TALK 4: VOJTA'S INEQUALITY

21 MAY
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§ Warm-up

Theorem (Thue 1909) $b>1$ squarefree, $\beta = b^{1/3}$, $c>0$. Then

$$(*) \quad \{ p/q \in \mathbb{Q} : |\tfrac{p}{q} - \beta| \leq c/q^3 \} \text{ is finite.}$$

Sketch of proof: let $H(P_q) = \max |p_i|$ $(p_i \in \mathbb{Z})$
$h(P) = \log H(P)$.

Claim 1 if $m+1 > \frac{2}{3} n \geq m > 3$ integer, then $\exists F(x,y) = P(x) + yQ(x)$ w/ $\deg_x F \leq m+n$ such that $\partial_k F = \frac{1}{k!}\frac{\partial^k}{\partial x^k} F$ vanishes at (β, β) for all $0 \leq k \leq n$ and $h(F) \leq c_1 (m+n)$

constant that only depends on b

This is proved using Siegel's lemma: let $N > M$ and take $A: \mathbb{Z}^N \to \mathbb{Z}^M$, then $\exists x \in \mathbb{Z}^N - 0$, $Ax = 0$ with

$$h(x) \leq \frac{M}{N-M} (\log N + h(A)).$$

(Follows from Minkowski's theorem applied to ker A)

How does it imply the claim?

Have $3n$ equations in $2(m+n+1)$ variables w/ coeffs bounded by $(4b)^{n+m}$

<u>Claim 2</u> $\exists\, c_2$ s.t. if $p_1/q_1, p_2/q_2 \in \mathbb{Q}$, then $\exists\, 0 \leq t \leq 1 + c_2 n / \log q_1$ such that

$$\partial_t F(p_1/q_1, p_2/q_2) \neq 0 \qquad (\text{"Roth's lemma"})$$

Idea: one-variable case: $P(X) \in \mathbb{Z}[X]$. If $\partial_t P(p/q) = 0$ for $t = 0, \dots, r$, then $(qX - p)^r \mid P$.

$$\Rightarrow r \leq h(P)/h(p/q)$$

General case: use induction on P. ✓

Conclusion of proof: Let $p_1/q_1, p_2/q_2$ be "good" rational approximations to β.

Then Claim 2 gives t s.t. $\partial_t F(p_1/q_1, p_2/q_2) \neq 0$, so it has absolute value $\underset{①}{\geq} \dfrac{1}{q_1^{m+n} q_2}$

Expand using Taylor series $\Rightarrow |\partial_t F(p_1/q_1, p_2/q_2)|$

$$\underset{②}{\leq} c_3^n \left(\left(\frac{c}{q_1^3}\right)^{n-t} + \left(\frac{c}{q_2^3}\right) \right)$$

If our set $(*)$ is infinite, it's possible to choose p_i, q_i, m, n so these inequalities ① & ② contradict each other

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§ Today

$C/\mathbb{Q}$ curve of genus $g \geq 2$

$J = \mathrm{Jac}(C)$

Assume $\exists\, P_0 \in C(\mathbb{Q})$ s.t. $(2g-2)P_0$ is canonical

Then $j : C \hookrightarrow J$, $P \mapsto [P - P_0]$

$$\Theta = \underbrace{C + \cdots + C}_{g-1 \text{ times}} \quad \text{is symmetric}$$

<u>Theorem (Vojta)</u>: $\exists\, \kappa_1, \kappa_2 > 0$ s.t. if $z, w \in C(\overline{\mathbb{Q}})$

$|z| = \hat{h}_\Theta(z)^{1/2} > \kappa_1$, $|w| > \kappa_2 |z|$, then

$$\frac{\langle z, w\rangle}{|z||w|} \leq 3/4 \quad (\text{so angle is "big"})$$

Basic idea: consider the divisor

$$\Omega = \Omega(d_1, d_2, d) = (d_1 - d)\,(P_0 \times C) + (d_2 - d)(C \times P_0) + d\Delta$$

on $C \times C$. Then a calculation shows

$$\circledast \quad h_\Omega(z, w) = \frac{d_1 |z|^2}{g} + \frac{d_2 |w|^2}{g} - 2d\langle z, w\rangle + O(1)$$

Consider the quadratic form

$$\frac{d_1}{g} x^2 + \frac{d_2}{g} y^2 - 2dxy$$

it fails to be positive definite when $d_1 d_2 < g^2 d^2$.

Case of Mumford gap principle: $d_1 = d_2 = d = 1$, $\Omega = \Delta$ & $h_\Omega \geq 0$ when $z \neq w$.

Vojta's inequality: choose

$d_1 \sim \sqrt{g}\, |w|^2$

$d_2 \sim \sqrt{g}\, |z|^2$

$d \sim |z|\,|w|$

Rearranging (*):

$$\frac{\langle z, w \rangle}{|z|\,|w|} \lessapprox \frac{1}{\sqrt{g}} + O\left(\frac{1}{|z|^2} \frac{1}{|w|^2}\right) - \frac{1}{2} \underbrace{h_\Omega(z,w)}_{\text{Should be } \geq O(1)} \quad (**)$$

If $d_1 d_2 > g\, d^2$, then RR shows $h^0(\Omega) > 0$. $\Omega \sim V$, V eff.

Would like to plug this in (**) & use $h_\Omega(z,w) \geq 0$.

But many difficulties!

