$

$[K:\mathbb{Q}]$ multiples of Weil heights s.t.

<u>thm</u> <u>2.7</u> ① $\forall Q \in C(K)$,
$$h_{\overline{\omega}_a}(Q) = \frac{2g-2}{g} |Q|^2 + \frac{\overline{\omega}_a^{\,2}}{4g(g-1)}, \quad \text{and}$$

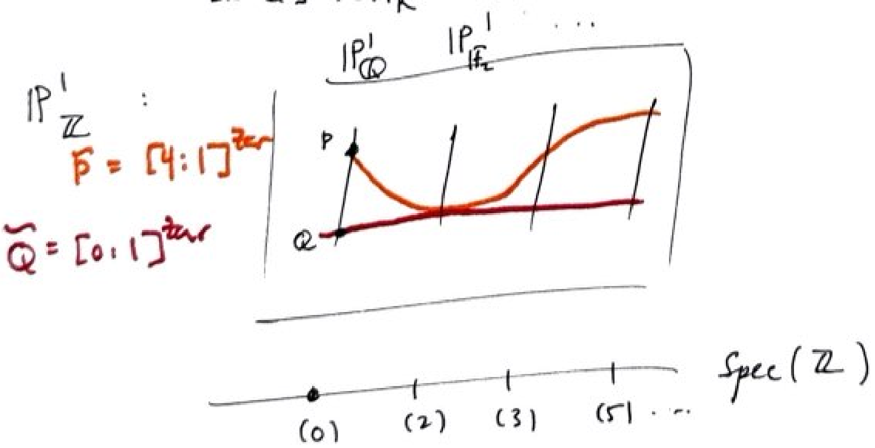
YYZ etc: bounding this using φ-invariant

② $\forall P, Q \in C(K)$,
$$h_{\overline{\mathcal{O}(\Delta)}_a}(P,Q) = \frac{1}{g}\left(|P|^2 + |Q|^2\right) - 2\langle P, Q\rangle - \frac{\overline{\omega}_a^{\,2}}{4g(g-1)}.$$

---

§1 <u>Motivating</u> <u>the</u> theory

On $\mathbb{P}^1$: $h(P) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} \varepsilon_v \log \max\{|x_0(P)|_v, |x_1(P)|_v\}$, $P \in \mathbb{P}^1(K)$.



$\log |x_0(P)|_2 = -\underbrace{2}\log 2$

intersection multiplicity, over (2).

Arakelov: add data at ∞ to get a good pairing $(\cdot,\cdot)_{Ar}$, on an arithmetic surface.

<u>Faltings</u>, <u>Hriljac</u>: If $D, E \in C(K)$ and $\mathcal{C}$ is an integral model of C over $\mathcal{O}_K$. $\exists$ vertically corrected $\hat{D}, \hat{E}$ of D^{zar}, E^{zar} in $\mathcal{C}$ s.t. $(\hat{D} \cdot \hat{E})_{Ar} = -[K:\mathbb{Q}]\langle D, E\rangle_{NT}$.

rmk: Bhicy CL, removes model cc.

§2 Metrized and adelic line bundles.

$\boxed{\text{def}^n}$ X proj var ~~/ # field K~~, L line bundle on X / # field K

$v \in \mathcal{M}_K$. A K_v-metric on L is a collection of norms

$\|\cdot\|_{v,x}$ on L_x resp., cts in the sense that $\forall\ (\mathcal{U}, s)$

rational ~~regular~~ section of L, $x \longmapsto \|s(x)\|_{v,x}$, $\underbrace{\mathcal{U}^{an}_{K_v}}_{\text{analytification}} \to \mathbb{R}$

is cts in the analytic topology.

