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Holly

TALK 5: Adelic line bundles

<u>Setup</u> K # field, C/K curve of genus $g \geq 2$
$\alpha_0 \in \mathrm{Pic}\, C$ s.t. $(2g-2)\alpha_0 \sim \omega_C$, $j = j_{\alpha_0} : C \hookrightarrow \mathrm{Jac}(C)$
Get NT pairing $\langle \cdot , \cdot \rangle_{NT}$ & $\hat{h}$ &

⚠ YYZ normalizes by $[K:\mathbb{Q}]$

<u>Goal</u>: construct Weil heights for $\mathcal{O}(\Delta)$ on $C \times C$ & ω on C such that

<u>Theorem (2.7 of YYZ)</u>

① $$h_{\overline{\omega}_a}(Q) = \frac{2g-2}{g} |Q|^2 + \frac{\overline{\omega}_a^2}{4g(g-1)} \quad \forall Q \in C(K)$$

② $$h_{\overline{\mathcal{O}(\Delta)}_a}(P,Q) = \frac{1}{g}\left(|P|^2 + |Q|^2\right) - 2\langle P,Q \rangle - \frac{\overline{\omega}_a^2}{4g(g-1)}$$
$\forall P,Q \in C(K)$

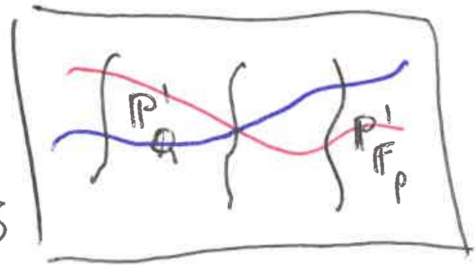
§1. Motivation

On $\mathbb{P}^1_K$, $$h(P) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} \varepsilon_v \log \max\left(|x_0(P)|_v, |x_1(P)|_v\right)$$

Geometric interpretation for at finite v:

✓ 1

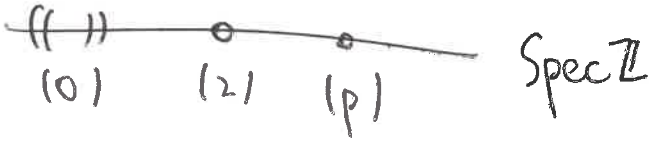
Consider $\mathbb{P}^1_{\mathbb{Z}}$:



Say $P, Q \in \mathbb{P}^1(\mathbb{Q})$

$\tilde{P}, \tilde{Q} \in \mathbb{P}^1(\mathbb{Z})$: Zariski closures

$i_v(P,Q) =$ v-adic intersection of $\tilde{P}, \tilde{Q}$



Taking $Q = \infty$ recovers

$$\max(|x_0(P)|_v, |x_1(P)|_v)$$

Arakelov's idea: add info at infinite places, get Arakelov intersection pairing $\langle \cdot , \cdot \rangle_{Ar}$

It has the property:

<u>Faltings - Hriljac</u>: If $D, E \in C(K)$, $g \geq 2$, $\mathcal{C}$ regular model of $C/\mathcal{O}_K$, then $\exists\ \tilde{D}, \tilde{E} \in$ ~~Pic~~ $\mathrm{Div}(\mathcal{C})_{Ar}$ with

$$(\tilde{D}, \tilde{E})_{Ar} = -[K:\mathbb{Q}]\langle D, E\rangle_{NT}$$

<u>Zhang</u>: ~~uses~~ adelic line bundles give good intersection theory without depending on models.

§2. Metrized / adelic line bundles

<u>Def</u>^{n}: (X, L) proj var + line bundle $/K$

If $v \in M_K$, a K_v-metric on L is a collection of norms $\|\cdot\|_{x,v}$ on L_x, varying "continuously" in x: we require for all (U, s), $U \subset X$ Zo open, $s \in H^0(U, L|_U)$, then

$$x \longmapsto \|s(x)\|_{x,v}$$

extends to a cts map $U^{an}_{K_v} \longrightarrow \mathbb{R}$, where $U^{an}_{K_v}$ is the "analytification" of U.

(In the sense of Berkovich for v finite)

If v is finite & $(\mathcal{X}, \mathcal{L}) / \mathcal{O}_{K_v}$ is an integral model of $(X, L^{\otimes e})$ for some $e \geq 1$, the associated "model metric" is

$$\|s\|_{x,v} = \inf \left\{ |a|_v^{1/e} : \begin{array}{l} a \in H(x) \\ \text{s.t. } a^{-1} s^{\otimes e} \in \mathcal{L}_{\cancel{\mathcal{X}_v}} \mathcal{O}_{\mathcal{X},x} \end{array} \right\}$$

So H^0 ~~$(\mathcal{X}) \subset H^0$~~ integral sections have norm ≤ 1.