<u>Problems</u>: (a) In (*), the $O(1)$ depends on d, d_1, d_2

(b) V depends on d_1, d_2, d

(c) Might have (z,w) on V

Solutions: (a) Need to explicitly normalize h_Ω.

Choose $N \geq 1$ s.t. NP_0 very ample, w/ embedding

$$\phi : C \hookrightarrow \mathbb{P}^n$$

Choose M such that $(M+1)(P_0 \times C + C \times P_0) - \Delta$ is very ample on $C \times C$, corr. to

$$\psi : C \times C \hookrightarrow \mathbb{P}^m$$

Given d, d_1, d_2, let

$$h_\Omega(z,w) = \delta_1 h_\phi(z) + \delta_2 h_\phi(w) - d\, h_\psi(z,w)$$

where $\delta_1 = (d_1 + Md)/N$, $\delta_2 = (d_2 + Md)/N$ are assumed integers. ~~So we fixed (a)!~~

Let $x_0, \ldots, x_n \in H^0(\mathbb{P}^n, \mathcal{O}(1))$

$y_1, \ldots, y_m \in H^0(\mathbb{P}^m, \mathcal{O}(1))$

& write ~~their restrictions~~ to C & $C \times C$ by same letters

$$x_i = \cancel{p_2^*}\, p_1^*(\mathrm{res}|_C), \quad x_i' = p_2^*(\mathrm{res}|_C)$$

$$y_i = \mathrm{res}|_{C \times C}$$

lift all sections on $C \times C$.

If δ_1, d_2, d are large enough, then

$$H^0(C\times C, \mathcal{O}(\delta_1, d_2))$$

is spanned by monomials in x_i, x'_j

Theorem: if $s \in H^0(C\times C, \mathcal{O}(\Omega))$ (recall $\mathcal{O}(\Omega) = \mathcal{O}(d_1, d_2)\otimes\mathcal{O}(d)^{-1}$) then $\forall j$, $y_j^d s \in H^0(C\times C, \mathcal{O}(\delta_1, \delta_2))$, so can rewrite

$$s = F_j(x, x') / y_j^d \Big|_{C\times C}, \quad \underset{\deg = (\delta_1, \delta_2)}{F_j \in \mathbb{Q}[\{x_i\}, \{x'_j\}]}$$

and $F_i / y_i^d = F_j / y_j^d$ on $C\times C$.

Conversely, given $\{F_i\}_{i=0,\dots,m}$ s.t. $F_i / y_i |_{C\times C}$ indep. of i, then they give a section s on $H^0(C\times C, \mathcal{O}(\Omega))$

Proposition:

$$h_\Omega(z, w) \leq \frac{d_1 |z|^2}{g} + \frac{d_2 |w|^2}{g} - 2d\langle z, w\rangle + \underset{\text{indep. of } d_i, d}{c_1} (d_1 + d_2 + d)$$

$\rightsquigarrow$ Problem (a) solved

(b) Suppose $\underset{\in H^0(\mathcal{O}(\Omega))}{s} \longleftrightarrow \underset{\overset{\shortparallel}{F}}{(F_i)}$, $s \neq 0$, $\mathrm{div}(s) = V \geq 0$

Then if $s(z,w) \neq 0$,

$$h_\Omega(z,w) \geq -h(F) - n\log((\delta_1+n)(\delta_2+n))$$

(Proof elementary).

To control $h(F)$, use Siegel's lemma!

<u>Prop</u>: Fix $\gamma > 0$ & suppose $d_1 d_2 - g d^2 \geq \gamma d_1 d_2$

Then $\exists\, s \in H^0(\mathcal{O}(\Omega)) - 0$ s.t. $h(F) \leq c_2\left(\frac{d_1+d_2}{\gamma}\right) + o(d_1+d_2)$

(Proof technical ~~...~~ but ultimately uses Siegel)

(c) Suppose $(z,w) \in V$. Let $\partial_t = \frac{1}{t!}\partial^t$ where ∂ is a derivation at z. Similarly define ∂'_t at w.

Then we define the <u>index</u> of s at (z,w)

$$\mathrm{Ind}_{(z,w)}(s) = \min\left\{ \frac{i_1}{\delta_1} + \frac{i_2}{\delta_2} \;\middle|\; \begin{array}{l} i_1, i_2 \geq 0 \\ \partial_{i_1}\partial'_{i_2}(s)(z,w) \neq 0 \end{array} \right\}$$

Say (i_1, i_2) is admissible ~~for~~ if it achieves the minimi

<u>Proposition</u>: $\exists$ a finite subset $Z \subset C(\overline{\mathbb{Q}})$ s.t. if $z, w \notin Z$, then

$$h_\Omega(z,w) \geq -h(F) - c_3\left(i_1|z|^2 + i_2|w|^2\right) - c_4\left(i_1 + i_2 + d_1 + d_2 + 1\right)$$

<u>Proposition</u>: Let $0 < \varepsilon, \gamma < 1$ and suppose

$$\begin{cases} \varepsilon^2 d_1 \geq d_2 \\ \min\left(d_2|w|^2, d_1|z|^2\right) \geq \dfrac{c_5}{\gamma\varepsilon^2} d_1 \\ d_1 d_2 - g d^2 \geq \gamma d_1 d_2 \end{cases}$$

Then $\exists\, (i_1, i_2)$ admissible for $s, (z,w)$ s.t.

$$\frac{i_1}{d_1} + \frac{i_2}{d_2} \leq 12N\varepsilon$$

$\leadsto$ Enough to prove Vojta