Nov 12 @ Eisenstein Ideal Seminar

A Criterion for rank 0 & Modular Curves$/_{\mathbb{Q}}$, Misc.

Theorem B Let $N \neq 2$, $p \nmid N$ be primes, $A/\mathbb{Q}$ be an abelian variety

(1) with good reduction away from N,

(2) completely toric reduction at N,

(3) the Jordan-Hölder factors of $A[p]_{\bar{\mathbb{Q}}}$ all being $\mathbb{Z}/p$ or μ_p. and thus $A[p^n]$

Then $A(\mathbb{Q})$ has rank 0.

Rmk 0. reduction of AVs:

$$A/\mathbb{Q} \rightsquigarrow \text{Néron model } \mathcal{A}/\mathbb{Z}$$

$$\rightsquigarrow \mathcal{A}^{\circ}_{\mathbb{F}_\ell} = \begin{cases} \text{AV} & \text{- good reduction} \\ \text{torus} & \text{- completely toric} \\ \cdots & \end{cases}$$

1. Fontaine: there's no abelian variety $/\mathbb{Z}$ (could exist if $\mathbb{Z} \mapsto \mathcal{O}_K$)

A in practice: certain quotient of $\mathrm{Jac}(X_0(N))$

2. ((p,N)-) admissible group schemes:

Comm. group $G/\mathbb{Z}$ that's separated, flat, "pf", quasi-finite, killed by some p^n, finite$/\mathbb{Z}[\frac{1}{N}]$, such that $G_{\mathbb{Z}[\frac{1}{N}]}$ is an iterated extension of $\mathbb{Z}/p$ and μ_p.

via Raynaud's thm

Prototype: $A[p^n]$ for $A/\mathbb{Q}$ satisfying (1) and (3) in Theorem B.

4 Building blocks: $(\mathbb{Z}/p)^b \subseteq \mathbb{Z}/p$, $\mu_p^b \subset \mu_p$

Key property from last talk:

$\delta(G) := \log_p \# G_{\mathbb{Q}} - \log_p \# G_{\mathbb{F}_N}$, $\alpha(G) := \log_p \# G(\bar{\mathbb{F}}_p)$

$h^i(G) := \log_p \# H^i_{fppf}(\mathbb{Z}, G)$

$\Rightarrow h^1 - h^0 \leq \delta - \alpha$ for any admissible G

(Reason: δ, α additive, $h^1 - h^0$ subadditive, and ineq. holds for building blocks)

Proof of Theorem B

- Take $G = A^0[p]$.
- A has good reduction at p.

Admissibility + self-duality $\Rightarrow \alpha(G) = g$

- $A_{\overline{\mathbb{F}}_v} \cong \mathbb{G}_m^g \Rightarrow \log_p \# \mathcal{G}_{\overline{\mathbb{F}}_v} = g$, $\delta(G) = 2g - g = \alpha(G)$
- Additivity $\Rightarrow \delta = \alpha$ for $\mathcal{A}^\circ[p^n]$
- Key property $\Rightarrow$

$$\forall n,\ h^1(\mathcal{A}^\circ[p^n]) \leq \log_p \# \mathcal{A}^\circ(\mathbb{Z})[p^n]$$
$$\leq \log_p \# \mathcal{A}(\mathbb{Z})[p^\infty]$$
$$\overset{\text{NMP}}{=} \log_p \# A(\mathbb{Q})[p^\infty].$$

- Kummer sequence for fppf sheaves

$$0 \to \mathcal{A}^\circ[p^n] \to \mathcal{A}^\circ \xrightarrow{p^n} \mathcal{A}^\circ \to 0$$

$\leftarrow$ p-divisible, hence $[p^n]$ surj. for fppf sheaves.

$$\Rightarrow \mathcal{A}^\circ(\mathbb{Z})/p^n \hookrightarrow H^1_{\text{fppf}}(\mathbb{Z}, \mathcal{A}^\circ[p^n])$$

$\Rightarrow$ $\mathcal{A}^\circ(\mathbb{Z})/p^n$ is bdd as $n \to +\infty$, $\mathcal{A}(\mathbb{Z}) = A(\mathbb{Q})$ has rank 0

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Modular Curves Misc.s

$/\mathbb{C}$: $\Gamma \subset SL_2(\mathbb{Z})$ congruence subgp.

$\rightsquigarrow$ quasi-proj. curve$/\mathbb{C}$ $Y_\Gamma := \Gamma \backslash \mathcal{H}$, $j(z) \in \mathcal{O}_{Y_\Gamma}$

<u>Compactification</u>: $X_\Gamma = Y_\Gamma \sqcup \Gamma \backslash \mathbb{P}^1(\mathbb{Q})$ proj. curve $\quad \mathbb{P}^1(\mathbb{Q}) = GL_2(\mathbb{Q})/B(\mathbb{Q}) = \{\mathbb{Q}\text{-parabolics of } GL_2\}$

<u>Enumeration of cusps</u>: cf. Diamond–Shurman, §3.8

$$\#(X_0(N) - Y_0(N)) = \sum_{d \mid N} \phi(\gcd(d, N/d)) = 2 \text{ when } N \text{ is prime}$$

$$\#(X_1(N) - Y_1(N)) = \frac{1}{2} \sum_{d \mid N} \phi(d)\phi(N/d) \text{ when } N=3 \text{ or } \geq 5$$
$$= N-1 \qquad \text{when } N \text{ is an odd prime}$$

<u>Example</u>: $X_1(11)$ has 10 cusps

<u>Manin–Drinfeld</u>:

Differences of cusps are always torsions in $\mathrm{Jac}(X_\Gamma)$.

$/\mathbb{Q}$:

- <u>$Y_0(N)$</u>: the (monic) minimal polynomial of $j(Nz)$ over $\mathbb{C}(j)$ lies in $\mathbb{Z}[j, t]$. This polynomial defines $Y_0(N)/_{\mathbb{Q}}$.
- General principle: moduli interpretation $\rightsquigarrow$ natural $\mathbb{Q}$-structure & integral models

(cf. for general Shimura varieties, rational structure $\leftarrow$ Galois action on special points)

<u>Example</u>: 5 of the 10 cusps of $X_1(11)$ are rational and they make the M-W group $\cong \mathbb{Z}/5$ of this EC.

The other 5 make a $\mathrm{Gal}(\mathbb{Q}(\zeta_{11})^+/\mathbb{Q})$-orbit.