Example: $K=\mathbb{Q}$, ~~&~~ $\mathcal{X}=\mathbb{P}^1$, $\mathcal{L}=\mathcal{O}(1)$, $e=1$, $s=x_0$

Then

$$\|s\|_{[t:u]} = |t|_p \qquad (t, u \text{ } p\text{-integral coprime})$$

Def'n: An adelic metric on (X, L) is a collection ~~$(\|\cdot\|_{x,v})$~~ of metrics $(\|\cdot\|_v)_v$ such that ~~$\|\cdot\|_v$~~ ~~=~~ $\exists$ model $(\mathcal{X}, \mathcal{L})$ over $\mathcal{O}[S^{-1}]$ & that induces $\|\cdot\|_v$ for all but finitely many v.

Example: $X=\mathbb{P}^n_{\mathbb{Q}}$, $L=\mathcal{O}(1)$, define $(\|\cdot\|_v)_{v\in M_K}$ by

$$\|s\|_{P,v} = \frac{|s(P)|_v}{\max_i |x_i(P)|}.$$

This is adelic. ~~cand~~

Arakelov, C, $g \geq 1$ over $\mathbb{C}$. Let $\alpha_1, \ldots, \alpha_g$ be a basis for $H^0(C, \omega_C)$, orthonormal wrt

$$\langle \alpha, \beta \rangle = \frac{i}{2} \int \alpha \wedge \overline{\beta}$$

Then we can define the Arakelov measure on $C(\mathbb{C})$

$$\mu_{Ar} = \frac{i}{2g} \sum_{i=1}^{g} \alpha_i \wedge \overline{\alpha}_i$$

The metric on ω is the unique one satisfying

$$c_1(\omega, \|\cdot\|_{Ar}) = (2g-2)\,\mu_{Ar}$$

roughly $\frac{1}{\pi i}\partial\bar{\partial}\log\|s\|_v$ on local charts

Globally, have Poincaré-Lelong formula

$$c_1(\omega, \|\cdot\|_{Ar}) = \frac{1}{\pi i}\partial\bar{\partial}\log\|s\|_v + \delta_{div(s)}$$

if s is a rational section.

A "model arithmetic metric" on (C, ω_C) is a model metric at finite places and $\|\cdot\|_{Ar}$ at infinite places.

There are various notions of positivity for ~~measure~~ metrics on L.

① If $v \mid \infty$, $(L, \|\cdot\|_v)$ semipositive if $c_1(L|_Y, \|\cdot\|_v)$ is nonnegative for all $Y \subset X$.

② If $(\mathcal{C}, \mathcal{L})$ model for $(C, L^{\otimes e})$, the model adelic metric is nef if it is semipositive for $v \mid \infty$ & $\mathcal{L}$ relatively nef for $v \nmid \infty$.

③ An adelic metric is nef if it is a "uniform limit" of nef model adelic metrics. It is integrable if it is a quotient of such.

④ Given $(L_i, \|\circ\|_i)_{1\leq i\leq n}$, $n = \dim X$ nef adelic metrics, then $\forall\, v \in M_K$, $\exists$ measure

$$c_1(L_1, \|\circ\|_1) \wedge \cdots \wedge c_1(L_n, \|\circ\|_n)$$

on $X_{K_v}^{an}$ ("Monge-Ampère measure")

Let

$\widehat{\mathrm{Pic}}(X)_{int}$ = integrable adelic line bundles.

If $n = \dim X$, there's a multilinear pairing

$$(\widehat{\mathrm{Pic}}\, X_{int})^{n+1} \longrightarrow \mathbb{R}$$

via inductive procedure.

