Study group Michaelmas 2020: ‘Modular curves and the Eisenstein Ideal’

In this study group we will go through some parts of the paper [Maz77]. We follow the strategy outlined by Andrew Snowden which can be found here. The lectures to which we reference below correspond to the lectures from that link.

October 8th: Overview

Discuss the aims of the study group, give an overview of the proof, place the result in a broader context and distribute the talks.
Speaker: Jef

October 15th: Finite group schemes over a field

Define a finite group scheme over a field and discuss their basic properties. Discuss the étale case, the connected-étale sequence, Cartier duality. Then discuss finite group schemes over a finite field: introduce Frobenius and Verschiebung. Give examples, notably the p-torsion of an elliptic curve in characteristic p.
References: Lecture 5 and 6 of Snowden.
Speaker: Marley

October 22nd: Finite group schemes over a DVR

Discuss finite flat group schemes over a general base, the finite étale case, the case of a DVR. Discuss quasi-finite étale group schemes over a DVR. State without proof Raynaud's prolongation theorem. Prove the corollary that the reduction map of an abelian variety is injective on torsion if $e < p - 1$.
References: Most of Lecture 7, start of Lecture 9 and end of Lecture 19 for last part.
Speaker: Amy

October 29th: Neron models

Define a Néron model of an abelian variety over a local field. Use it to define the reduction type of such an abelian variety. State (and maybe sketch a

proof?) of the Néron-Ogg-Shafarevich theorem, and its generalization to semistable reduction. Time permitting, give examples of the Néron model of $y^2 = x^3 + p^n$ over $\mathbb{Q}_p$ for various $n = 1, 2$.
References: Lecture 9, also the general reference [BLR90] for Néron models.
Speaker: Lukas

November 5th: Admissible group schemes

The background lectures have now ended, and we start by discussing actual content of the paper. Define admissible and pre-admissible group schemes. Describe the invariants of elementary admissible groups.
References: Lecture 11, as well as the original paper [Maz77].
Speaker: Guillem

November 12th: A criterion for rank zero

Complete the proof of 'Theorem B', which gives a criterion for an abelian variety to have rank zero. Start with the theory of modular curves, as much as time permits.
References: end of Lecture 11, Lecture 12 and 14 for the modular curve business.
Speaker: Jun

November 19th: Modular curves and their integral models

Continue the discussion of modular curves. In particular, describe the special fibre of $X_0(N)$ (for N prime) over $\mathbb{F}_N$. (Including the obligatory picture.) Mention its consequences for the Néron model of the Jacobian of $X_0(N)$. Define Hecke operators and state the Eichler-Shimura relation.
References: Lecture 14 and 15. The first sections of [Rib90] might also be useful.
Speaker: Dmitri

November 26nd: A criterion for non-existence of torsion points

Prove Theorem A from the introduction.
References: Lecture 18.
Speaker: Jef

December 3th: the Eisenstein ideal

Beginning of Step 3 from the introduction: define the Eisenstein ideal and prove some of its properties.
References: Lecture 20.
Speaker: Guillem

December 10th: Finishing up

Show that the Eisenstein quotient has rank zero, thereby completing the proof of the theorem.
References: Lecture 21.
Speaker: ???

References

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990.

[Maz77] Barry Mazur. Modular curves and the eisenstein ideal. *Publications Mathématiques de l'IHÉS*, 47:33–186, 1977.

[Rib90] K. A. Ribet. On modular representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ arising from modular forms. *Invent. Math.*, 100(2):431–476, 1990.