Mordell Seminar Talk 1

$C/\mathbb{Q}$: curve of genus $g \geq 2$ (smooth, projective, geom. integral)

Example (Diophantus) $y^2 = x^6 + x^2 + 1$ genus 2

Conjecture (Mordell 1922): $C(\mathbb{Q})$ is finite

Proven by

- Faltings (1983)
- Vojta (1991), simplified by Bombieri
- Lawrence - Venkatesh (2020)

Open questions

- How to compute $C(\mathbb{Q})$?
- How large is $\# C(\mathbb{Q})$?

Uniformity Conjecture: For all $g \geq 2$, $\exists\, B_g > 0$ such that $\# C(\mathbb{Q}) \leq B_g$ $\forall\, C/\mathbb{Q}$ of genus g.

Remarks 1) Implied by Bombieri - Lang conjecture

2) If B_g exists then $B_g \geq 8g + 12$ (Mestre) & $B_2 \geq 642$ (Stoll)

What if we allow B_g to depend on more data?

Let

$J = \mathrm{Pic}^0_C$ = Jacobian variety of C,

an abelian variety of dimension g.

Mordell-Weil : $J(\mathbb{Q}) \simeq \mathbb{Z}^{\mathrm{rk}\, J} \oplus (\text{finite group})$

If $\alpha \in \mathrm{Pic}^1_C(\mathbb{Q}) = \mathrm{Pic}^1(C)$, get embedding

$$i_\alpha : C \hookrightarrow J, \quad P \mapsto [P] - \alpha$$

$i_\alpha(C) =$ "$C - \alpha$".

Theorem (Dimitrov-Gao-Habegger 2021): for every $g \geq 2$, there are $c_1, c_2 > 0$ (only depending on g) such that if

- ~~$/\overline{\mathbb{Q}}$~~ $C/\mathbb{C}$ curve of genus g
- $\Lambda \subset J(\mathbb{C})$ subgroup with $\mathrm{rk}(\Lambda) = \dim_{\mathbb{Q}}(\Lambda \otimes \mathbb{Q}) < \infty$
- $\alpha \in \mathrm{Pic}^1(C)$

Then

$$\#((C-\alpha) \cap \Lambda) \leq c_1(g) \cdot c_2(g)^{\mathrm{rk}\, \Lambda} \qquad (*)$$

Special cases:

1) C defined over $\mathbb{Q}$, $\Lambda = J(\mathbb{Q})$, then get

$$\# C(\mathbb{Q}) \leq c_1(g) \cdot c_2(g)^{\mathrm{rk}\, J}$$

2) $\Lambda = \cancel{J(\bar{\mathbb{Q}}}\, J(\mathbb{C})_{tors}$, so $rk(\Lambda)=0$ and get

$$\#((C(\mathbb{C})-\alpha)\cap J_{tors}) \leq c_1(g)$$

"Uniform Manin-Mumford".

Over function fields, Looper-Silverman-Wilms (2022) made $c_1(g)$ explicit. More recently

Theorem (Yu-Yuan-Zhou, 2026): in the above theorem, we can take

$$c_1(g) = 10^{13} g^8 \qquad c_2(g) = \cancel{(2}\ 1+\frac{1.25}{\sqrt{g}}$$

$$\Rightarrow \quad \#C(\mathbb{Q}) \leq 10^{13} g^8 \left(1+\frac{1.25}{\sqrt{g}}\right)^{rk J}$$

