TALK 2 : Weil height machine & Canonical Heights

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Last time: embed $C(\mathbb{Q}) \hookrightarrow J(\mathbb{Q}) \otimes \mathbb{R}$

Today: define inner product / height on

§ Heights on $\mathbb{P}^n$

If $x \in \mathbb{Q}$, $x = a/b$ $\gcd(a,b)=1$, $H(x) = \max(|a|,|b|)$

$$h(x) = \log H(x)$$

For $P \in \mathbb{P}^n(\mathbb{Q})$, $P = (a_0 : \cdots : a_n)$, a_i coprime integers

$$H(P) := \max |a_i|$$

$$h(P) = \log H(P)$$

If k is a number field, Σ_k = normalized abs. values such that $\prod_{v \in \Sigma_k} |x|_v = 1 \quad \forall\, x \in k^\times$

Given $P = (a_0 : \cdots : a_n) \in \mathbb{P}^n(k)$,

$$H(P) := \left(\prod_{v \in \Sigma_k} \max_i (|a_i|_v) \right)^{1/[k:\mathbb{Q}]} \qquad h = \log H$$

Agrees with previous defn for $k = \mathbb{Q}$.

Extends to $H, h : \mathbb{P}^n(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}$ "absolute Height"

<u>Theorem (Northcott)</u>: $\forall B, d \geq 1$:

$$\#\{P \in \mathbb{P}^n(\bar{\mathbb{Q}}) : H(P) \leq B,\ \deg P \leq d\} < \infty$$

(Idea: if $P = (1 : x_1 : \cdots : x_n)$, bound coefficients of the min polys of x_i)

<u>Theorem</u>: (i) If $\phi : \mathbb{P}^N \to \mathbb{P}^n$ morphism of deg d / $\bar{\mathbb{Q}}$
Then $\exists\, C = C(\phi)$ such that

$$|h(\phi(P)) - d\,h(P)| \leq C \qquad (*)$$

for all $P \in \mathbb{P}^n(\bar{\mathbb{Q}})$

(ii) Given $X \subset \mathbb{P}^N$ closed & $\phi : \mathbb{P}^N \dashrightarrow \mathbb{P}^n$ a rational map whose indeterminacy locus does not meet X, then $(*)$ holds for all $P \in X(\bar{\mathbb{Q}})$.

⌈Idea: if $\phi = (f_0 : \cdots : f_n)$, $P \in X(\bar{\mathbb{Q}})$, want

$$c_1 H(P)^d \leq H((f_0(P) : \cdots : f_n(P))) \leq c_2 H(P)^d$$

triangle inequality

Lower bound: if X defined by $G_1, \ldots, G_r$
Nullstellensatz: $\exists\, t \geq d$, hom polys Q_{ij} with

$$x_i^t = f_0 Q_{i0} + \cdots + f_n Q_{in} + \cdots G_r \cdot Q_{in+r}$$

Use triangle inequality to these equations! ⌋

§ Heights on varieties

k number field

X/k smooth projective variety

Given $f : X \to \mathbb{P}^n$, can define $h_f = h \circ f : X(\bar{k}) \to \mathbb{R}_{\geq 0}$

There's a bijection

$$\left\{ \begin{array}{c} \text{Morphisms} \\ X \to \mathbb{P}^n \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{line bundle } L \text{ on } X \\ + \text{ generating sections} \\ s_0, \dots, s_n \end{array} \right\} \Big/ \sim$$

$$f \longmapsto (f^*\mathcal{O}(1), f^*(x_0), \dots, f^*(x_n))$$

<u>Key lemma</u>: Suppose ~~$f, g : X \to \mathbb{P}$~~ $f : X \to \mathbb{P}^n$, $g : X \to \mathbb{P}^m$ has $f^*\mathcal{O}(1) \simeq g^*\mathcal{O}(1)$, then $\exists\, c = c(f,g)$ with

$$|h_f - h_g| \leq c$$

[Idea: may assume f associated to complete linear system

So

$$\begin{tikzcd} & \mathbb{P}^n \arrow[d, dashed] \\ X \arrow[ur] \arrow[r] & \mathbb{P}^m \end{tikzcd}$$

Now apply previous theorem to $\mathbb{P}^n \dashrightarrow \mathbb{P}^m$ ~~wt~~ which has degree 1