If v is finite, $(\mathcal{X}, \mathcal{L})$ integral model of $(X, L^{\otimes e})$

over $\mathcal{O}_{K_v}$, the model metric of $(\mathcal{X}, \mathcal{L})$ on L is

$\|s\|_{v,x} = \inf \{ |a|_v^{1/e} : a \in H(x),\ a^{-1} s^{\otimes e} \in \mathcal{L}_{R(x)} \}$

completion of ~~ff(R)~~ residue field of $\mathcal{O}_{X,\mathrm{red}\,x}$ of x; $\mathcal{L}_{R(x)}$ ← valuation ring

<u>example</u> $K_v = \mathbb{Q}_p$, $X = \mathbb{P}^1$, $L = \mathcal{O}(1)$, $\mathcal{X} = \mathbb{P}^1_{\mathbb{Z}_p}$, $\mathcal{L} = \mathcal{O}(1)$, $e = 1$, $s = x_0$, $x = [t : u]$ coprime p-integral. Then $\|s(x)\|_{p,x}$ is the minimal valuation of an element a s.t. $a^{-1}t$ is integral i.e. $|t|_p = p^{-v_p(t)}$.

$\boxed{\text{def}^n}$ (X, L) as above. An <u>adelic metric</u> <u>on L</u> is a collection $(\|\cdot\|_v)_{v \in M_K}$ of K_v-metrics on L which are the model metric for some $(\mathcal{X}, \mathcal{L})$ away from fin. many places.

<u>example</u> $X = \mathbb{P}^n / \mathbb{Q}$, $L = \mathcal{O}(1)$. Define $\|\cdot\|_v$:

$$\|s\|_{v,P} = \frac{\cancel{|x_j(P)|}\ |s(P)|_v}{\max_i |x_i(P)|_v}.$$

This is adelic as it agrees with the model metric for $(\mathbb{P}^n_{\mathbb{Z}}, \mathcal{O}(1))$ at every finite place. n.b. can replace archimedean metrics as we like!

<u>Arakelov</u>: for a curve C of genus $g \geq 1$, there is an algebro-geometrically natural metric on canonical bundle ω. Let $\alpha_1, \dots, \alpha_g$ be a basis for $H^0(C(\mathbb{C}), \omega)$, orthonormal w/r/t $\langle \alpha, \beta \rangle = \frac{i}{2} \int_{C(\mathbb{C})} \alpha \wedge \bar{\beta}$. The <u>Arakelov measure</u> is

$$\mu_{Ar} := \frac{i}{2g} \sum_{j=1}^{g} \alpha_j \wedge \bar{\alpha}_j,$$

and the <u>Arakelov metric</u> is the unique smooth Hermitian metric on ω with

$$c_1(\omega, \|\cdot\|_{Ar}) = (2g-2)\mu_{Ar}.$$

First Chern form / current: if $(L, \|\cdot\|_v)$ is metrized / K_v, $v \in M_K^\infty$, $\exists$ ~~canonical~~ natural closed $(1,1)$-current whose dR class is the algebraic $c_1(L)$: if $s \neq 0$ is a rational section of L,

dual to $(n-1,n-1)$-forms

$$c_1(L, \|\cdot\|_v) = \frac{1}{i\pi} \partial\bar\partial \log \|s\|_v + \delta_{div(s)} .$$

(Poincaré - Lelong - not def^n)

def^n Notions of positivity:

- $v|\infty$: $(L, \|\cdot\|_v)$ is <u>semipositive</u> if $c_1(L|_Y, \|\cdot\|_v)$ is nonneg measure for any analytic curve $Y \subseteq X$.
- if $(\mathcal{X}, \mathcal{L})$ is a model of (X, L^{e}), the "model adelic metric" is <u>nef</u> if archimedean metrics are semipos and $\mathcal{L}$ is relatively nef at finite places → takes e^{th} root of Arakelov metric at ∞.
- ~~a~~ an adelic metric is <u>nef</u> if unif limit of nef model adelic metrics, <u>integrable</u> if a quotient of nef. Call these $\widehat{\mathrm{Pic}}(X)_{int}$.