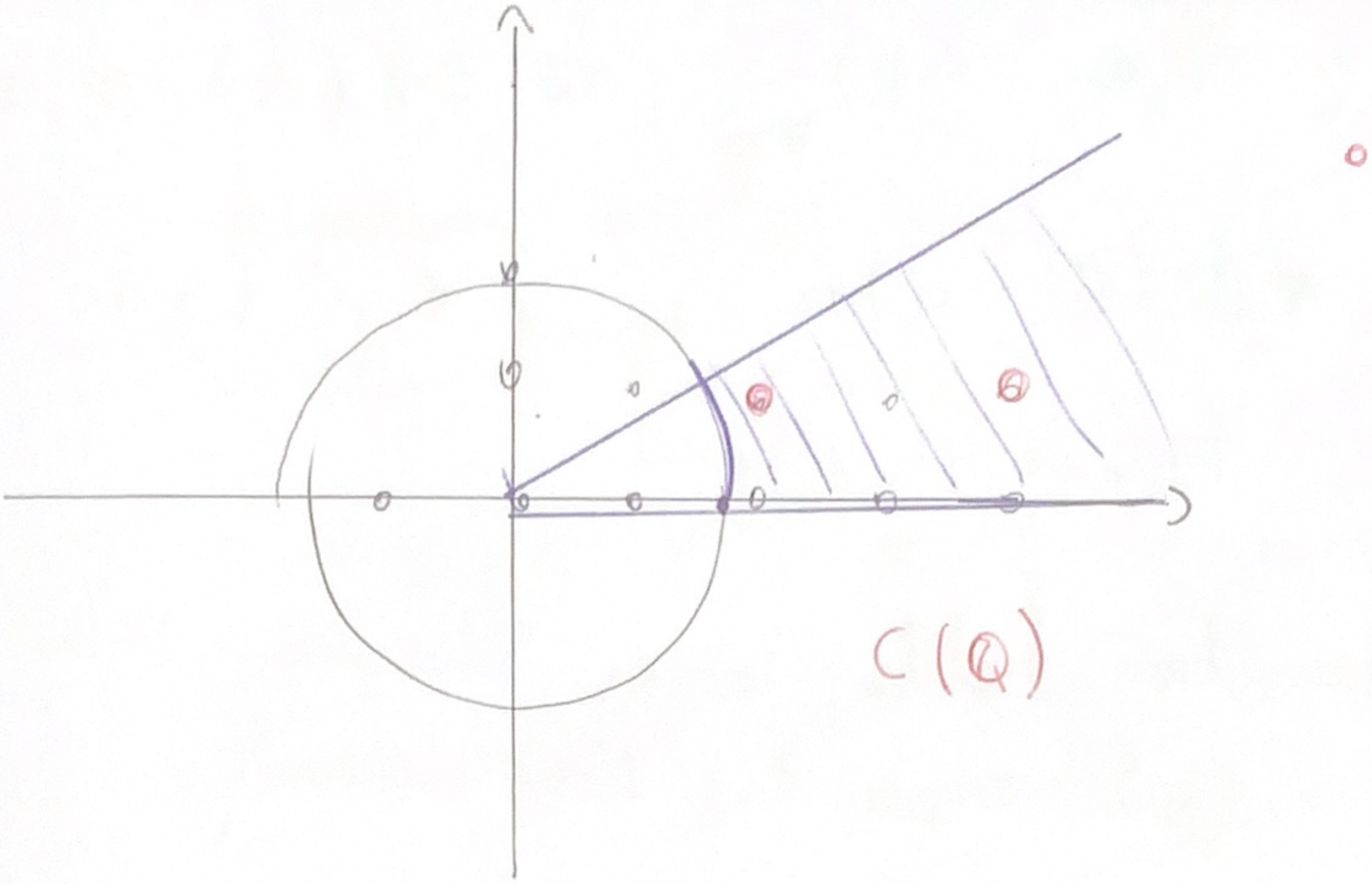
Proof method: refining Vojta's approach to Mordell conjecture.

Fix $\alpha \in Pic^1(C)$ to view $C \hookrightarrow J$. Have $J(\mathbb{Q})\otimes\mathbb{R} \simeq \mathbb{R}^{rkJ}$. The canonical height $\hat{h}: J(\bar{\mathbb{Q}}) \longrightarrow \mathbb{R}$ induces a pos. def. inner product

$$\langle D, E\rangle = \frac{1}{2}(\hat{h}(D+E) - \hat{h}(D) - \hat{h}(E))$$

on $J(\mathbb{Q})\otimes\mathbb{R}$. Let $|D| = \langle D,D\rangle^{1/2} = \hat{h}(D)^{1/2}$

Also $\theta_{D,E} \in [0, 180°)$ is so that $\cos\theta_{D,E} = \frac{\langle D,E\rangle}{|D||E|}$



Key idea: "if $P \neq Q \in C(\mathbb{Q})$ have $|P|, |Q|$ large and $\theta_{P,Q}$ small, then points "repel each other".

<u>Theorem</u>: Given $C/\mathbb{Q}$, there exist $R = R(C) > 0$ and $k = k(C) > 0$ such that for all $P \neq Q \in C(\overline{\mathbb{Q}})$ with $|Q| \geq |P| \geq R$ and

$$\cos\theta_P \frac{\langle P, Q\rangle}{|P||Q|} \geq \frac{3}{4} \qquad (\Leftrightarrow \theta_{P,Q} \leq 41.41^\circ)$$

we both have

- $|Q| \geq 2|P|$ (Mumford inequality '64)
- $|Q| \leq k|P|$ (Vojta's inequality '91)

Since $\mathbb{R}^{rk(J)}$ is covered by $\ll 7^{rk J}$ cones with angle 41° we get

$$\#\{P \in C(\mathbb{Q}) : |P| \geq R\} \ll 7^{rk J} \cdot (1 + \log_2 k)$$

"large points"

Since $\#\{P \overset{\text{small}}{\in} C(\mathbb{Q}) : |P| \leq R\} < \infty$ finite by Northcott, get $\#C(\mathbb{Q}) \leq \infty$.

Dimitrov - Gao - Habegger: give better bound for

$\#C(\mathbb{Q})$ ~~small~~ not-large

using uniform Bogomolov conjecture.

Yu - Yuan - Zhou: make everything $(R(C), \kappa(C), \ldots)$ explicit.

↑ Crucial technical tool: adelic line bundles and their heights / intersections.

Talks

7 May: Weil / canonical heights (Adam)

14 May: Mumford gap principle (Martin)

21 May: Vojta (Jack)

] pre - YYZ

28 May: Admissible adelic line bundles on curves (Holly)

4 June: Quantitative Mumford / Vojta (Bence)

11 June: Bogomolov. (Dmitri)

18 June: Putting everything together, sphere packing (??)