Modular Curves and their integral models

Dmitri Whitmore

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Quick recap of modular curves over C

- Recall for each $N \geq 1$ we have the following subgroups of $\Gamma(1) = SL_2(\mathbb{Z})$:
  $\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | c \equiv 0 \mod N\}$
  $\Gamma_1(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | c \equiv 0 \mod N, a, d \equiv 1 \mod N\}$
  $\Gamma(N) = \{A \in \Gamma(1) | A \equiv I \mod N\}$
- We say $\Gamma \subset \Gamma(1) = SL_2(\mathbb{Z}))$ is a congruence subgroup if it contains $\Gamma(N)$ for some $N \geq 1$
- $\Gamma(1)$ and hence any congruence subgroup Γ naturally acts on the upper half-plane $\mathcal{H}$ by Mobius transformations, and we can give the quotient $Y(\Gamma) = \mathcal{H}/\Gamma$ the structure of a (non-compact) Riemann surface
- We can compactify this by adding a finite number of points 'cusps' - one way to do this is by starting with the extended upper half-plane $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$ and defining a suitable structure on the quotient $X(\Gamma) = \mathcal{H}^*/\Gamma$ so that it becomes a compact Riemann surface

Moduli-theoretic interpretation

- Via the j-function, $Y(1)$ parameterises isomorphism classes of elliptic curves over $\mathbb{C}$, and similarly we have interpretations of the modular curves arising from the congruence subgroups we mentioned:
  - $Y_0(N)$ parameterises pairs (E, C) where E an elliptic curve over $\mathbb{C}$ and C a cyclic subgroup of $E(\mathbb{C})$ order N
  - $Y_1(N)$ parameterises pairs (E, P) where E an elliptic curve over $\mathbb{C}$ and P a point of order N
  - $Y(N)$ parameterises pairs $(E, (P, Q))$ where E an elliptic curve over $\mathbb{C}$ and (P, Q) is a pair of points of order N generating $E[N](\mathbb{C})$ (with Weil pairing a fixed Nth root of unity?)
- This suggests defining these modular curves algebraically by viewing them as moduli-spaces. This doesn't quite work however...

Moduli-theoretic interpretation

- Consider when $N = 1$, so we are interested in whether the functor $F_{\Gamma(1)} : S \mapsto \{\text{iso classes of elliptic curves } /S\}$ (defined on the fppf site of $\operatorname{Spec} \mathbb{Z}$ say) is representable by a scheme (here we take an elliptic curve over a general scheme S to be a smooth proper scheme E/S equipped with a section $0 \in E(S)$ such that each geometric fiber is a geometrically connected genus 1 curve)
- Recall however that there are elliptic curves which are not isomorphic over $\mathbb{Q}$ but are isomorphic over $\mathbb{C}$ (these quadratic twists biject with $\mathbb{Q}^*/(\mathbb{Q}^*)^2$)
- This implies our functor is not even a sheaf (as the map $F_{\Gamma(1)}(\mathbb{Q}) \to F_{\Gamma(1)}(\mathbb{C})$ is not injective), so can't be representable by any scheme

Some solutions

- Define for $N \geq 2$, invertible on S, a $\Gamma(N)$-structure on an elliptic curve E/S to be a pair (P, Q) where $P, Q \in E(S)[N]$ such that the pair (P, Q) defines an isomorphism of group schemes from $(\mathbb{Z}/N\mathbb{Z})_S$ to $E[N]$ (or equivalently that they give a basis for the N-torsion in each geometric fibre of S)
- It turns out if $N \geq 3$ then any automorphism which fixes E and the $\Gamma(N)$-structure is the identity
- Using this, one can show that the functor $F_{\Gamma(N)}$ (defined analogously on the fppf site of $\operatorname{Spec} \mathbb{Z}[1/N]$ say by sending S to iso classes of $\Gamma(N)$-structures) is representable by a smooth affine scheme $Y(N)$ for $N \geq 3$

Representability of $F_{\Gamma(N)}$

A sketch of the argument Snowden presents is as follows:

- we construct this affine scheme explicitly for some small values of N (e.g. $N = 3, 4$) together with a universal elliptic curve and $\Gamma(N)$-structure
- Use a relative representability lemma and the observation that to give a $\Gamma(3N)$ structure is the same as giving a $\Gamma(3)$ and $\Gamma(N)$ structure for $(N, 3) = 1$ to show $F_{\Gamma(3N)}$ is representable over $\mathbb{Z}[1/3N]$
- Observe that for $N \geq 4, (N, 3) = 1$ the action of $GL_2(\mathbb{Z}/3\mathbb{Z})$ on the $\Gamma(3)$-structure part is free, and that the quotient scheme represents $F_{\Gamma(N)}$ on $\mathbb{Z}[1/3N]$
- Use the $N = 4$ case to similarly show $F_{\Gamma(N)}$ is representable on $\mathbb{Z}[1/N]$ whenever $(N, 6) = 1$ (patch together along the open subschemes) - and then do some more work to remove this condition (details are omitted in his lecture here)