Key pt: $n = \dim X$, $(L_i, \|\cdot\|_{v,i})_{i=1}^n$ nef adelic metrics $\Rightarrow$ at each place v we have a <u>nonnegative</u> <u>measure</u>:

$$c_1(L_1, \|\cdot\|_{v,1}) \wedge \ldots \wedge c_1(L_n, \|\cdot\|_{v,n}) .$$

"Monge-Ampère"

§3 Intersecting adelic line bundles

Inductive $\widehat{\mathrm{Pic}}(X)_{\mathrm{int}}^{n+1} \to \mathbb{R}$ $\quad (n = \dim X)$ symmetric, multilinear;

$n=0$: $X = \mathrm{Spec}(K)$, $L = \langle s \rangle$, $\widehat{\deg}$ map is arith degree:

$$\widehat{\deg}(\overline{L}) := -\sum_{v \in M_K} \varepsilon_v \log \|s\|_v .$$

$n>0$: s_{n+1} nonzero rational section of L_{n+1} on X. Then

$$\overline{L}_1 \cdot \ldots \cdot \overline{L}_{n+1} = \overline{L}_1|_{\mathrm{div}(s_{n+1})} \cdot \ldots \cdot \overline{L}_n|_{\mathrm{div}(s_{n+1})} - \sum_v \varepsilon_v \int_{X_{K_v}^{an}} \log \|s_{n+1}\|_{n+1,v} \wedge c_1(\overline{L}_1)_v \wedge \ldots \wedge c_n(\overline{L}_n)_v .$$

$\boxed{\text{def}^n}$ The *height function* associated to an adelic line bundle $\overline{L}$ on X is ($\tilde{x}$ the closed pt $\leftrightarrow x$)

$$h_{\overline{L}}(x) := \frac{1}{|\mathrm{Gal}(\overline{K}/K)\tilde{x}|} \overline{L} \cdot \tilde{x} = \frac{1}{|\mathrm{Gal}(\overline{K}/K)\tilde{x}|} \widehat{\deg}(\overline{L}|_{\tilde{x}}).$$

$[K:\mathbb{Q}]^{-1} h_{\overline{L}}(x)$ is a Weil height for L.

$\boxed{\text{def}^n}$ The set of *effective sections* of $\overline{L}$ is

$$\widehat{H}^0(X,\overline{L}) := \left\{ s \in H^0(X,L) \;\middle|\; \sup_{x \in X_{K_v}^{an}} \|s(x)\|_v \le 1 \;\; \forall v \in M_K \right\}$$

n.b. bdd lattice pts so finite.

The *volume* of $\overline{L}$ is

$$\widehat{\mathrm{vol}}(\overline{L}) := \limsup_{k \to \infty} \frac{(n+1)!}{k^{n+1}} \log \# \widehat{H}^0(X,L).$$

Facts

1. $s \in \widehat{H}^0(X,\overline{L}) \Rightarrow h_{\overline{L}}(x) \ge 0 \quad \forall x \in (X \setminus \mathrm{div}(s))(\overline{K})$
2. arithmetic Siu [Yuan]: $\overline{L}_1, \overline{L}_2$ nef adelic line bundles $\Rightarrow$

$$\widehat{\mathrm{vol}}(\overline{L}_1 - \overline{L}_2) \ge \overline{L}_1^{n+1} - (n+1)\overline{L}_1^n \cdot \overline{L}_2 .$$

§4 Heights from admissible metrics

Now $X = C \times C$, $L = \mathcal{O}(\Delta)$. Any K_v-metric $\|\cdot\|_v$ on $\mathcal{O}(\Delta)$ induces:

ⓐ Green function $G_a = -\log \| 1_\Delta \|_v : (C \times C \setminus \Delta)^{an}_{K_v} \to \mathbb{R}$

ⓑ metric of ω on C via

$$\mathcal{O}(-\Delta)|_\Delta \xrightarrow{\sim} I_\Delta / I_\Delta^2 \xrightarrow{\sim} \omega$$

$$f \longmapsto f \bmod I_\Delta^2 \longmapsto g(z,z)\,dz$$

locally, where

$$f(x,y) = g(x,y)(x-y).$$

ⓒ a metric of $\mathcal{O}(x)$ on C for any $x \in C(K')$, K'/K finite, via

$(x, id)^* \mathcal{O}(\Delta) \simeq \mathcal{O}(x)$ $\to$ n.b. $G_{\mathcal{O}(x), a}(y) = G_a(x,y)$.

