The Mumford gap principle

Recall: K number field

Given X/K smooth proj. variety, $\mathcal{L} \in \mathrm{Pic}(X)$, define height $h_{\mathcal{L}}$ s.t.:

① if $\mathcal{L} \sim \mathcal{L}'$, then $h_{\mathcal{L}} = h_{\mathcal{L}'} + O(1)$

② $h_{\mathcal{L}\otimes\mathcal{L}'} = h_{\mathcal{L}} + h_{\mathcal{L}'} + O(1)$

③ $D \in \mathrm{Div}(X)$ effective, B base locus of $|D| \Rightarrow \exists c \in \mathbb{R} : h_{\boxed{D}}(P) \geq c \quad \forall P \in (X \setminus B)(\overline{K})$

④ $h_{f^*\mathcal{L}} = h_{\mathcal{L}} \circ f + O(1)$

If A/K abelian variety, $\mathcal{L} \in \mathrm{Pic}(A)$:

→ if $\mathcal{L}$ symmetric, get $\hat{h}_{\mathcal{L}}$ canonical height s.t. $\hat{h}_{\mathcal{L}}$ is a quadratic form & ⑤

⑥ $\hat{h}_{A,\mathcal{L}} = h_{\mathcal{L}} + O(1)$

→ if $\mathcal{L}$ antisymmetric, get $\hat{h}_{\mathcal{L}}$ w/

⑦ $\hat{h}_{\mathcal{L}}$ linear

→ in general: ⑧ $\hat{h}_{\mathcal{L}} := \frac{1}{2}\left(\hat{h}_{\mathcal{L}+[-1]^*\mathcal{L}} + \hat{h}_{\mathcal{L}-[-1]^*\mathcal{L}}\right)$ also satisfying ⑥

Prop: $\mathcal{L}' \in \mathrm{Pic}^0(A)$ antisymmetric, $\mathcal{L} \in \mathrm{Pic}(A)$ symmetric, ample

$\Rightarrow \hat{h}_{\mathcal{L}'} = O\left(\hat{h}_{\mathcal{L}}^{1/2}\right)$ ⑨

§ Theta divisors

$C/_K$ curve of genus $g \geq 2$, $P_0 \in C(K)$ $\rightsquigarrow$ $j: C \longrightarrow J := \mathrm{Jac}(C)$
$P \longmapsto [P - P_0]$

Let $\Theta = \underbrace{j(C) + \cdots + j(C)}_{g-1}$, a subvariety of codimension 1.

$\longrightarrow$ Θ is an irreducible ample divisor, called the <u>theta divisor</u>

Prop:

(i) Let $\Theta^- = [-1]^* \Theta$. Then

(10) $$j^* \Theta^- \sim g \cdot [P_0] \quad \text{in} \quad \mathrm{Div}(C)$$

(ii) Define maps $s, p_1, p_2 : J \times J \to J$ by $\begin{cases} s(P,Q) = P+Q \\ p_1(P,Q) = P \\ p_2(P,Q) = Q \end{cases}$

$\Delta \subset C \times C$ diagonal

Then

(11) $$(j \times j)^* \big(\underbrace{s^*\Theta - p_1^*\Theta - p_2^*\Theta}_{\delta} \big) \sim -\Delta + (C \times \{P_0\}) + (\{P_0\} \times C)$$

in $\mathrm{Div}(C \times C)$

We will consider the height $\hat{h}_{\Theta + \Theta^-}$ on J (and C, by restriction)

Get a bilinear pairing by (5):

$$\langle P, Q \rangle = \frac{1}{2} \big(\hat{h}_{\Theta+\Theta^-}(P+Q) - \hat{h}_{\Theta+\Theta^-}(P) - \hat{h}_{\Theta+\Theta^-}(Q) \big)$$

$$|P| = \langle P, P \rangle^{1/2}, \quad \cos \theta_{P,Q} = \frac{\langle P, Q \rangle}{|P|\,|Q|}$$

Thm (Mumford's gap principle): Fix $\varepsilon > 0$. $\exists B = B_{C,\varepsilon} > 0$ s.t. for all $P, Q \in C(\overline{K})$ satisfying:

- $P \neq Q$; • $|P| \geq |Q| > B$; • $\cos\theta_{P,Q} \geq \frac{3}{4} + \varepsilon$

then $|P| \geq 2|Q|$.

§ Mumford's formula

Lemma: Let $P, Q \in C(\overline{K})$. Then

$$h_{C\times C,\Delta}(P,Q) = \frac{1}{2g}|P|^2 + \frac{1}{2g}|Q|^2 - \langle P, Q\rangle + O(|P|+|Q|+1)$$

Proof:

By (11) $\Delta \sim -(j\times j)^*(\delta) + (C\times\{P_0\}) + (\{P_0\}\times C)$.

$$\Rightarrow h_{C\times C,\Delta}(P,Q) \overset{(1)+(2)}{=} \underbrace{-h_{(j\times j)^*(\delta)}(P,Q)}_{A} + \underbrace{h_{C\times\{P_0\}}(P,Q)}_{B} + \underbrace{h_{\{P_0\}\times C}(P,Q)}_{C} + O(1)$$