If $\underline{n=0}$: $X = \operatorname{Spec} K$, $L = \langle s \rangle$. Then this is

$$\widehat{\deg}\, \overline{L} = -\sum_v \varepsilon_v \log \|s\|_v$$

If $n>0$, choose a rational section s_{n+1} of $\overline{L}_{n+1}$ and define

defined by induction

$$\overline{L}_1 \cdots \overline{L}_{n+1} := \overline{L}_1|_{\operatorname{div}(s_{n+1})} \cdots \overline{L}_n|_{\operatorname{div}(s_{n+1})} - \sum_{v \in M_K} \varepsilon_v \int_{X^{an}_{K_v}} \log \|s_{n+1}\|_{n+1}\, c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_n)$$

Def: The height associated to $\overline{L}$ for $x \in X(\overline{K})$ is

$$h_{\overline{L}}(x) = \frac{1}{[K(x):K]} \widehat{\deg}\,(x^* \overline{L})$$

Fact $[K:\mathbb{Q}]^{-1} h_{\overline{L}}(x)$ is a Weil height for L

Def^n

$$\hat{H}^0(X, \overline{L}) = \{ s \in H^0(X, L) : \sup_{x \in X^{an}_{K_v}} \|s(x)\| \leq 1 \;\; \forall v \in M_K \}$$

$$\widehat{\mathrm{Vol}}(\bar{L}) = \limsup_{k\to\infty} \frac{(n+1)!}{k^{n+1}} \cdot \# \hat{H}^0(X, \bar{L}^{\otimes k})$$

The point: if $0 \neq s \in \hat{H}^0(X, \bar{L})$,

$$h_{\bar{L}}(x) \geq 0 \quad \forall x \in (X \setminus \mathrm{div}\, s)(\bar{K})$$

~~Mo~~ Moreover, there's the arithmetic Siu inequality due to X. Yuan: if $\bar{L}_1, \bar{L}_2$ nef,

$$\widehat{\mathrm{Vol}}(\bar{L}_1 - \bar{L}_2) \geq \bar{L}_1^{n+1} - (n+1)\bar{L}_1^n \cdot L_2$$

§4. Heights ~~fr~~ from admissible metrics

A canonical metric on $\mathcal{O}(\Delta)$ on $C \times C$.

Let $X = C \times C$, $L \overset{1_\Delta \in}{=} \mathcal{O}(\Delta)$. Then any K_v-metric of $\mathcal{O}(\Delta)$ induses:

ⓐ $G_a = -\log \|1_\Delta\|_v$, $G_a : (C \times C \setminus \Delta)^{an}_{K_v} \to \mathbb{R}$

ⓑ A metric of ω on C, via the adjunction iso

$$\mathcal{O}(-\Delta)|_\Delta \xrightarrow{\sim} \omega$$
$$f \longmapsto g(x,x)dx \quad \text{where } f = g(x,y)(x-y)$$

(c) For any $x \in C(K')$, get a metric of $\mathcal{O}(x)$ on C, via $(x, \mathrm{Id}) : C \to C \times C$

$\hookrightarrow$ $G_{\mathcal{O}(x)}(y) = G_a(x, y)$

<u>Definition / Theorem</u>: there exists a unique metric on $\mathcal{O}(\Delta)$, the "admissible metric" $\|\cdot\|_a$, such that

(1) It is symmetric under $(x,y) \mapsto (y,x)$

(2) $\forall x \in C(K')$, $c_1(\mathcal{O}(x), \|\cdot\|_a) = \frac{1}{2g-2} c_1(\omega, \|\cdot\|_a)$

(3) $\forall x \in C(k')$:

$$\int G_a(x, \cdot)\, \mu_{Ar} = 0$$

This is the one whose height functions we'll be using. So get $\overline{\mathcal{O}}(\Delta)_a$, $\overline{\mathcal{O}}(-\Delta)_a$, $\overline{\omega}_a$ on C, $\overline{\mathcal{O}}(P)_a$

<u>Theorem</u>: $\overline{\omega}_a^2 \geq \frac{g-1}{2g+1} \sum_{v \in M_K} \varepsilon_v \underbrace{\left(-\int G_{a,v}\, c_1(\overline{\mathcal{O}}(\Delta)_a)^{n-?}\right)}_{\substack{\shortparallel \\ \varphi_v(C)}}$ local Zhang's invariant

Theorem 2.5 If $D, E \in \mathrm{Pic}^{\circ} C$,

$$\overline{\mathcal{O}(D)}_a \cdot \overline{\mathcal{O}(E)}_a = -2\langle D, E\rangle_{NT}$$

Theorem 2.6 [Arithmetic adjunction]: If $P \in C(K)$,

$$\overline{\mathcal{O}(P)}_a \cdot \overline{\mathcal{O}(P)}_a = -\overline{\omega}_a \cdot \overline{\mathcal{O}(P)}_a$$
$$= h_{\overline{\omega}_a}(P)$$