<u>thm</u> At each place $v \in \mathcal{M}_K$ $\exists!$ "admissible" metric $\|\cdot\|_a$ of $\mathcal{O}(\Delta)$ that satisfies:

① (symmetry) invariance under $\begin{array}{c}(x,y) \mapsto (y,x) \\ C \times C \to C \times C\end{array}$

② (curvature translation-invariance): $\forall x \in C(K')$,

$$c_1(\mathcal{O}(x), \|\cdot\|_a) = \frac{1}{2g-2} c_1(\omega, \|\cdot\|_a) \quad (= \mu_{Ar}, \text{ if } v \mid \infty)$$

③ (zero average): $\forall x \in C(K')$,

$$\int_{C^{an}_{K'}} G_a(x, \cdot)\, \underbrace{c_1(\mathcal{O}(x), \|\cdot\|_a)}_{\text{n.b. } \frac{1}{2g-2} c_1(\overline{\omega}_a)} = 0.$$

$\boxed{\text{def}^n}$ The <u>admissible</u> <u>adelic</u> <u>metric</u> on $\mathcal{O}(\Delta)$ is the admissible metric at each place v (resp. $\overline{\omega}_a$, $\overline{\mathcal{O}(x)}_a$) write $\overline{\mathcal{O}(\Delta)}_a$

integrable

nef

on ω: places of good reduction: model metric, arch: Arakelov

thm 2.4

$$\underset{\overline{\omega}_a\cdot\overline{\omega}_a}{\overline{\omega}_a^2} \geq \frac{g-1}{2g+1}\sum_{v\in M_K}\varepsilon_v\cdot\underbrace{\underbrace{\left(-\int G_{a,v}\, c_1\left(\overline{\mathcal{O}(\Delta)_a}\right)^{\wedge 2}\right)}_{=:\ \varphi_v(C)}}_{=:\ \varphi(C)} > 0$$

$\uparrow$ Zhang, Cinkir.

thm 2.5 $D, E\in \mathrm{Pic}^\circ(C)$. $\rightarrow$ generally, $D = P-\alpha_0$, $(2g-2)\alpha_0 \sim \omega$

Then

$$\overline{\mathcal{O}(D)_a}\cdot\overline{\mathcal{O}(E)_a} = -2\langle D, E\rangle_{NT}$$

$\sim \langle D-\alpha_0, E-\alpha_0\rangle\cdot[K:\mathbb{Q}]$.

$\uparrow$ extend def^n of $\mathcal{O}(x)$ to $\mathrm{Pic}^\circ(C)/\bar{K}$.

thm 2.6 (arithmetic adjunction) $P\in C(K)$.

$$\overline{\mathcal{O}(P)_a}\cdot\overline{\mathcal{O}(P)_a} = -\overline{\omega}_a\cdot\overline{\mathcal{O}(P)_a} = -\overline{\omega}_a\cdot P \quad (= -h_{\overline{\omega}_a}(P))$$

proof by induction formula with $s = \mathbb{1}_P$, (suppressing $\sum_v$)

$$\overline{\mathcal{O}(P)_a}\cdot\overline{\mathcal{O}(P)_a} = \overline{\mathcal{O}(P)_a}\cdot P - \int_{\mathcal{C}} \log\|\mathbb{1}_P\|\, c_1(\mathcal{O}(P)_a)$$

$$= \widehat{\deg}\left(\overline{\mathcal{O}(P)_a}|_P\right) + \frac{1}{2g-2}\int G_a(P,\cdot)\, c_1(\overline{\omega}_a),$$

while

$$\overline{\omega}_a\cdot\overline{\mathcal{O}(P)_a} = \overline{\omega}_a\cdot P - \int\log\|\mathbb{1}_P\|\, c_1(\overline{\omega}_a)$$

$$= \widehat{\deg}\left(\overline{\omega}_a|_P\right) + \int G_a(P,\cdot)\, c_1(\overline{\omega}_a).$$