(A) $h_{(j\times j)^*\delta}(P,Q) \overset{(4)}{=} h_\delta(j(P), j(Q)) + O(1)$

$$\overset{(2)+(4)+(6)}{=} \hat{h}_{\Theta}(\overbrace{\delta(P,Q)}^{P+Q}) - \hat{h}_{\Theta}(P) - \hat{h}_{\Theta}(Q) + O(1)$$

$$\overset{(8)}{=} \frac{1}{2}\Big(\hat{h}_{\Theta+\Theta^-}(P+Q) - \hat{h}_{\Theta+\Theta^-}(P) - \hat{h}_{\Theta+\Theta^-}(Q)$$

$$+ \cancel{\text{odd}}(\dots)\Big) \quad {=0 \text{ by } (7)}$$

$$= \langle P, Q\rangle$$

(B) $h_{C\times\{P_0\}}(P,Q) = h_{P_2^\alpha(P_0)}(P,Q) \overset{④}{=} h_{[P_0]}(Q)+O(1)$

recall: (10) $g[P_0]\sim j^\alpha \Theta^-$

so $g\, h_{[P_0]}(Q) \overset{⑦+②+⑥}{=} \hat h_{\Theta^-}(Q)+O(1)$

$$\overset{⑧}{=} \frac12\left[\hat h_{\Theta+\Theta^-}(Q)+\hat h_{\Theta^- - \Theta}(Q)\right] + O(1)$$

$$\overset{⑨}{=} \frac12|Q|^2 + O(|Q|+1)$$

Conclusion: $h_{C\times\{P_0\}}(P,Q) = \frac{1}{2g}|Q|^2+O(|Q|+1)$

& $h_{\{P_0\}\times C}(P,Q)=\frac{1}{2g}|P|^2+O(|P|+1)$ □

Proof of gap principle:

$P\neq Q \Rightarrow$ by (3) assume $h_\Delta(P,Q)\geq 0$ &

$$\Rightarrow \frac{1}{2g}|P|^2+\frac{1}{2g}|Q|^2+O(|P|) \geq \langle P,Q\rangle \geq \left(\frac34+\varepsilon\right)|P||Q|$$

$$\Rightarrow \frac{1}{2g}\left(\frac{|P|}{|Q|}+\frac{|Q|}{|P|}\right)+O\left(\frac{1}{|Q|}\right)\geq \frac34+\varepsilon$$

choose B large enough

$$\Rightarrow \frac{1}{2g}\left(\frac{|P|}{|Q|}+1\right)\geq\frac34 \Rightarrow \frac{|P|}{|Q|}\geq\frac{3g}{2}-1 \overset{g\geq 2}{\geq} 3-1=2$$

□

Cor: C/K curve of genus $g \geq 2$. $\exists c > 0$ s.t

$$\#\{P \in C(K) \mid \hat{h}(j(P)) \leq \log T\} \leq c \log\log T \quad \forall T \gg 0$$

or $H_D(j(P)) \leq T$ for *any* Weil height H_D

D ample, $\deg > 0$.

Use general lemma:

Lemma: V finite dim'l real v.s., $\|\cdot\|$ Euclidean norm, $\Lambda \leq V$ lattice, $S \subseteq \Lambda$ subset s.t $\exists a, b > 0$:

$$\|x-y\|^2 \geq a(\|x\|^2 + \|y\|^2) - b \quad \forall x, y \in S, x \neq y.$$

Then, $\exists c > 0$ s.t $\#\{x \in S \mid \|x\| \leq T\} \leq c \log(T) \quad \forall T \gg 0$.

Lemma $\Rightarrow$ Cor:

$$\langle P, Q \rangle \leq \frac{1}{2g}(|P|^2 + |Q|^2) + O(|P|)$$

alt: $\underset{\frac{|P|^2+|Q|^2-|P-Q|^2}{2}}{\langle P, Q \rangle} \leq \frac{1+\varepsilon}{2g}(|P|^2 + |Q|^2) + c_1$, $\varepsilon > 0$, $c_1 \in \mathbb{R}$.

$$\Rightarrow \underset{\varepsilon = \frac{1}{2}}{\cdots} \Rightarrow |P-Q|^2 \geq \frac{1}{4}(|P|^2 + |Q|^2) - \underset{0}{\underset{\vee}{c_2}}$$

Note: $\ker(J(K) \to J(K) \otimes \mathbb{R}) = J(K)_{tors}$, finite.

pf lemma (sketch):

$$S(u,v) = \{x \in S \mid u \le \|x\| \le v\}$$

choose $u \ge \alpha = \sqrt{\frac{b}{a}}$ $\Rightarrow$ $\underbrace{B_x(\tfrac{1}{2}u\sqrt{a})}_{\cap} \cap \underbrace{B_y(\tfrac{1}{2}u\sqrt{a})}_{\cap} = \emptyset$

$x \neq y \in S$

$B_0(v + \tfrac{1}{2}u\sqrt{a})$

contains disjoint union $\left(B_x(\tfrac{1}{2}u\sqrt{a})\right)_{x \in S(u,v)}$

$\Rightarrow \overset{\text{trust me}}{---} \Rightarrow \# S(u,v) \le \left(\frac{2v}{u\sqrt{a}} + 1\right)^n$ $n = \dim V$.

$$\# S(0,T) = \# S(0,\alpha) + \sum_{0 \le i \le \log(T/\alpha)} \# S(\alpha e^i, \alpha e^{i+1}) \cdots \le \left(1 + \log\frac{T}{\alpha}\right)\left(\frac{2e}{\sqrt{a}} + 1\right)^n + C_2$$

$\square$