Coarse moduli space

- We define for $N \geq 2$ a $\Gamma_0(N)$ structure on E to be a closed etale subgroup $G \subset E$, which in every geometric fibre is cyclic of cardinality N
- The corresponding functor $F_{\Gamma_0(N)}$ can't be representable by a scheme then, as we always have a non-trivial automorphism of a $\Gamma_0(N)$ structure given by $[-1]$
- However, the sheafification of this functor is in fact representable by a smooth affine scheme $Y_0(N)$ defined over $\mathbb{Z}[1/N]$ (the coarse moduli space) and $Y_0(N)(\mathbb{C})$ can be identified with $\mathcal{H}/\Gamma_0(N)$
- The construction presented in Snowden's lectures went via stacks (a kind of object which will be able to represent the kind of functors we discussed) and then quotienting by the action of some finite group

Compact modular curves

- The curve we just constructed has a couple of problems for what we want to do: its defined over $\mathbb{Z}[1/N]$ rather than $\mathbb{Z}$ and is not compact (it doesn't contain the cusps)
- To fix this we'll define a slightly more general functor - we first need to make a couple of definitions:
- Let $n \geq 1$ and k a field. The standard n-gon over k, denoted C_n, is the quotient of $\mathbb{P}^1_k \times \mathbb{Z}/n\mathbb{Z}$ given by identifying (∞, i) with $(0, i+1)$. Note that the smooth locus is the group scheme $\mathbb{G}_m \times \mathbb{Z}/n\mathbb{Z}$, and this acts naturally on C_n
- A generalised elliptic curve over a scheme S is a triple $(E, e, +)$ where E/S a finite flat scheme, $e \in E(S)$ a section and $+ : E^{\text{sm}} \times E \to E$ such that $(E^{\text{sm}}, e, +)$ is a group scheme with $+$ defining an action on E, and all geometric fibres of E are elliptic curves or n-gons

$X_0(N)$

- We are now ready to define our functor - for simplicity in the definition we will now assume N is squarefree
- If E/S is a generalised elliptic curve, a $\Gamma_0(N)$-structure is given by $G \subset E^{\text{sm}}$ a closed subgroup of order N finite and flat over S; then our functor takes a scheme S to (the groupoid of) such pairs (E, G)
- Again this functor is representable only by a stack (this time defined over $\mathbb{Z}$ however); if we then take the coarse moduli space of the functor we now get a scheme $X_0(N)$ defined over $\mathbb{Z}$

The special fibre

- Now further assume N is prime. Since our scheme was defined over $\mathbb{Z}$ we can ask what does the special fibre $X_0(N)_{\mathbb{F}_N}$ look like
- We recall firstly the possibilities for the N-torsion of E an elliptic curve over $\overline{\mathbb{F}_N}$: either its ordinary and $E[N] \cong \mu_N \times \mathbb{Z}/N\mathbb{Z}$ or its supersingular and $E[N] \cong \alpha_{N^2}$ - the supersingular case occurs only for finitely many curves up to isomorphism
- It turns out to be given by gluing two copies of $X_0(1)_{\mathbb{F}_N}$ along the supersingular loci via the Frobenius map - we also recall that $X_0(1)$ is just $\mathbb{P}^1$ so this is two copies of $\mathbb{P}^1_{\mathbb{F}_N}$ glued along a finite number of points

Hecke operators

- Recall that $S_2(N)$ is the space of weight 2 cusp forms for the congruence subgroup $\Gamma_0(N)$, which may naturally be identified with $H^0(X_0(N), \Omega^1)$
- For $(n, N) = 1$ we can most easily define the Hecke operators T_n as endomorphisms of $S_2(N)$ by $T_n(f)(\Lambda) = \sum_{|\Lambda:\Lambda'|=n} f(\Lambda')$ (where we have identified points of $X_0(N)$ as lattices defining elliptic curves)
- We have the familiar facts that these family of operators commute, $T_{mn} = T_m T_n$ for $(m, n) = 1$, $T_{p^{n+1}} = T_p T_{p^n} - pT_{p^{n-1}}$ for p prime, $n \geq 1$ so the T_p for $(p, N) = 1$ prime generate them all, and we can compute the effect on $f = \sum_{n\geq 1}(a_n)q^n$ in terms of fourier expansions:
$$T_p f = \sum_{n\geq 1}(a_{pn} + pa_{n/p})q^n$$