By defn, $\overline{\mathcal{O}(P)_a}|_P = \overline{\mathcal{O}(\Delta)_a}|_{(P,P)}$ while $\overline{\omega}_a|_P = \overline{\mathcal{O}(-\Delta)_a}|_{(P,P)}$,

so 2.6 follows from zero-average condition. $\square$

thm 2.7 ① $Q\in C(K)$. $h_{\overline{\omega}_a}(Q) = \dfrac{2g-2}{g}|Q|^2 + \dfrac{\overline{\omega}_a^2}{4g(g-1)}$.

② $P_1, P_2\in C(K)$.

$$h_{\overline{\mathcal{O}(\Delta)_a}}(P_1,P_2) = \overline{\mathcal{O}(P_1)_a}\cdot\overline{\mathcal{O}(P_2)_a} = \frac{1}{g}\left(|P_1|^2+|P_2|^2\right) - 2\langle P_1,P_2\rangle - \frac{\overline{\omega}_a^2}{4g(g-1)}.$$

proof also formally, noting Q embeds via $Q-\alpha_0$, $(2g-2)\alpha_0\sim\omega$.

① $$-2|Q|^2 \overset{2.5}{=} \left(\overline{\mathcal{O}(Q)}_a - \frac{\overline{\omega}_a}{2g-2} \right)^2$$

$$= \overline{\mathcal{O}(Q)}_a^{\,2} - \frac{1}{g-1} \overline{\mathcal{O}(Q)}_a \cdot \overline{\omega}_a + \frac{\overline{\omega}_a^{\,2}}{4(g-1)^2}$$

$$\overset{2.6}{=} \left(-1 - \frac{1}{g-1}\right) \overline{\mathcal{O}(Q)}_a \cdot \overline{\omega}_a + \frac{\overline{\omega}_a^{\,2}}{4(g-1)^2}$$

$$= -\frac{g}{g-1} h_{\overline{\omega}_a}(Q) + \frac{\overline{\omega}_a^{\,2}}{4(g-1)^2}, \quad \text{rearrange}$$

② $$\overline{\mathcal{O}(P_1)}_a \cdot \overline{\mathcal{O}(P_2)}_a = \overline{\mathcal{O}(P_1)}_a \cdot P_2 - \oint \sum_{v \in M_K} \varepsilon_v \int \log \| s_2 \|_2 \, c_1(\overline{\mathcal{O}(P_1)})_a$$

$$c_1(\overline{\mathcal{O}(P_1)})_a = c_2(\overline{\mathcal{O}(P_2)})_a$$

$$= \overline{\mathcal{O}(P_1)}_a \cdot P_2$$

$$= \sum_{v \in M_K} \varepsilon_v G_{a,v}(P_1, P_2) = \overline{\mathcal{O}(\Delta)}_a \cdot (P_1, P_2).$$

So $$-2\langle P_1, P_2 \rangle \overset{2.5}{=} \left(\overline{\mathcal{O}(P_1)}_a - \frac{\overline{\omega}_a}{2g-2} \right)\left(\overline{\mathcal{O}(P_2)}_a - \frac{\overline{\omega}_a}{2g-2} \right)$$

$$= \overline{\mathcal{O}(P_1)}_a \cdot \overline{\mathcal{O}(P_2)}_a - \frac{1}{2g-2} \overline{\omega}_a \cdot \left(\overline{\mathcal{O}(P_1)}_a + \overline{\mathcal{O}(P_2)}_a \right) + \frac{\overline{\omega}_a^{\,2}}{4(g-1)^2}$$

$$\overset{2.6}{=} h_{\overline{\mathcal{O}(\Delta)}_a}(P_1, P_2) - \frac{1}{2g-2}\left(h_{\overline{\omega}_a}(P_1) + h_{\overline{\omega}_a}(P_2) \right) + \frac{\overline{\omega}_a^{\,2}}{4(g-1)^2}.$$