$\boxed{\text{Weil height machine}}$: there's a unique homomorphism

$$\mathrm{Pic}(X) \longrightarrow \left\{ \begin{array}{c} \text{Functions} \\ X(\bar{k}) \to \mathbb{R} \end{array} \right\} \Big/ \left\{ \begin{array}{c} \text{bounded} \\ \text{functions} \end{array} \right\}$$

$$L \longmapsto h_L$$

Such that if $f : X \to \mathbb{P}^n$ is associated to L, then $h_L = h_f + O(1)$

Moreover, ~~if~~

(i) If $\alpha: X \to Y$ is a morphism, $L \in \mathrm{Pic}(Y)$ then

$$h_{\phi^* L} = h_L \circ \phi + O(1)$$

(ii) If L is ample, $\forall B, d \geq 1$

$$\#\{P \in X(\bar{k}) : h(P) \leq B \ \& \ \deg P \leq d\} < \infty$$

Proof idea: write L as difference of two very ample line bundles $M_1 - M_2$ & set $h_L = h_{M_1} - h_{M_2}$

Shows uniqueness. Existence follows from

$$h_{M_1} - h_{M_2} = h_{M_1'} - h_{M_2'} + O(1) \quad (*)$$

But $h_{M_1} + h_{M_1'} = h_{M_1 + M_1'} + O(1)$ (Segre embedding)

and $M_1 + M_1' \cong M_2 + M_2'$ so $(*)$ follows

Heights on abelian varieties

A: abelian variety over a number field k

Aside: M, N abelian groups, a function $q: M \to N$ is a "quadratic map" if

$$(-,-)_q : M \longrightarrow N, \ (x,y) \mapsto q(x+y) - q(x) - q(y)$$

is bilinear. It's a quadratic form if $q(nx) = n^2 q(x)$ $\forall n \in \mathbb{Z}$ (equivalent to require $q(-x) = q(x)$)

$(-,-)_q$ bilinear means

$$q(x+y+z) - q(x+y) - q(x+z) - q(y+z) + q(x) + q(y) + q(z) = 0 \quad \forall x, y, z$$

Theorem of the cube: If $L \in \mathrm{Pic}(A)$, $p_i : A^3 \to A$

$p_{ij} = p_i + p_j$

$s = p_1 + p_2 + p_3$

Then

$$s^* L - p_{12}^* L - p_{13}^* L - p_{23}^* L + p_1^* L + p_2^* L + p_3^* L$$

is trivial

Consequence: $[m]^* L \simeq L^{\frac{m(m+1)}{2}} \otimes ([-1]^* L)^{\frac{m(m-1)}{2}}$

So if L is symmetric, $[m]^* L \simeq L^{m^2}$

anti-symmetric, $[m]^* L \simeq L^{m}$

So if L is symmetric, then h_L is (quadratic form) + (bounded function)

$\Rightarrow h_L(mP) = m^2 h_L(P) + O(1)$

Definition: If L on A symmetric, define

$$\hat{h}_L(P) = \lim_{n \to \infty} \frac{1}{4^n} h_L(2^n P)$$

Theorem: If L is symmetric,

(i) $\left\{ \frac{1}{4^n} h_L(2^n P) \right\}$ is Cauchy, so the limit exists.

(ii) $\hat{h}_L$ is a quadratic form $A(\bar{k}) \longrightarrow \mathbb{R}$

(iii) $\hat{h}_L = h_L + O(1)$

(iv) If L is ample, $\hat{h}_L$ satisfies a Northcott property, and

$$\hat{h}_L(P) \geqslant 0 \quad \forall P \in A(\bar{k})$$

with equality if and only if P is torsion.

Remarks:
- (iv) $\Rightarrow$ for L ample, then $\hat{h}_L$ induces a pos. def. quad form on $A(k) \otimes_{\mathbb{Z}} \mathbb{R}$
- For L antisymmetric, instead set $\hat{h}_L(P) = \lim \frac{1}{2^m} h_L(2^m P)$. Get linear $\hat{h}_L$. For general L, set

$$\hat{h}_L = \frac{1}{2} \hat{h}_{L+(-1)^*L} + \frac{1}{2} \hat{h}_{L-(-1)^*L}$$