Eigenforms

- The Hecke operators are self-adjoint with respect to the Peterssen inner product on $S_2(N)$, so this leads to a decomposition of $S_2(N)$ as a direct sum of eigenspaces for the Hecke operators
- We have multiplicity one, which states that if $f, g \in S_2(N)$ are normalised eigenforms (so $a_1(f) = a_1(g) = 1$) with the same eigenvalues on T_p for $(p, N) = 1$ then $f = g$ (recall we are still supposing N prime so we only have newforms appearing)
- If we let $\mathbb{T}$ be the $\mathbb{Z}$-subalgebra of $\mathtt{End}(S_2(N))$ generated by the T_p for $(p, N) = 1$ prime, then eigenforms for the Hecke operators correspond to algebra homomorphisms $\mathbb{T} \to \mathbb{C}$ by sending T_p to its eigenvalue
- We deduce $\mathbb{T}_{\mathbb{C}} = \mathbb{T} \otimes_{\mathbb{Z}} \mathbb{C}$ $S_2(N)$ is a direct product of copies of $\mathbb{C}$ indexed by the normalised eigenforms and that $S_2(N)$ is free of rank 1 as a $\mathbb{T}_{\mathbb{C}}$-module

Correspondences

- If C is a smooth projective curve, a correspondence $C \dashrightarrow C$ is a pair of finite maps $p_1, p_2 : C' \to C$ from a smooth projective curve C'
- If C is defined over $\mathbb{C}$ then a correspondence induces a map on both the singular cohomology $H^1(C, \mathbb{Z})$ via the composition $(p_2)_* \circ (p_1)^*$ (where the pushforward is defined via the pushforward on homology using Poincare duality)
- They also act on differential forms by a similar formula, and the isomorphism $H^1(C, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^0(C, \Omega^1) \oplus \overline{H^0(C, \Omega^1)}$ arising from Hodge theory is compatible with the map induced by a correspondence

Hecke Correspondences

- In our setting, if we take $C = X_0(N)$ and $(p, N) = 1$ a prime, then we define the Hecke correspondence by letting $C' = X_0(Np)$ and viewing a $\Gamma_0(Np)$ structure on an elliptic curve E as corresponding to a pair consisting of an N-isogeny $E' \to E$ (the kernel of which will be the subgroup of order N) and a subgroup $G \subset E$ of order p, then we define the maps p_1, p_2 as follows:

$$p_1(f : E' \to E, G) = (E, G)$$
$$p_2(f : E' \to E, G) = (E', f(G))$$

- Viewing $H^1(C, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = S_2(N) \oplus \overline{S_2(N)}$, the map induced by these Hecke correspondences matchup with the action of the Hecke operators

Consequences for the Hecke algebra

- If we let $\tilde{\mathbb{T}} = \mathbb{Z}[T_p : (p, N) = 1]$ be the polynomial algebra acting on $H^1(X_0(N), \mathbb{Z})$ by these correspondences, we see that an element acts as 0 iff it acts as 0 on $S_2(N)$ and hence its image in $\mathrm{End}(H^1(X, \mathbb{Z}))$ is isomorphic to $\mathbb{T}$ and (using the freeness result of $S_2(N)$ as a $\mathbb{T}_{\mathbb{C}}$-module) we deduce $H^1(X_0(N), \mathbb{Q})$ is a free $\mathbb{T}_{\mathbb{Q}} = \mathbb{T} \otimes \mathbb{Q}$-module of rank 2

Action on Jacobians and Eichler-Shimura

- Our next observation is that a correspondence $C \dashrightarrow C$ also induces a map on divisors, again by the same kind of formula, and hence a map on the Jacobian variety
- Even nicer, our above Hecke correspondences induce maps on the Jacobian $J_0(N)$ of $X_0(N)$, defined now over $\mathbb{Q}$
- For $(p, N) = 1$ prime then, $J_0(N)$ extends to an abelian scheme over $\mathbb{Z}_p$ and T_p also extends. If we then look in the special fibre, there is a formula for T_p known as the Eichler-Shimura relation:

$$T_p = F + V$$

  where F is the Frobenius and V the Verscheibung acting on $J_0(N)_{\mathbb{F}_p}$